§3.1 – ~~§3.2~~

linear

- The general PDE of order 2 in 2 variables

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$$

$A, B, \ldots, G$ are fcn's of (x, y).

- As discussed before, to find a unique sol'n.

Need $\begin{cases} 2 \text{ boundary conditions} \\ 2 \text{ initial conditions} \end{cases}$

~~1 bdy + 1 initial~~ initial value problem

⇝ boundary value problem / (I.V.P.)
(B.V.P.)

- Most common PDE of the above type.

- Laplacian or Laplace operator

$$\left.\begin{array}{l} \Delta u(x,y) \\ \text{or } \nabla^2 u(x,y) \end{array}\right\} := \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u$$

(i) ~~2~~1D wave eq'n

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad u = u(x, t)$$

(ii) 1D heat eq'n $\quad \dfrac{\partial u}{\partial t} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0$

(iii) Laplace eq'n $\quad \Delta u = 0$

(iv) Poisson eq'n $\quad \Delta u(x,y) = G(x,y)$

Superposition principal (recalled)

- If the PDE is linear & homogeneous then (a sol'n) + (another sol'n) = (a new sol'n)

e.g. If $\begin{cases} \Delta u_1(x,y)=0, \\ \Delta u_2(x,y)=0 \end{cases} \Rightarrow \Delta(u_1+u_2)=0$

- However if $\begin{cases} \Delta u_1 = G_1 \\ \Delta u_2 = G_2 \end{cases}$ nonhomog. $\Rightarrow \Delta(u_1+u_2) = G_1+G_2$

- If the eq'n is non-linear, superposition fails.

Note: Linearity means: In the PDE, each term involve only 1 factor of u or its derivative.

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \\ &\frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial x} + u = 0 \end{aligned}\right\} \text{ Non linear;}$$

$$\frac{\partial^2 u}{\partial x^2} + e^t \frac{\partial^2 u}{\partial t^2} = \cos(x)\, u \quad \leftarrow \text{linear!}$$

HW: §3.1 # 1, 4, 5, 6, 9.

§3.3 Method of separation of Variables | Sol'n of 1D wave eq...

§3.3 Method of separation of Variables

Sol'n of 1D wave eqn

1D Wave eq'n $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.

b.c.

subject to boundary conditions : $\left\{ u(0,t)=0,\ u(L,t)=0.\ \forall t \right.$

ends of string are fixed

& initial condition : $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$

i.c.

initial shape

$0 < \forall x < L$

initial velocity

In this section we'll solve this w/ sep. of var. (SOV)

Step 1 : $u(x,t) = X(x)\,T(t)$

$\Rightarrow \frac{\partial^2 u}{\partial t^2} = X\,T''$, $\frac{\partial^2 u}{\partial x^2} = X''\,T$

$\Rightarrow$ PDE becomes $X\,T'' = c\,X''\,T$

$\Leftrightarrow \frac{T''}{cT} = \frac{X''}{X} =: k$

$\Rightarrow \boxed{\begin{cases} X'' = kX \\ T'' = ckT \end{cases}}$ ← 2 ODE's ! (NOT PDE)

$\boxed{\text{b.c.}}$ $X(0)\,T(t) = 0 = X(L)\,T(t)$

$\Rightarrow \boxed{X(0) = X(L) = 0}$

N.B. $\begin{cases} X'' = \frac{d^2 X(x)}{dx^2} \\ T'' = \frac{d^2 T(t)}{dt^2} \end{cases}$

$\boxed{\text{Step 2}}$ Solving ODE's

$X'' - kX = 0$ (similarly $T'' - kc^2T = 0$)

(2nd order homog.)

$\lambda^2 - k = 0$

$\boxed{\text{Case (i)} \quad \text{If } k > 0}$: $\lambda = \pm\sqrt{k}$

$X = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}.$

$\boxed{\text{b.c}}$ $\begin{cases} X(0) = c_1 + c_2 = 0 \\ X(L) = c_1 e^{\sqrt{k}L} + c_2 e^{-\sqrt{k}L} = 0 \end{cases}$

$\Rightarrow (-c_2)(\underset{\substack{\neq \\ 0}}{e^{\sqrt{k}L}})(\underset{\substack{\times \\ 0}}{1 - e^{-kL}}) = 0$ $k \neq 0$. $L \neq 0$

$\Rightarrow c_2 = 0 \Rightarrow X = 0 \rightarrow \Leftarrow$

$\boxed{\text{Case (ii)}}$ $\boxed{k = 0}$ $X'' = 0 \Rightarrow X = c_1 x + c_2$

$\boxed{\text{b.c.}}$ $\Leftrightarrow$ $c_2 = 0$ & $c_1 = 0$. $\Leftrightarrow X = 0 \rightarrow \Leftarrow$

$\boxed{\text{Case (iii)} \quad k < 0}$ $X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$

i.e. $k = -\mu^2$ $\boxed{\text{b.C}}$ $\begin{cases} X(0) = 0 \Leftrightarrow c_1 = 0 \\ X(L) = 0 \Leftrightarrow c_2 \sin \mu L = 0 \end{cases}$

$\Rightarrow \mu = \frac{n\pi}{L} =: \mu_n$

$\Rightarrow$ gen. soln : $X_n(x) = \underbrace{c_2 \sin \frac{n\pi}{L} x}_{\substack{\parallel \\ X_n}}$ $n = \pm1, \pm2, \pm3, \ldots$

Similarly : $T'' - kc^2 T = 0$ $k < 0$.

Similarly: $T'' - kc^2T = 0 \quad k < 0$
$k = -\mu^2$

in book b_n / in book b_n^*

$$T = T_n = a_n \cos\left(c\frac{n\pi}{L}t\right) + b_n \sin\left(c\frac{n\pi}{L}t\right)$$

Fourier series?

Summary: ① All possible sol'ns are of the form

$$u_n(x,t) = \sin\frac{n\pi}{L}x\left(\overset{b_n}{a_n}\cos c\frac{n\pi}{L}t + \overset{b_n^*}{b_n}\sin\left(c\frac{n\pi}{L}t\right)\right)$$

② By superposition principle, a general sol'n

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

arb. const. Need to fix them

Step 3 Fourier Series sol'n of the Entire Problem

$\boxed{\text{I.C}}$ (1) $u(x,0) = f(x)$ ~~Fourier series~~ ~~$\sum a_n \sin\left(\frac{n\pi}{L}x\right)$~~

~~b.c. $\Rightarrow$ x~~ dir.

$$= \sum \overset{}{a_n} \sin\left(\frac{n\pi}{L}x\right)$$

Fourier series

b_n

$$\boxed{\overset{b_n}{a_n} = \frac{2}{L}\int_0^L f(x)\left(\sin\frac{n\pi}{L}x\right)dx}$$

(2) $u'(x,0) = g(x)$

~~b.c. $\Rightarrow$ t~~ dir.

$$= \sum_n \left(\sin\frac{n\pi}{L}x\right)\left(\overset{b_n^*}{b_n}\, c\frac{n\pi}{L}\right)$$

$$\Rightarrow \overset{b_n^*}{b_n}\left(c\frac{n\pi}{L}\right) = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$\Rightarrow \boxed{\underset{b_n^*}{b_n} = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx}$$

HW # 1(a), ~~2(a)~~, 13.
4(a), (5)(a)

Def'n: From the above, we've seen

nth normal mode $\rightarrow$ $u_n(x,t) = \sin\frac{n\pi}{L}x\,(b_n\cos\lambda_n t + b_n^*\sin\lambda_n t)$ $\quad c\cdot\frac{n\pi}{L}$

• the general sol'n

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

no const coeff as b_n, b_n^* are arbitrary already

is a superposition of all normal modes.

• $\boxed{n=1}$ $\boxed{\text{fundamental mode}}$

$\boxed{\text{Example 1}}$: $\begin{cases} \text{A string w/ same b.c. \& i.c.} \\ f(x) = \sin\frac{m\pi}{L}x, \quad g(x)=0 \ (\text{initial velocity}=0) \end{cases}$

$\leftarrow$ initial shape

i.e. $\begin{cases} b_n = 0 & \forall n \neq m \ \cancel{\text{except}} \\ b_m = 1 & (n=m). \end{cases}$ $\quad \Downarrow \ b_n^* = 0$

$\Rightarrow u = u_m(x,t) = \sin\frac{m\pi}{L}x\cos c\frac{m\pi}{L}t.$

Starting m^{th} mode w/ initial velocity $=0$

$\Longrightarrow$ Continue to vibrate in m^{th} mode

$\boxed{\text{Example 2}}$: $f(x)=0$, $g(x) = x\cos x$

$$u(x,t) = \sum_{n=1}^{\infty}\frac{8}{\pi}\frac{(-1)^{n+1}}{(4n^2-1)^2}\sin 2nx\,\sin 2nt$$

All possible modes!

Rmk: Fundamental mode $\Rightarrow$ "pitch"

$n\geq 2$ modes $\Longrightarrow$ overtones / harmonics.

$\boxed{\text{Exercise 4(a)}}$ $f(x) = \sin\pi x + \frac{1}{2}\sin 3\pi x + 3\sin 7\pi x$ ...

$\boxed{\text{Exercise 4(a)}}$

$$\begin{cases} f(x) = \sin \pi x + \frac{1}{2} \sin 3\pi x + 3 \sin 7\pi x \\ g(x) = \sin 2\pi x, \quad c = 1 \end{cases} \Rightarrow L = 1$$

$$u(x,t) = \sin n\pi x \left(b_n \cos(n\pi t) + b_n^* \sin(n\pi t) \right)$$

$$f(x) = u(x,0) = \sum_n b_n \sin n\pi x \Rightarrow b_1 = 1, \quad b_3 = \frac{1}{2}, \quad b_7 = 3$$

all other b_n's $= 0$

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_n \left(b_n^* n\pi \right) \sin n\pi x \Rightarrow b_2^* = \frac{1}{2\pi}$$

all other $b_n^* = 0$.

$$\Rightarrow u(x,t) = \ldots$$

[redacted]

[redacted] But the "real material" just started!.

§3.4 D'Alembert's Method

Recall PDE for vibrating strings w/ b.c.'s + i.c.'s

can be solved by Fourier series method

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \,(b_n \cos \lambda_n t + b_n^* \sin \lambda_n t) \\ b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x\, dx \Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x + f(\ldots \\ b_n^* = \frac{2}{cn\pi}\int_0^L g(x) \sin \frac{n\pi}{L} x\, dx \Rightarrow g(x) = \sum_{n=1}^{\infty} \left(b_n^* \frac{cn\pi}{L}\right) \sin \frac{n\pi}{L} x \end{cases}$$

λ_n ||

Thm: [D'Alembert]

$G := \cancel{\tfrac{1}{2c}} \int g^*(s)\, ds + \text{cos's}$

$$u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\, ds$$

where f^* & g^* are odd extn's of f & g defined $0 \le x \le L$

$G(x+ct) - G(x-ct)$

Reasons for this soln:

Exc T4

~~① $u(x,t)$ = fn of $(x-ct)$ fn $(x+ct)$~~

① The soln to $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is of the form

$$u = \mathcal{F}(x-ct) + \mathcal{G}(x+ct)$$

$\mathcal{F}, \mathcal{G}$: arb. fn's

R, L

Pf: Let $\begin{cases} w = x - ct \\ z = x + ct \end{cases}$

~~PDE~~ $\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial w^2} + \frac{\partial^2 u}{\partial z^2} - 2\frac{\partial^2 u}{\partial w \partial z}\right]$

PDE $= c^2 \frac{\partial^2 u}{\partial x^2} = c^2\left[\frac{\partial^2 u}{\partial w^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial w \partial z}\right]$

$\Leftrightarrow \frac{\partial^2 u}{\partial z \partial w} = 0 \Leftrightarrow u$ is ~~indep of~~ $\mathcal{F}(z) + \mathcal{G}(w)$

(9) ② Need to check D'Alembert's soln satisfies b.c.

(a) ② Need to check D'Alembert's soln satisfies b.c.

(i) $f^*(x)$ is odd + 2L-periodic by defn

(ii) $g^*(x)$ " "

(ii) $G(x)$ is $\underline{2L\text{-periodic}}$, $\underline{\text{Even fn}}$

$$G(x) := \int_a^x g^*(x)\, dx$$

$$G(x+2L) = \int_a^x g^*(x+2L)\, d(x+2L) = \int g^*(x)\, dx = G(x)$$

$$G(-x) = \int g^*(-x)\, d(-x) = \int (-g^*(x))(-dx) = \int g^*(x)\, dx$$

$$= G(x)$$

(iii) $u(0,t) = \frac{1}{2}[\underbrace{f^*(-ct) + f^*(ct)}_{=0}] + \frac{1}{2c}[\underbrace{G(ct) - G(-ct)}_{=0}]$

odd   even

$u(L,t) = \frac{1}{2}[f^*(\underbrace{L-ct}_{-2L}) + f^*(L+ct)] + \frac{1}{2c}[G(L+ct) - G(\underbrace{L-ct}_{-2L})]$

$= 0$

$f^*(-L-ct) = -f^*(L+ct)$

$G(-L-ct) = G(L+ct)$

(b) ③ i.c.

$$u(x,0) = \frac{1}{2}[f^*(x) + f^*(x)] + \frac{1}{2c}[G(x) - G(x)]$$

$$= f^*(x) = f(x) \quad \text{when } 0<x<L$$

$$\frac{\partial}{\partial t} u(x,0) = \frac{1}{2}[\underbrace{(-c) f^{*\prime}(x) + (c) f^{*\prime}(x)}_{=0}] + \frac{1}{2c}[c\, G'(x) - (-c)\, G'(x)]$$

$$= \frac{1}{2}[2G'(x)] = G'(x) = g^*(x)$$

$$= g(x) \quad \text{if } 0<x<L$$

Geometric meaning

- Wave goes to right or to left at velocity $= c$
  (right moving) (left moving)

$$R(x-ct) \qquad L(x+ct)$$

$$\frac{1}{2}\left[f^*(x-ct)-\frac{1}{c}G(x-ct)\right] \qquad \frac{1}{2}\left[f^*(x+ct)+\frac{1}{c}G(x+ct)\right]$$

coming from initial position / shape

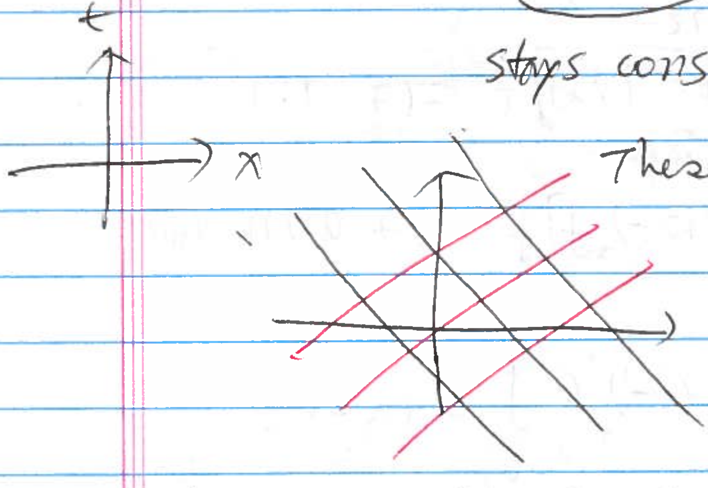
integration over initial velocity.

Characteristic lines

$$u(x,t) = \underbrace{R(x-ct)}_{\text{stays const on a line } x-ct=\text{const}} + \underbrace{L(x+ct)}_{\text{stays const on the line } x+ct=\text{const}}$$

stays const on the line $x+ct=\text{const}$, slope $= -\frac{1}{c}$

stays const on a line $x-ct = \text{const}$, slope $= \frac{1}{c}$

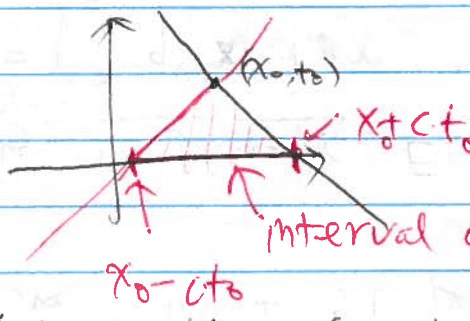


These lines = $\boxed{\text{characteristic line}}$

Interval of dependence

Given (x_0, t_0)

$\exists$ 2 chara. lines thru (x_0, t_0)



From D'Alembert's sol'n. the value of f^*, g^* outside interval of dep does not affect $u(x_0,t_0)$

"Causality" of wave fcn's

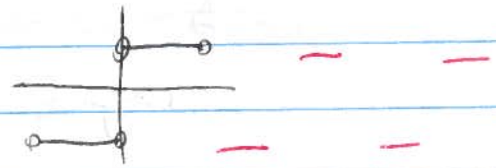
"Causality" of wave fcn's &

Exc: #1, 3, 6, /14

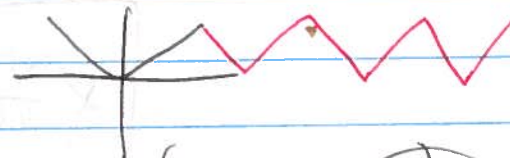
~~Example~~ $L=1$ $\quad f(x)=0 \quad g(x)=1, \quad c=1$

Exc #4. $\quad f^*(x)=0$

$$g^*(x)=\begin{cases} 1 & 0\le x<1 \\ -1 & -1<x<0 \end{cases}$$



$$G(x)=\begin{cases} x & 0\le x\le 1 \\ -x & -1\le x\le 0 \end{cases}$$



$$u(x,t)=\frac{1}{2(1)}\left[G(x+t)-G(x-t)\right] =$$

§3.5 1D heat eq'n

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

1st order in t — temperature

b.c. $\begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases}$ both ends kept const temp $=0$

2 b/c 2nd order in x

homog. b.c.

i.c. $u(x,0)=f(x)$ $\quad 0<x<L$

initial temp distrib.

only one b/c 1st order in t

- **Separation of variables**

$$u(x,t) = X(x)\,T(t)$$

heat eq'n $\Rightarrow \dfrac{T'}{c^2T} = \dfrac{X''}{X} = k$

$\boxed{X}$: $\begin{cases} X'' - kX = 0 \\ X(0)=0 \\ X(L)=0 \end{cases}$ Same as in the wave eq'n $\boxed{\begin{array}{l} k=-\mu^2 \\ \mu_n = \frac{n\pi}{L} \end{array}}$

$$\Rightarrow \boxed{X_n(x) = \sin \frac{n\pi}{L} x}$$

$\boxed{T}$: $T' - kc^2T = T' + \left(\frac{cn\pi}{L}\right)^2 T = 0$ 1st order homog.

$=: \lambda_n$

$$\Rightarrow T_n(t) = b_n e^{-\lambda_n^2 t}$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x}$$

$\boxed{\text{i.c.}}$: $u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$

$$\Rightarrow \boxed{b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)dx}$$

- **Steady-state temperature distribution**

- Steady-State temperature distribution

$u(x,t)$ = temp does not vary w/ time

$\Rightarrow \frac{\partial u}{\partial t} = 0$

$\Rightarrow$ heat eq $\Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = Ax + B$

- Nonzero boundary conditions

Dirichlet type

$$\begin{cases} \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} & 0 < x < L, \quad t > 0 \\ u(0,t) = T_1 \neq 0, \quad u(L,t) = T_2 \neq 0 \\ u(x,0) = f(x) \end{cases}$$

$\boxed{\text{nonhomog. b.c.}}$

Sep. of variables won't work! (Try it to find out!)

heat eq'n is a homog. eq'n $\Rightarrow$ if u_1 = a sol'n, u_2 = - -

then $u_1 + u_2$ = a sol'n.

$u_1(x,t) \equiv$ Steady-state sol'n $= \frac{T_2 - T_1}{L} x + T_1$

$\boxed{\text{b.c. for } u_1}$ $u_1(0,t) = T_1, \quad u_1(L,t) = T_2$ ✓

$u(0,t) = u_1(0,t) + u_2(0,t) = T_1 \Rightarrow \boxed{u_2(0,t) = 0}$

(with $u_1(0,t) = T_1$)

Similarly $\Rightarrow \boxed{u_2(L,t) = 0}$

$\Rightarrow u_2$ can be solved via S.O.V.

$$u_2(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right) e^{-\lambda_n^2 t}$$

$\boxed{\text{HW: \#1, 2, 5, 9, 11, 13}}$

$\boxed{\text{i.c.}}$ $u_1(x,0) + u_2(x,0) = f(x)$

$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = u_2(x,0) = f(x) - u_1(x,0)$ ← new i.c. for $u_2(x,t)$

$\Rightarrow \boxed{b_n = \frac{2}{L} \int_0^L (f(x) - u_1(x)) \sin\left(\frac{n\pi}{L} x\right) dx}$

$\boxed{\text{Exc 2}}$ $\boxed{\text{homog. b. c.}}$ $L=\pi,\ c=1,\ f(x)=30\sin x$

$\Rightarrow b_1 = 30 \quad b_n = 0 \quad \forall n \neq 1, \quad \lambda_1 = 1$

$u(x,t) = 30\sin(x)\, e^{-t}$

$\boxed{\text{Exc 11}}$ ~~u(x,t)~~

$\boxed{\text{nonhomog. b. c.}}$ $u(0,t)=100,\ u(1,t)=0$

$f(x) = 30\sin(\pi x),\ L=1,\ c=1$

$$u_1(x,t) = \frac{0-100}{1}x + 100 = 100(1-x)$$

$$u_2(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

$= n\pi$

$$\lambda_n = \frac{cn\pi}{L} = n\pi$$

$$b_n = 2\int_0^1 \left[30\sin(\pi x) - 100(1-x)\right] \sin(n\pi x)\, dx$$

$$= (30)(2)\int_0^1 (\sin \pi x)\sin(n\pi x)\,dx = \begin{cases} 30 & \text{if } n=1 \\ 0 & n\neq 1 \end{cases}$$

$$-(100)(2)\int_0^1 (1-x)\sin(n\pi x)\,dx = -\frac{200}{n\pi}$$

integ. by parts

$$\left[\int_0^1 (1-x)\sin ax\, dx = -\frac{1}{a}\cos(ax) - \frac{1}{a^2}\sin(ax) + \frac{x}{a}\cos ax \right.$$

$a = n\pi$

$$= -\frac{(-1)^n}{n\pi} + 0 + \frac{(-1)^n}{n\pi} + \frac{1}{n\pi}$$

§3.6 Heat Conduction in Bars

§3.6 Heat conduction in Bars

$\boxed{\text{Varying b.c.'s}}$ $\boxed{\text{HW \#1, 2, 4}}$

$\boxed{\text{Example 1}}$ $\boxed{\text{only}}$ $\boxed{\text{Insulated ends}}$

(Neumann type) b.c. $\begin{cases} \frac{\partial u}{\partial x}(0,t)=0, \\ \frac{\partial u}{\partial x}(L,t)=0 \end{cases}$ $t>0$

c.f. Dirichlet type $\begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases}$

i.c. $u(x,0)=f(x)$

$\boxed{\text{Sol'n}}$: Sep. of Var: $u(x,t)=X(x)T(t)$

same way $\Rightarrow \begin{cases} X''-kX=0 \\ T'-kc^2T=0 \end{cases}$ b.c $\Rightarrow$ $X'(0)=0$, $X'(L)=0$

$\boxed{\text{For } X}$: diff. b.c.

$k>0 \Rightarrow X=0$ as before

$k=0 \Rightarrow X(x)=c_1x+c_2$ as before

w/ the new $\boxed{\text{b.c.}}$ $\Rightarrow c_1=0$, c_2 = arbitrary.

$\Rightarrow X(x)=c_2$ = a constant.

$k<0$, Let $k=-\mu^2$. $\Rightarrow X=c_1\cos\mu x+c_2\sin\mu x$

$\cancel{\boxed{\text{b.c}}}\ \cancel{\Rightarrow}$ $X'=-c_1\mu\sin\mu x+c_2\mu\cos\mu x$

b.c. $X'(0)=c_2\mu=0 \Rightarrow c_2=0$

$X'(L)=-c_1\mu\sin\mu L=0 \Rightarrow \mu L=n\pi$

$\Rightarrow \mu_n=\frac{n\pi}{L}$

$\boxed{T}$: $T_n=a_ne^{-\lambda_n^2t}$ $\lambda_n:=c\frac{n\pi}{L}$

$\Rightarrow u(x,t)=a_0+\sum_{n=1}^{\infty}a_ne^{-\lambda_n^2t}\cos\frac{n\pi}{L}x.$

$\boxed{\text{i.c.}}$: $f(x)=u(x,0)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{L}x$ cosine series of $f(x)$

$\Rightarrow a_0=\frac{1}{L}\int_0^L f(x)\,dx$, $a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx$

$\boxed{\text{Ex 2}}$ Bar w/ 1 ~~insulated~~ fixed temperature & 1 radiating ends

$K>0$

same others. $\boxed{\text{b. c.}}$ $u(0,t)=0$ $\frac{\partial u}{\partial x}(L,t)=-Ku(L,t)$

Robin conditions — fixed temp at 0 — radiating end

radiation proportional to the temperature

Sol'n $u(x,t)=X(x)T(t)$ S.O.V.

$$\rightarrow \begin{cases} X''-kX=0 \\ X(0)=0 \\ X'(L)=-KX(L) \end{cases} + \boxed{T'-kc^2T=0}$$

$\boxed{\text{Exc \#4}}$

$L=1$
$c=1$
$f(x)=\begin{cases}100 & 0<x<\frac{1}{2} \\ 0 & \frac{1}{2}<x<1\end{cases}$

$$f(x)=100 \overset{\text{need}}{=} a_0+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

cosine series

$$a_0 = \int_0^{\frac{1}{2}} 100\,dx = \frac{100}{2}$$

$$a_n = 2\int_0^{1/2} 100 \cos(n\pi x)\,dx = \frac{200}{n\pi}\sin(n\pi x)\Big|_0^{1/2}$$

$$=\frac{200}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$=\begin{cases} 0 & n=\text{even} \\ \frac{200(-1)^k}{(2k+1)} & n=\text{odd}=2k+1 \end{cases}$$

$$\Rightarrow u(x,t)=50+\frac{200}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\cos((2k+1)\pi x)$$

$\boxed{\text{Laplace's Eqn}}$ $\boxed{\text{I. Rectangular Coordinates}}$ §3.8

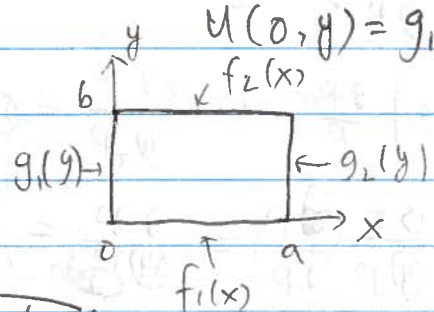
Laplace's Eq'n I. Rectangular Coordinates ← §3.8

- No time variable 2 space variables (x, y).

Laplace eq'n: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $0 < x < a$, $0 < y < b$

Dirichlet b.c $u(x,0) = f_1(x)$ $u(x,b) = f_2(x)$, $0 < x < a$

$u(0,y) = g_1(y)$, $u(a,y) = g_2(y)$, $0 < y < b$



specifies fcn value on boundary

normal derivative

Neumann b.c instead of $u(x,0)$ use $\frac{\partial u}{\partial x}(x,0)$ etc.

i.e. specify $\frac{\partial u}{\partial n}$ on the boundary

Fact: One can solve Laplace eq'n w/ Dirichlet b.c. via Sep. of var. on a rectangle (p.168)

II. Laplace eq'n in cylindrical coordinates §4.5

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}.$$

(ρ, ϕ, z) If u is indep of ϕ. i.e. radially symmetric

$x = \rho\cos\phi$ $u = R(\rho)Z(z)$

$y = \rho\sin\phi$

$z = z$

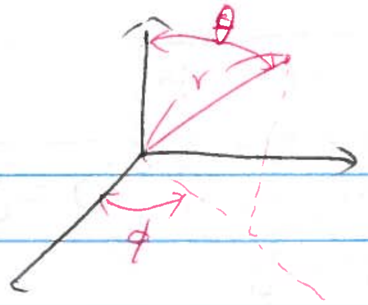
c.f. Bessel's eq'n

$$\Rightarrow \begin{cases} \boxed{\rho^2 R'' + \rho R' - k\rho^2 R = 0} \\ Z'' + kZ = 0 \end{cases}$$

parametric form of modified Bessel eq'n of order 0.

§5.1

III In spherical coordinate



$$\Delta u \overset{\text{in}}{\underset{\text{sph. coord}}{=\!=}} \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}$$

$$+\frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\,\frac{\partial u}{\partial \theta} + \csc^2\theta\,\frac{\partial^2 u}{\partial \phi^2}\right) = 0$$

If u is radially symmetric, i.e. indep of ϕ.

$$\Rightarrow \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\,\frac{\partial u}{\partial \theta}\right) = 0$$

$\boxed{\text{S.O.V}}$: $u(r,\theta) \overset{\text{no } \phi}{=} R(r)\,\Theta(\theta)$

$$\Delta u = 0 \iff \begin{cases} r^2R'' + 2rR' - \mu R = 0 & \leftarrow \boxed{\text{Euler's eqn}} \\ \Theta'' + \cot\theta\,\Theta' + \mu\Theta = 0 & \leftarrow (*) \end{cases}$$

related to Legendre's eqn via a change of variables

$$s = \cos\theta \qquad \frac{ds}{d\theta} = -\sin\theta$$

$$\Theta' = \frac{d\Theta}{d\theta} = \frac{ds}{d\theta}\frac{d\Theta}{ds} = -\sin\theta\,\frac{d\Theta}{ds}$$

$$\Theta'' = \frac{d^2\Theta}{d\theta^2} = -\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{ds}\right) = -\underbrace{\cos\theta}_{s}\,\frac{d\Theta}{ds} - \sin\theta\,\frac{d}{ds}\left(\frac{d\Theta}{ds}\right)\frac{ds}{d\theta}$$

$1-s^2$

$$= -s\frac{d\Theta}{ds} + (1-s^2)\frac{d^2\Theta}{ds^2}$$

$$(*) \iff \boxed{(1-s^2)\frac{d^2\Theta}{ds^2} - 2s\frac{d\Theta}{ds} + \mu\Theta = 0} \qquad \boxed{\text{Legendre's D.E}}$$

IV Rmk: The heat & wave eqn's can be generalized

$\boxed{\text{IV}}$ ~~Rmk~~: The heat & wave eq'n's can be generalized to higher dimensions

e.g. $\dfrac{\partial^2 u}{\partial t^2} = c^2 (\Delta u)$ wave eq'n

$\dfrac{\partial u}{\partial t} = c^2 (\Delta u)$ heat eq'n

All the above can be applied to the higher dim. generalization.

§4.6

$\boxed{\text{V}}$ $\boxed{\text{Poisson eq'n}}$ & $\boxed{\text{Helmholtz eqn}}$

$\Delta u = f$. $\quad \Delta u = -k u$

For radially symmetric case in spherical coord (r, θ) (indep of ϕ)

$u = R(r)\,\Theta(\theta)$

$$\Delta u = -k u \iff \begin{cases} \Theta'' + m^2 \Theta = 0 \\ r^2 R'' + rR' + (kr^2 - m^2) R = 0 \end{cases}$$

modified Bessel eqn of order m

e.g. wave, heat, §3.3 #13

- MT: Do SOV. Solve Equations starting from scratch. Can ask for integration formulas (NOT sol'ns to ODE, PDE!)
- Do change of variables from rectangular to spherical / polar coordinates for Laplace eq'n.

$\to$ get ODEs w/o solving them