The general PDE of order 2 in 2 variables
\[ A U_{xx} + 2B U_{xy} + C U_{yy} + D U_x + E U_y + F U = G \]

A, B, ..., G are fun's of (x, y).

As discussed before, to find a unique soln.
Need 2 boundary conditions

2 initial conditions

\[ \text{boundary value problem (B.V.P.)} \]

\[ \text{initial value problem (I.V.P.)} \]

Most common PDE of the above type.

\( \Delta U(x, y) := \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \)

(i) \[ \text{2D wave eqn} \]
\[ \frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} = 0 \]
\[ U = U(x, t) \]

(ii) \[ \text{2D heat eqn} \]
\[ \frac{\partial U}{\partial t} - c^2 \frac{\partial^2 U}{\partial x^2} = 0 \]

(iii) \[ \text{Laplace eqn} \]
\[ \Delta U = 0 \]

(iv) \[ \text{Poisson eqn} \]
\[ \Delta U(x, y) = G(x, y) \]
Superposition principal (recalled)

- If the PDE is **linear** & **homogeneous**
  
  then \( (\text{a soln}) + (\text{another soln}) \)
  
  = \( (\text{a new soln}) \)

  e.g. If \( \begin{cases} \Delta u_1(x,y)=0, \\ \Delta u_2(x,y)=0 \end{cases} \Rightarrow \Delta(u_1+u_2)=0 \)

  - However. If \( \begin{cases} \Delta u_1 = G_1, \\ \Delta u_2 = G_2 \end{cases} \) \( \Rightarrow \Delta(u_1+u_2) \neq G_1 + G_2 \)

  - If the eqn is **non-linear**, superposition fails

Note: **Linearity means:** In the PDE, each term involves only 1 factor of \( u \) or its derivative.

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad \text{Non-linear} \]

\[
\frac{\partial^2 u}{\partial x^2} + u = 0
\]

\[
\frac{\partial^2 u}{\partial x^2} + e^u \frac{\partial^2 u}{\partial t^2} = \cos(x+y)u \quad \text{Linear}
\]

HW: 83.1 # 1, 4, 5, 6, 9
§3.3 Method of Separation of Variables

1D Wave eqn
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

subject to boundary conditions:
- \( u(0,t) = 0 \), \( u(L,t) = 0 \)
- ends of string are fixed

& initial condition:
- \( u(x,0) = f(x) \)
- \( \frac{\partial u}{\partial t}(x,0) = g(x) \)
- initial shape \( 0 < x < L \)

In this section we'll solve this w/ separation of variables (SOV)

Step 1:
- \( u(x,t) = X(x)T(t) \)

\[ \frac{\partial^2 u}{\partial t^2} = X T'' \quad \frac{\partial^2 u}{\partial x^2} = X'' T \]

\( \Rightarrow \) PDE becomes
- \( X T'' = c X'' T \)

\[ \Rightarrow \frac{T''}{c T} = \frac{X''}{X} = k \]

\( \Rightarrow \)
- \( X'' = kX \)
- \( T'' = c k T \)

b.c. \( X(0) T(t) = 0 = X(L) T(t) \)

\( \Rightarrow \)
- \( X(0) = X(L) = 0 \)

N.B.
- \( X'' = \frac{d^2 X(x)}{dx^2} \)
- \( T'' = \frac{dT(t)}{dt^2} \)
Step 2: Solving ODE's

\( x'' - kx = 0 \) \hspace{1cm} \text{(similarly \quad T'' - kc^2T = 0)}

(2nd order homog.)

\( x'' - kx = 0 \)

Case (i) If \( k > 0 \) : \quad x = \pm \sqrt{k} x

\[ x = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}. \]

b.c. \( x(0) = c_1 + c_2 \Rightarrow 1 \]

\[ x(L) = c_1 e^{\sqrt{k}L} + c_2 e^{-\sqrt{k}L} = 0 \]

\[ \Rightarrow (c_1)(e^{\sqrt{k}L})(1 - e^{-\sqrt{k}L}) = 0 \]

\[ \Rightarrow c_2 \Rightarrow x = 0 \quad \Rightarrow x \]

Case (ii) \( k = 0 \)

\[ x'' = 0 \Rightarrow x = c_1 x + c_2 \]

b.c. \( c_2 = 0 \Rightarrow c_1 = 0 \quad \Rightarrow x = 0 \Rightarrow \]

Case (iii) \( k < 0 \)

\( x(x) = c_1 \cos \frac{\pi x}{L} + c_2 \sin \frac{\pi x}{L} \quad \Rightarrow \]

\[ x(0) = 0 \Rightarrow c_1 = 0 \]

\[ x(L) = 0 \Rightarrow c_2 \sin \frac{\pi L}{L} \Rightarrow \]

\[ \Rightarrow M = \frac{h_{\text{max}}}{L} = \lambda_n \]

\[ \Rightarrow \quad \text{gen. soln}: \quad x(x) = c_2 \sin \frac{n\pi x}{L} \quad n = 1, 2, 3, \ldots \]

Similarly \( \quad T'' - kc^2T = 0 \quad k < 0 \)
Similarly: a T'' - kc^2T = 0, k < 0

\[ T = T_n = a_n \cos \left( \frac{n\pi}{L} t \right) + b_n \sin \left( \frac{n\pi}{L} t \right) \]

Fourier series?

Summary: All possible solutions are of the form

\[ u_n(x,t) = \sin \frac{n\pi}{L} x \left( a_n \cos \frac{n\pi}{L} t + b_n \sin \frac{n\pi}{L} t \right) \]

2. By superposition principle, a general solution

\[ u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \]

arb. const. [Need to fix them]

---

Step 3: Fourier Series solutions of the Entire Problem

1. C(1)

\[ u(x,0) = f(x) \]

\[ \sum_{n=1}^{\infty} \left( a_n \sin \left( \frac{n\pi}{L} x \right) \right) \]

Fourier series

\[ a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \]

(a) \[ u(x,0) = g(x) \]

\[ \sum_{n=1}^{\infty} \left( b_n \cos \left( \frac{n\pi}{L} x \right) \right) \]

\[ b_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx \]

HW: 16b, 16c, 13, 496d, 51g
**Defn.:** From the above, we've seen

\[ \text{Un}(x,t) = \sum_{n=1}^{\infty} \left( b_n \cos \frac{n\pi x}{L} + b_n^* \sin \frac{n\pi x}{L} \right) \]

The general sol'n,

\[ U(x,t) = \sum_{n=1}^{\infty} \text{Un}(x,t) \]

is a superposition of all normal modes.

\[ n=1 \] fundamental mode

**Example 1:** A string w/ some b.c. & i.c.

\[ f(x) = \sin \frac{m\pi x}{L} \]

\[ g(x) = 0 \] (initial velocity = 0)

\[ b_n = 0 \quad \forall \quad n \neq m \]

\[ b_m = 1 \quad (n=m) \]

\[ \Rightarrow U = U_m(x,t) = \sin \frac{m\pi x}{L} \cos \frac{m\pi t}{L} \]

Starting \( m \)th mode w/ initial velocity = 0

\[ \Rightarrow \text{continue to vibrate in } m \text{th mode} \]

**Example 2:**

\[ f(x) = 0 \quad , \quad g(x) = x \cos x \]

\[ U(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi} \frac{1}{(4n^2-1)^2} \sin 2n\pi x \cos 2n\pi t \]

All possible modes!

**Remark:** Fundamental mode \( \Rightarrow \) "pitch"

\[ n \geq 2 \text{ modes } \Rightarrow \text{overtone / harmonics} \]

**Exercise 4:**

\[ f(x) = \sin \pi x + \frac{1}{2} \sin 3\pi x + 3 \sin 7\pi x \]
Exercise 4(a) \[ f(x) = \sin \pi x + \frac{1}{2} \sin 3 \pi x + 3 \sin 7 \pi x \] \[ g(x) = \sin 2 \pi x, \quad c = 1 \]

\[ u(x, t) = \sum b_n \sin (n \pi x) + b_n^* \sin (n \pi (x + t)) \]

\[ f(x) = u(x, 0) = \sum b_n \sin n \pi x \Rightarrow b_1 = 1, \quad b_2 = \frac{1}{2}, \quad b_7 = 3 \]

\[ g(x) = \frac{2}{n \pi} u(x, 0) = \sum b_n^* \sin n \pi x \Rightarrow b_2^* = \frac{1}{2} \pi \]

all other \( b_n^* = 0 \)

\[ \Rightarrow u(x, t) = \ldots \]

Phase 1

But the 'real material' just started!
§3.4 D'Alambert's Method

Recall PDE for vibrating string w/ b.c.'s + i.c.'s can be solved by Fourier series method

\[ U(x,t) = \sum_{n=1}^{\infty} \left( b_n \cos \frac{n\pi x}{L} + b_n^* \sin \frac{n\pi x}{L} \right) e^{-\alpha n^2 \pi^2 t} \]

\[ b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \]

\[ b_n^* = \frac{2}{cn^\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx \Rightarrow g(x) = \sum_{n=1}^{\infty} b_n^* \cos \frac{n\pi x}{L} \]

**Thm:** [D'Alambert]

\[ u(x,t) = \frac{1}{2} \left( f(x - ct) + g(x + ct) \right) + \int_{x-ct}^{x+ct} g'(s) \, ds \]

where \( f \) & \( g \) are odd ext's of \( f \times g \) defined \( 0 \leq x \leq L \)

Returning to this soln:

\[ u(x,t) = \frac{1}{2} \left( f(x - ct) + g(x + ct) \right) + \int_{x-ct}^{x+ct} g'(s) \, ds \]

The soln to \( \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \) is of the form

\[ u = f(x - ct) + g(x + ct) \]

\[ \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} = 0 \]

\[ \frac{\partial f}{\partial x} - \frac{\partial g}{\partial x} = 0 \]

\[ \frac{2u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial^2 u}{\partial x \partial y} \right) \]

\[ \frac{\partial^2 u}{\partial x^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial x \partial t} \right) \]

Subject to \( \frac{\partial u}{\partial x} = 0 \) \( \Rightarrow u \) is even & periodic

\[ f(z) + g(w) \]

(0) Need to check D'Alambert's soln satisfies b.c.
(a) Need to check D'Alembert's soln satisfies bc.

(i) \( f^*(x) \) is odd + 2L-periodic by defn.

\[ g^*(x) = \int_a^x f^*(x) \, dx \]

\[ G(x+2L) = \int_a^x g^*(x+2L) \, dx = \int_a^x g^*(x) \, dx = G(x) \]

\[ G(-x) = \int_a^{-x} g^*(-x) \, dx = \int_a^{x} g^* (x) \, (-dx) = \int_a^{x} g^*(x) \, dx \]

(ii) \( G(x) \) is 2L-periodic, Even fun

\[
\begin{align*}
\text{(iii)} & \quad u(0, t) = \frac{1}{2} \left[ f^*(-ct) + f^*(ct) \right] + \frac{1}{2c} \left[ G(ct) - G(-ct) \right] \\
& \quad \text{if } t \text{ odd} \\
& \quad \text{if } t \text{ even}
\end{align*}
\]

\[ u(L, t) = \frac{1}{2} \left[ f^* (L-ct) + f^* (L+ct) \right] + \frac{1}{2c} \left[ G(L+ct) - G(L-ct) \right] \]

(iii) \( u(x, 0) = \frac{1}{2} \left[ f^*(x) + f^*(L-x) \right] + \frac{1}{2c} \left[ g(x) - g(x) \right] \)

\[ \begin{align*}
\frac{\partial}{\partial t} u(x, 0) &= \frac{1}{2} \left[ f''(x) + f''(L-x) \right] + \frac{1}{2c} \left[ c G'(x) - (-c) G'(x) \right] \\
&= \frac{1}{2} \left[ f''(x) \right] \quad \text{when } 0 < x < L
\end{align*} \]

\[ u(x, 0) = \frac{1}{2} \left[ f''(x) + f''(L-x) \right] + \frac{1}{2c} \left[ c G'(x) - (-c) G'(x) \right] \]

\[ = \frac{1}{2} \left[ f''(x) \right] \quad \text{when } 0 < x < L \]

(b) i.e.

\[ -f^*(L+ct) \]

\[ g(L+ct) \]

\[ f(x) \quad \text{when } -L < x < L \]
Geometric meaning

- **Wave goes to right** on to left at velocity $c$

\[ R(x-c) \quad L(x+c) \]

\[ \frac{1}{2} \left[ (f_2(x-c) - \frac{1}{2} G(x+c)) \right] \quad \frac{1}{2} \left[ (f_2(x+c) + \frac{1}{2} G(x+c)) \right] \]

- **Stays const on the line** $x+c=\text{const}$

\[ U(x,t) = R(x-c) + L(x+c) \]

\[ \text{slope} = -\frac{1}{2} \]

- **Stays const on a line** $x-c=\text{const}$

\[ \text{These lines} = \text{characteristic line} \]

Interval of dependence

**Given** $(x_0,t_0)$

\[ x_t < t_0 \]

From ODE/PDE, the value of $f^*, g^*$ outside interval of dep. does not affect $u(x_0,t_0)$

"Causality" of wave fun's &
"Causality" of wave func's

Ex. #1. \( f(x) = 0 \) \( g(x) = 1 \), \( c = 1 \)

\[ L = 1 \]

\[ f^*(x) = 0 \]

\[ g^*(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \end{cases} \]

\[ G(x) = \begin{cases} x & 0 \leq x \leq 1 \\ -x & -1 \leq x \leq 0 \end{cases} \]

\[ u(x, t) = \frac{1}{2(1)} \left[ G(x+t) - G(x-t) \right] \]
§3.5 1D heat eqn

\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

b.c. \( u(0, t) = 0 \) \( \text{both ends} \)

b.c. \( u(L, t) = 0 \) \( \text{kept constant temp} \)

i.c. \( u(x, 0) = f(x) \) \( 0 < x < L \)

\text{only one b/c 1st order in } t

\text{Separation of variables}

\[ u(x, t) = X(x) T(t) \]

heat eqn \( \Rightarrow \frac{T'}{c^2 T} = \frac{X''}{X} = k \)

\[
\begin{cases}
X'' - kX = 0 \\
X(0) = 0 \\
X(L) = 0
\end{cases}
\]

\[ X_n(x) = \sin \frac{n\pi x}{L} \]

\[ T' - k c^2 T = T' + \left( \frac{cn\pi}{L} \right)^2 T = 0 \]

1st order \( \text{homog.} \)

\[ T_n(t) = b_n e^{-\lambda_n t} \]

\[ u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin \frac{n\pi x}{L} \]

\[ u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right) = f(x) \]

\[ b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) dx \]

\text{Steady-State temperature distribution}
Steady-State Temperature Distribution

\[ u(x, t) = \text{temp does not vary w/ time} \]

\[ \frac{\partial u}{\partial t} = 0 \]

\[ \Rightarrow \text{eq heat eq} \Rightarrow \frac{\partial^2 u}{\partial x^2} > 0 \Rightarrow u = Ax + B \]

Non-zero boundary conditions

\[ \begin{cases} \frac{\partial u}{\partial x}(0, t) = 0, & 0 < x < L, \ t > 0 \\ u(0, t) = T_1, & u(L, t) = T_2 \end{cases} \]

non-homog. b. c.

Sep. of variables won't work! (Try it to find out)

heat eqn is a homog. eqn \[ \Rightarrow \text{if } u_1 = \text{a soln,} \]

then \[ u_1 + u_2 = \text{a soln}. \]

\[ u_1(x) = \text{steady-state soln} = \frac{T_1 - T_2}{L} x + T_1 \]

\[ u_1(0) = T_1, \ u_1(L) = T_2. \]

\[ u(0, t) = u_1(0, t) + u_2(0, t) = T_1 \Rightarrow u_2(0, t) = 0. \]

Similarly \[ u(L, t) = 0. \]

\[ u_2(x, t) = \sum b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2 t} \]

HW: 1, 2, 5, 9, 11, 13

Using i.c. for \[ u_2(x, t) \]

\[ u_1(x, 0) + u_2(x, 0) = f(x) \]

\[ \sum b_n \sin\left(\frac{n\pi x}{L}\right) = u_2(x, 0) = f(x) - u_1(x, 0) \]

\[ \Rightarrow b_n = \frac{2}{L} \int_0^L f(x) - u_1(x, 0) \sin\left(\frac{n\pi x}{L}\right) dx \]
Exercise 2

\[ \text{homog. b.c. } \quad L=1, \quad c=1, \quad f(x)=30 \sin x \]

\[ \implies b_1 = 30, \quad b_n = 0 \quad \text{for } n \neq 1, \quad x_1 = 1 \]

\[ U(x, t) = 30 \sin (x) e^{-t} \]

Exercise 11

\[ \text{nonhomog. b.c. } \quad U(0, t) = 100, \quad U(1, t) = 0 \]

\[ f(x) = 30 \sin (\pi x), \quad L=1, \quad c=1 \]

\[ U_1(x, t) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi}{L} x \right) e^{-\left( \frac{c \pi^2}{L^2} \right) t} \]

\[ \frac{c \pi^2}{L^2} = \frac{n \pi^2}{L^2} \]

\[ b_n = 2 \int_0^1 [30 \sin (\pi x) - 100 (1-x)] \sin (n \pi x) \, dx \]

\[ = 60 \left( \frac{1}{2} \right) \int_0^1 (\sin n \pi x) \sin (n \pi x) \, dx \]

\[ = 0 \quad \text{if } n \neq 1 \]

\[ = 100 \left( \frac{1}{2} \right) \int_0^1 (1-x) \sin (n \pi x) \, dx \]

\[ = \frac{200}{n \pi} \]

\[ \left[ \int_0^1 (1-x) \sin ax \, dx \right] \]

\[ = - \frac{\cos (ax)}{a} + \frac{x \cos ax}{a^2} \]

\[ a = \frac{n \pi}{L} \]

\[ = \frac{1}{n \pi} \left( - \frac{1}{n \pi} \cos ax \right) + \frac{x \cos ax}{a^2} \]

\[ = - \frac{1}{n \pi} + \frac{1}{n \pi} \]

\[ + \frac{1}{n \pi} \]

\[ \text{Reg. by parts} \]

\[ \text{§3.6 Heat Conduction in Bars} \]

Example 1: Insulated end

b.c. \( \frac{\partial u}{\partial x}(L,t) = 0 \)

(Neumann type)

r.c. \( u(x,0) = f(x) \)

Solution: Separation of Variables \( u(x,t) = X(x)T(t) \)

Same way \( \Rightarrow \begin{cases} \frac{X''}{X} = kX \\ \frac{T'}{kT} = \gamma \end{cases} \)

b.c. \( X(0) = 0 \Rightarrow X'(0) = 0 \)

For \( X \): diff. b.c.

\( k > 0 \Rightarrow X = 0 \) as before

\( k < 0 \Rightarrow X = c_1 \cos \mu x + c_2 \sin \mu x \)

With the new b.c. \( \Rightarrow c_1 = 0, c_2 = \text{arbitrary} \)

\( \Rightarrow X(x) = c_2 = \text{a constant} \)

\( k < 0, \) let \( k = -\mu^2 \Rightarrow X = c_1 \cos \mu x + c_2 \sin \mu x \)

When \( X' = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x \)

b.c. \( X(0) = c_1 \mu = 0 \Rightarrow c_2 \neq 0 \)

\( X'(L) = -c_1 \mu \sin n\pi L = 0 \Rightarrow mL = n\pi \)

\( \Rightarrow \mu_n = \frac{n\pi}{L} \)

(7) \( T_n = a_n e^{-\lambda_n^2 t} \) \( \lambda_n : = \frac{c_n}{L} \)

\( \Rightarrow u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2 \pi^2}{L^2} t} \cos \frac{n\pi x}{L} \)

r.c.: \( f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \)

\( \Rightarrow a_0 = \frac{1}{L} \int_0^L f(x) dx \), \( a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \)
Bar with fixed temperature at one end and radiating ends at the other.

Some others, b.c. \( u(0,t) = 0 \), \( \frac{\partial u}{\partial x} (L,t) = -R u(L,t) \).

Robin conditions at fixed temperature 0, radiating end, radiation proportional to the temperature.

\[ u(x,t) = X(x) T(t) \] S.O. U.

\[ X'' - kX = 0 \]
\[ X(x) = \begin{cases} \sin \omega x & \text{if } \omega > k \frac{L}{2} \\ 0 & \text{if } \omega = k \frac{L}{2} \\
 0 & \text{if } \omega < k \frac{L}{2} \end{cases} \]

\[ T' - k(2T - T_0) = 0 \]

**Exercise 4**

\[ f(x) = 100 = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right) \]

\[ a_0 = \frac{1}{L} \int_{0}^{L} f(x) \cos \left( \frac{\pi n x}{L} \right) dx = \frac{100}{2} \]

\[ a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{\pi n x}{L} \right) dx = \frac{200}{n\pi} \sin \left( \frac{n\pi}{2} \right) \]

\[ c = \begin{cases} 0 & \text{if } n \text{ even} \\
\frac{200(-1)^{k+1}}{(2k+1)\pi} & \text{if } n = 2k+1 \end{cases} \]

\[ u(x,t) = 50 + \frac{200}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \cos \left( (2k+1)\pi x \right) \]

Laplace's Eq in Rectangular Coordinates.
Laplace's Eqn in Rectangular Coordinates

No time variable. 2 space variables \((x, y)\).

Laplace eqn: \( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad 0 < x < a \)
\( 0 < y < b \)

Dirichlet b.c:
\( u(x, 0) = f_1(x), \quad u(x, b) = f_2(x), \quad 0 < x < a \)
\( u(0, y) = g_1(y), \quad u(a, y) = g_2(y), \quad 0 < y < b \)

\( g(y) = f(y) \quad \text{on boundary} \)

Neumann b.c: instead of \( u(x, 0) \) use \( \frac{\partial u}{\partial x} \) etc.
\( \text{i.e.'specify } \frac{\partial u}{\partial n} \text{ on the boundary} \)

Fact: one can solve Laplace eqn w/ Dirichlet b.c. via

separation of variables on a rectangle (p. 168)

Laplace eqn in cylindrical coordinates

\( \Delta U = \nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \phi^2} \)
\( \frac{\partial^2 U}{\partial \phi^2} + \frac{1}{r} \frac{\partial U}{\partial \phi} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \phi^2} \)

\((r, \phi, z)\) if \( U \) is indep of \( \phi \), i.e. radially symmetric

\( x = r \cos \phi \quad U = R(r) Z(z) \)
\( y = r \sin \phi \quad \Rightarrow \quad \frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \frac{1}{r^2} \frac{\partial^2 R}{\partial \phi^2} - k R = 0 \)
\( Z'' + k Z = 0 \)

\( k \) Bessel's eqn
\( \text{parametric form of modified Bessel eqn of order 0} \)
In spherical coordinates

\[ \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2 u}{\partial \theta^2} + \cot \theta \frac{\partial u}{\partial \theta} + \frac{u}{\sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right) = 0 \]

If \( u \) is radially symmetric, i.e., independent of \( \phi \),

\[ \Rightarrow \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2 u}{\partial \theta^2} + \cot \theta \frac{\partial u}{\partial \theta} \right) = 0 \]

**S.O.V.**

\[ u(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \]

\[ \Delta u = 0 \iff \begin{cases} r^2 \Phi'' + 2r \Phi' - m^2 R = 0 \quad \text{Euler's eqn} \\ (\Theta'' + \cot \theta \Theta' + m^2 \Theta) = 0 \end{cases} \]

Related to Legendre's eqn via a change of variables

\[ s = \cos \theta \quad \frac{ds}{d \theta} = -\sin \theta \]

\[ \Theta' = \frac{d \Theta}{d \theta} = \frac{ds}{d \theta} \frac{d \Theta}{ds} = -\sin \theta \frac{d \Theta}{ds} \]

\[ \Theta'' = \frac{d^2 \Theta}{d \theta^2} = -\frac{d}{ds} \left( \sin \theta \frac{d \Theta}{ds} \right) = -\cos \theta \frac{d \Theta}{ds} - \sin \theta \frac{d^2 \Theta}{ds^2} \]

\[ \Phi = \frac{d \Phi}{ds} + \left( k s^2 \right) \frac{d^2 \Phi}{ds^2} \]

\( (X) \left\langle \begin{array}{l} (1-s^2) \frac{d^4 \Theta}{ds^4} - 2s \frac{d^3 \Theta}{ds^3} + m^2 \Theta = 0 \end{array} \right. \]

Legendre's DE

**Book:** The heat & wave eqns can be generalized
The heat & wave eqns can be generalized to higher dimensions.

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \left( \frac{\partial^2 y}{\partial x^2} \right) \]  

Wave eqn

\[ \frac{\partial u}{\partial t} = c^2 \left( \Delta u \right) \]  

Heat eqn

All the above can be applied to the higher-dimensional generalization.

Poisson eqn & Helmholtz eqn

\[ \Delta u = f, \quad \Delta u = -k u \]

For radially symmetric case in spherical coord. \((r, \theta)\)

\[ u = R(r) \Theta(\theta) \]

\[ \Rightarrow \quad \Theta'' + m^2 \Theta = 0 \]

\[ \Delta u = -k u \quad \Rightarrow \quad r^2 R'' + rR' + (kr^2 - m^2)R = 0 \]

Modified Bessel eqn of order \(m\)

E.g.: Wave, heat, §3.3 #13

MT: Do SOV. Solve Equations starting from scratch. Can ask for integration formulas (NOT solve for ODE, PDE!)

- Do change of variables from rectangular to spherical/polar coordinates for Laplace eqn.
- Get ODEs w/o solving them.