§7.1 Fourier integral representation | HW #1, 5, 13(a)

· $f(x)$ is T-periodic ~~with period~~ & piecewise smooth (부드럽게)

then $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi}{T}x + b_n \sin\frac{2n\pi}{T}x\right)$

where $a_0 = \frac{1}{L}\int_c^{c+T} f(x)\,dx$, $a_n = \frac{2}{L}\int_c^{c+T} f(x)\cos\frac{2n\pi}{T}x\,dx$

$b_n = \frac{2}{L}\int_c^{c+T} f(x)\sin\frac{2n\pi}{T}x\,dx$

$T = 2\pi$

· If f is NOT periodic, then

Thm f is piecewise smooth & $\int_{-\infty}^{\infty}|f(x)|\,dx < \infty$

then $f(x) = \int_0^{\infty}[A(w)\cos wx + B(w)\sin wx]\,dw$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)\,dx$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)\,dx$

Rmk This can be considered as a continuous version of Fourier series. (cf. Riemann sum & integral)

Example 1: $f(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{o.w.} \end{cases}$

· Find Fourier integral representation

Sol · $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos wx\,dx = \frac{1}{\pi}\int_{-1}^{1}\cos(wx)\,dx = \frac{1}{\pi}\left[\frac{\sin wx}{w}\right]_{-1}^{1}$

$= \frac{2\sin w}{\pi w}$ (if $w \neq 0$)

If $w=0$ $A(0) = \frac{2}{\pi} = \lim_{w\to 0} A(w)$

· $B(w) = 0$ since $f(x)$ is even.

$\Rightarrow f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin w}{w}\cdot\cos(wx)\,dw$ ※

Rmk: ① F.I.R. is continuous & will "smooth out" discontinuity. For example

F.I.R.:
$$\left[\frac{2}{\pi}\int_0^\infty \frac{\sin w \cos(wx)}{w}\,dw\right] = \begin{cases} 1 & \text{if } |x|<1 \\ \frac{1}{2} & \text{if } |x|=1 \\ 0 & \text{if } |x|>1 \end{cases} \quad \frac{f(x+)+f(x-)}{2}$$

② It can be used to evaluate some improper integrals. For example, set $x=0$

$$\int_0^\infty \frac{\sin w}{w}\,dw = \frac{\pi}{2} \quad \text{(Dirichlet integral)}$$

$x=1$
$$\int_0^\infty \frac{\sin w \cos w}{w}\,dw = \frac{\pi}{4}.$$

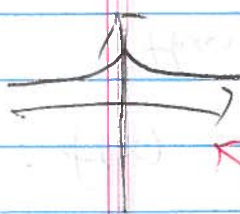
Exc 5. $f(x) = e^{-|x|}$

· even fcn. No sine $\Rightarrow B(w)=0$.

· $A(w) = \frac{2}{\pi}\int_0^\infty e^{-x}\cos(wx)\,dx = \frac{2}{\pi}\frac{1}{1+w^2}.$

$$\int e^{-x}\cos(wx)\,dx = \left[-e^{-x}\cos wx - w\int e^{-x}\sin(wx)\,dx\right]$$
$$= -e^{-x}\cos wx + w\,e^{-x}\sin(wx) - w^2\int e^{-x}\cos(wx)\,dx$$
$$\Rightarrow (1+w^2)\int e^{-x}\cos(wx)\,dx = -e^{-x}\cos(wx) + w\,e^{-x}\sin wx$$

$$\Rightarrow e^{-|x|} = \frac{2}{\pi}\int_0^\infty \frac{1}{1+w^2}\cos(wx)\,dx$$



since $e^{-|x|}$ is continuous everywhere. ※

§7.2 Cpx Form & Fourier transform
series

§7.2 Cpx Form & Fourier transform

~~Fourier transform~~ series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\frac{2\pi}{T}x}$$

$T \approx 2\pi$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x) e^{-in\frac{2\pi}{T}x}\,dx$$

$$= \sum_{n=-\infty}^{\infty}\left(\frac{1}{T}\int_{-T/2}^{T/2} f(x)\, e^{-in\frac{2\pi}{T}x}\,dx\right) e^{in\frac{2\pi}{T}x}$$

make "n" a continuous $\rightsquigarrow \omega$.

$$\sim \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \underbrace{\left(\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\,dx\right)}_{= \sqrt{2\pi}\,\hat{f}(\omega)} e^{i\omega x}$$

Not $0<\omega<\infty$

Fourier transform: $\hat{f}(\omega) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i\omega x}\,dx$ $\quad -\infty<\omega<\infty$

inverse Fourier transform $\to$ $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\,d\omega$

Ex 1 $\quad f(x) = \begin{cases} 1 & -a<x<a \\ 0 & \text{o.w.} \end{cases}$

$(\omega \neq 0)$ $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-i\omega x}\,dx$

$$= \left[\frac{-1}{\sqrt{2\pi}\, i\omega} e^{-i\omega x}\right]_{-a}^{a} \left(= \sqrt{\frac{2}{\pi}}\,\frac{\sin a\omega}{\omega}\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{ia\omega} - e^{-ia\omega}}{i\omega}\right)$$

$\omega = 0$ $\quad \hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} dx = a\sqrt{\frac{2}{\pi}} = \lim_{\omega\to 0}\hat{f}(\omega)$

$f(x) =$ $\Rightarrow \hat{f}(\omega)$ cts at $\omega = 0$ (& everywhere)

$$\left(f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \frac{1}{\sqrt{2\pi}} \frac{e^{ia\omega} - e^{-ia\omega}}{i\omega} \, d\omega \right.$$

$$\left. = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i(a+1)\omega x}}{\omega} \, d\omega - \int_{-\infty}^{\infty} \frac{e^{i(1-a)\omega x}}{\omega} \, d\omega \right)$$

In "real" world

$$\left.\begin{matrix} 1 \\ \frac{1}{2} \\ 0 \end{matrix}\right\} = \frac{f(x^+) + f(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\underbrace{\cos \omega x}_{\text{even}} + i \underbrace{\sin \omega x}_{\text{odd}}) \sqrt{\frac{2}{\pi}} \underbrace{\frac{\sin a\omega}{\omega}}_{\text{even}} \, d\omega$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(\omega x) \sin(a\omega)}{\omega} \, d\omega$$

(cf. Ex 1 of the previous section), where $a = 1$)

Ex2

$$f(x) = \begin{cases} e^{-x} & , \text{ if } x \geq 0 \\ 0 & , \text{ if } x \leq 0 \end{cases} \leftarrow \text{real-valued}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} e^{-i\omega x} \, dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(1+i\omega)x} \, dx$$

$$= \frac{-1}{\sqrt{2\pi}(1+i\omega)} e^{-x(1+i\omega)} \Big]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}(1+i\omega)} \leftarrow \text{cpx-valued}$$

Notations: $\mathcal{F}(f) := \hat{f}$

Facts ① $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$

Facts. ① $\mathcal{F}(af+bg) = a\mathcal{F}(f)+b\mathcal{F}(g)$

due to $\boxed{\text{linearity}}$ of integration

② $\mathcal{F}(f') = (i\omega)\mathcal{F}(f)$ $\quad \frac{d}{dx}$ derivative $\longleftrightarrow$ multiplication $i\omega$

either by integ. by parts OR inverse Fourier transf.

$\mathcal{F}(f'') = (i\omega)^2\mathcal{F}(f) = -\omega^2\mathcal{F}(f)$

$\mathcal{F}(f^{(n)}) = (i\omega)^n\mathcal{F}(f)$ $\qquad \frac{1}{i\cdot i} = \frac{1}{-1}$

Rmk: $i=\sqrt{-1}$, $\quad i^2=-1$, $\quad i^3=-i$, $\quad i^4=1$, $\quad \frac{1}{i} = \frac{i^4}{i} = i^3 = -i$

③ $\underbrace{\mathcal{F}(xf(x))}(\omega) = i[\hat{f}]'(\omega) = i\frac{d}{d\omega}\left(\mathcal{F}(f)(\omega)\right)$, $\quad \mathcal{F}(f) = \hat{f}$

$\mathcal{F}(x^n f(x))(\omega) = i^n[\hat{f}]^{(n)}(\omega)$ $\qquad x \longleftrightarrow i\frac{d}{d\omega}$

Pf: ② $\mathcal{F}(f') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\,dx$ $\quad$ (integ by parts)

$$= \frac{1}{\sqrt{2\pi}}\Big[\underbrace{f(x)}_{\substack{\downarrow\, x\to\pm\infty \\ 0}}e^{-i\omega x}\Big]_{x=-\infty}^{\infty} - \frac{-i\omega}{\sqrt{2\pi}}\int f(x)e^{-i\omega x}\,dx$$

integrable

③ $i[\hat{f}]'(\omega) = i\frac{d}{d\omega}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$

$= \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)(-ix)e^{-i\omega x}\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(xf(x))e^{-i\omega x}\,dx$

$= \mathcal{F}(xf(x))$. ✕

Convolution of fcns

usually don't have this

$$(f*g)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

Thm 4: $\mathcal{F}(f*g) = \mathcal{F}(f)\,\mathcal{F}(g)$

Pf: $\mathcal{F}(f*g)(\omega) = \mathcal{F}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t) g(t)\, dt\right)(\omega)$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{x=-\infty}^{\infty} \left(\int_{t=-\infty}^{\infty} f(\underbrace{x-t}_{=u})\, g(t)\, dt\right) e^{-i\omega \overset{(u+t)}{x}}\, \underbrace{dx}_{du}$$

$$= \left(\int_{u=-\infty}^{\infty} f(u) e^{-iu\omega}\, du\right)\left(\int_{t=-\infty}^{\infty} g(t) e^{-i\omega t}\, dt\right)$$

$$= \mathcal{F}(f)\,\mathcal{F}(g) \quad ※$$

Thm 5: $\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right) = \frac{1}{\sqrt{a}} e^{-\frac{\omega^2}{2a}}$

Pf: $\underbrace{e^{-\frac{ax^2}{2}}}_{=f(x)}$

$f(x)$ satisfies $f'(x) + a x f(x) = 0$

Fourier transf: $i\omega \hat{f}(\omega) + a \frac{d}{d\omega} \hat{f}(\omega) = 0$

i.e. $\frac{d}{d\omega} \hat{f}(\omega) + \frac{\omega}{a} \hat{f}(\omega) = 0$

$$\Rightarrow \hat{f}(\omega) = C e^{-\frac{\omega^2}{2a}}$$

To determine C. Note that $C = \hat{f}(0)$

$u = \sqrt{\frac{a}{2}}\, x$

$$\hat{f}(0) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx = \frac{1}{\sqrt{a\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-u^2} du}_{\sqrt{\pi}} = \frac{1}{\sqrt{a}}$$

$$\left\{ \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy \right.$$

$$\left. \underset{\text{polar coord}}{=} \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\phi = \frac{2\pi}{2} \int_{r=0}^{\infty} e^{-r^2} dr^2 = \pi \right\}$$

Rmk: when $a=1$ $\mathcal{F}\left(e^{-\frac{x^2}{2}}\right) = e^{-\frac{\omega^2}{2}}$.

Rmk: when $\underline{a=1}$ $\mathcal{F}(e^{-\frac{x^2}{2}}) = e^{-\frac{w^2}{2}}$.

identical.

$\boxed{\text{HW \# 1, 5}}$, ~~10 (9, 16)~~

D,3 case

Ex

$\boxed{\text{HW \# 10, 14, 20}}$

§7.3 Fourier transform method in PDE

~~HW §7.2 #10, 12, 20~~

- When the region (or 1 dim of the region) is very large for all practical purpose, it's ∞.
  $\Rightarrow$ Fourier series NOT so well-suited.
  Fourier transform is.

- $u(x,t)$ $\begin{cases} -\infty < x < \infty & \text{(entire real line)} \\ 0 < t \end{cases}$

  Define $\mathcal{F}(u(x,t))(\omega) = \hat{u}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx$

  DO NOTHING about t.

Partial derivatives

- $\mathcal{F}\left(\frac{\partial^n}{\partial t^n} u(x,t)\right)(\omega) = \frac{\partial^n}{\partial t^n} \hat{u}(\omega, t)$ ← Diff. under the integral sign

  ← write $\left(\frac{d}{dt}\right)^n$ Proof

- $\mathcal{F}\left(\frac{\partial^n}{\partial x^n} u(x,t)\right)(\omega) = (i\omega)^n \hat{u}(\omega, t)$ ← Integration by parts

Can use Fourier transform to solve PDE w/ "$L \to \infty$"

Ex 1 The wave eqn w/ "$L = \infty$" $\rightsquigarrow$ No b.c.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Assume f & g have Fourier transform.

$\rightarrow$ Solve B.V.P. via Fourier transform.

Sol'n. After Fourier transf $\Rightarrow$ t = variable

Sol'n. After Fourier transf. $\Rightarrow$

① Fourier transform
$$\begin{cases} \frac{d^2}{dt^2}\hat{u}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \\ \frac{d}{dt}\hat{u}(\omega,0) = \hat{g}(\omega) \end{cases}$$

$\leftarrow$ ODE in t (t = variable) w/ "parameter ω"

② solving ODE
$$\Rightarrow \hat{u}(\omega,t) = A(\omega)\cos(c\omega t) + B(\omega)\sin(c\omega t)$$

③ Impose i.c.
$$\left.\begin{array}{l} \hat{u}(\omega,0) = A(\omega) = \hat{f}(\omega) \\ \frac{d}{dt}\hat{u}(\omega,0) = c\omega B(\omega) = \hat{g}(\omega) \end{array}\right] \Rightarrow \hat{u}(\omega,t)$$

④ Inverse Fourier transf.
$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\hat{f}(\omega)\cos(c\omega t) + \frac{\hat{g}(\omega)}{c\omega}\sin(c\omega t)\right]e^{i\omega x}\,d\omega$$

\#

Ex2 heat eq'n $L=\infty$

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t>0$$

$$u(x,0) = f(x) \qquad \text{i.c.}$$

Sol'n:

① $\begin{cases} \frac{d}{dt}\hat{u}(\omega,t) = c^2(-\omega^2)\hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}$

② $\hat{u}(\omega,t) = e^{-c^2\omega^2 t}A(\omega)$

③ $\hat{u}(\omega,0) = A(\omega) = \hat{f}(\omega)$

④ $u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{-c^2\omega^2 t}\,e^{i\omega x}\,d\omega$

\#

HW	§7.2 #10, 19, 20
	§7.3 #1, 3, 9

$$\mathcal{F}\left(\sqrt{\tfrac{\pi}{2}}\, e^{-|x|}\right) = \frac{1}{2}\int_{-\infty}^{0} e^{x} e^{iwx}\,dx + \frac{1}{2}\int_{0}^{\infty} e^{-x} e^{iwx}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{(1+iw)x}}{1+iw}\right]_{-\infty}^{0} + \frac{1}{2}\left[\frac{e^{(-1+iw)x}}{-1+iw}\right]_{0}^{\infty}$$

$$= \frac{1}{2}\left[\frac{1}{1+iw} - \frac{1}{-1+iw}\right]$$

$$= \frac{-1+\cancel{iw}-1-\cancel{iw}}{-2(1+iw)(1-iw)} = \frac{1}{1+w^2}$$