PROBLEM 1

We will solve an initial value problem of the ODE

$$y'' + 4y = x \sin 2x.$$

(a) Find the general solution y_h for the homogeneous equation $y'' + 4y = 0$.. Check, using Wronskian, that your solutions are linearly independent. (10)

(b) Find a particular solution for the original nonhomogeneous equation. (15)

(c) Write down the general solution for the nonhomogeneous equation. (10)

(d) Solve the initial value problem for the above ODE with initial conditions $y(0) = 4$ and $y'(0) = 2$. (10)

(a) $y'' + 4y = 0$ char eq: $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

$y_h = c_1 \cos 2x + c_2 \sin 2x$

$W(y_1, y_2)(x) = y_1 y_2' - y_1' y_2 = 2\cos^2(2x) + 2\sin^2(2x) = 2 \neq 0$ ✓

(b) $y_p = (Ax^2 + Bx)\cos 2x + (Cx^2 + Dx)\sin 2x$

$y_p' = (2Ax + B)\cos 2x - 2(Ax^2 + Bx)\sin 2x + (2Cx + D)\sin 2x + 2(Cx^2 + Dx)\cos 2x$

$y_p'' = 2A x \cos 2x - 4(2Ax + B)\sin 2x - 4(Ax^2 + Bx)\cos 2x$
$\quad + 2Cx \sin 2x + 4(2Cx + D)\cos 2x - 4(Cx^2 + Dx)\sin 2x$

plug into PDE $\Rightarrow$

$(2A + 4(2Cx + D))\cos 2x + (2C - 4(2Ax + B))\sin 2x$

$= x \sin 2x \Rightarrow \begin{cases} -8A = 1 \\ 8C = 0 \\ -4B + 2C = 0 \\ 2A + 4D = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{8} \\ B = C = 0 \\ D = \frac{1}{16} \end{cases}$

$\Rightarrow y_p = -\frac{1}{8} x^2 \cos 2x + \frac{1}{16} x \sin 2x$

(c) $y = y_p + y_h = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{8} x^2 \cos 2x + \frac{1}{16} x \sin 2x$

(d) $\begin{cases} y(0) = c_1 = 4 \\ y'(0) = 2c_2 = 2 \end{cases} \Rightarrow \begin{cases} c_1 = 4 \\ c_2 = 1 \end{cases} \Rightarrow y = (4 - \frac{1}{8}x^2)\cos 2x + (1 + \frac{x}{16})\sin 2x$

PROBLEM 2

Solve the following ODE via power series method.

$$y' + y = x + x^2.$$

5 (a) Explain that 0 is an ordinary (regular) point. That is, the power series method applies to this ODE.

13 (b) Write down the recurrence relation for the coefficients.

5 (c) Since this is an ODE of order 1, how many arbitrary constants are there in the general solution?

12 (d) Compute the first 4 coefficients in terms of the arbitrary constant(s).

5 (a) The coeff's of $y \overset{=1}{}$ & $g(x) = x + x^2$ are both polynomials, hence power series.

(b) $y = \sum_{m=0}^{\infty} a_m x^m$

$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$ 5pt

LHS $= y' + y = \sum_{m=0}^{\infty} [(m+1) a_{m+1} + a_m] x^m$

RHS $= x + x^2$

$\Rightarrow \boxed{a_{m+1} = \frac{-a_m}{m+1} \quad \text{for} \quad m \geq 3}$ 5pt

5 (c) 1 constant a_0

(d)

coeff of	LHS	RHS
1	$a_0 + a_1$	0
x	$a_1 + 2a_2$	1
x^2	$a_2 + 3a_3$	1
x^3	$a_3 + 4a_4$	0

10 pt

$a_1 = -a_0$

$a_2 = \frac{1+a_0}{2}$ $\quad \frac{1-a_1}{2}$

$a_3 = \frac{1-a_0}{6}$ $\quad \frac{1-a_2}{3} = \frac{1-\frac{1}{2}(1+a_0)}{3}$

$a_4 = \left(\frac{-1}{4}\right) \frac{1-a_0}{6} = \frac{-1+a_0}{24}$

PROBLEM 3

Solve the following equation by the method of the characteristic curves.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Verify your solution (by plugging it back to the PDE.)

characteristic curve eq

15 5pt $\frac{dy}{dx} = \frac{y}{x} \Rightarrow y = cx$ 5pt

$\boxed{u(x,y) = f\left(\frac{y}{x}\right)}$ for any diff ~~eq~~ fun f 5pt

5 Verify. $u_x = -\frac{y}{x^2} f'\left(\frac{y}{x}\right)$, $u_y = \frac{1}{x} f'\left(\frac{y}{x}\right)$ $\Rightarrow x u_x + y u_y = 0$ ~~(5pt)~~

(5pt)

Another possibility.

$$u(c) = u(\ln y - \ln x)$$

$$\frac{\partial u}{\partial x} = u'\left(-\frac{1}{x}\right) \qquad \frac{\partial u}{\partial y} = u'\left(\frac{1}{y}\right)$$