- Can impose $\boxed{\text{initial condition}}$

$$\frac{\partial u}{\partial t}\Big|_{t=0} = 0. \Rightarrow b_n^* = 0$$, In this case

At $t=0$. $u(x,0) = \sum b_n \sin \frac{n\pi x}{L}$

HW §1.2 # 4

$\boxed{\text{Ch 2}}$ 2.1 Periodic fns   T-periodic fn $f(x)$

$$f(x+\underset{\substack{\uparrow \\ \text{period}}}{T}) = f(x)$$

e.g. $f(x) = \cos x$. $T = 2\pi$
$f(x) = \sin 2x$ $T = \pi$

- Piecewise cts / sm fns

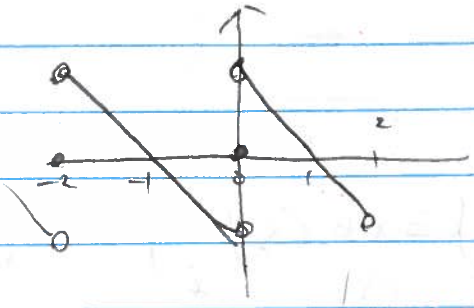
(i) $f(x)$ is said to be piecewise cts on $[a,b]$ if
- $f(a+)$ & $f(b-)$ exist.
- $f(x)$ defined & cts on (a,b) except at finite # of pts & at those pts the left & right limit exist.

(ii) $f(x)$ is piecewise sm on $[a,b]$ if
- $f(x)$ piecewise cts on $[a,b]$
- $f'(x)$ exists & cts in (a,b) except at finite # of pts & at those pts, the left & right limit of $f'(x)$ exist
- $\lim_{x \to a^+} f'(x)$ & $\lim_{x \to b^-} f'(x)$ exist.

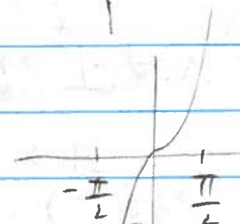
Thm1. If $f(x)$ is T-periodic & piecewise cts
$\forall a$.
$$\int_0^T f(x)\,dx = \int_a^{T+a} f(x)\,dx$$

e.g.

- $f(x) = \cos mx$, $\sin mx$ $\quad T = \frac{2\pi}{m}$

- $f(x) =$  $\quad T = 2$

Pw cts

- $f(x) = \tan x$  $T = \pi$ but NOT piecewise cts

- Round down / Gauss fn $f(x) = [x]$. P. sm but NOT periodic
- fractional part $f(x) = x - [x]$ is p. sm & periodic $T = 1$

2.2 Orthogonality

Lemma $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \end{cases}$ (sin sin; $\frac{T}{2}$, $\frac{2\pi}{T}x$)

$\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \quad \forall m, n$

Pf: $\cos mx \cos nx = \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right)$

RHS $= \frac{1}{2} \mathrm{Re}\left(e^{i(m+n)x} + e^{i(m-n)x}\right)$

$= \frac{1}{2} \mathrm{Re}\left[e^{imx}\left(e^{inx} + e^{-inx}\right)\right]$

$= \mathrm{Re}\left[e^{imx} \cdot \cos nx\right] = $ LHS

$\sin mx \sin nx = \frac{1}{2}\left(\cancel{\sin(m+n)x - \sin(m-n)x}\right)$

HW: 2.1 # 1 (c), (d), 2(b) # ~~3~~, 7, 8

do this in class

2.2 Fourier series

2.2

2.3

2.6

Fourier series

a vector $\vec{v} \in \mathbb{R}^n$

expand it in coordinates w.r.t. a basis

$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \cdots + v_n \vec{e}_n$

E.g. $\cdot \cos(n\frac{2\pi}{T}x)$, $\cdot \sin(n\frac{2\pi}{T}x)$, $\cdot e^{in\frac{2\pi}{T}x}$

A fn can be expanded in terms of a "basis" as well

Thm: For a $\boxed{T\text{-periodic}}$ fns $f(x)$, piecewise sm

$$\frac{f(x^+)+f(x^-)}{2} = \sum_{n=0}^{\infty}\left(a_n \cos\left(n\frac{2\pi}{T}x\right) + b_n \sin\left(n\frac{2\pi}{T}x\right)\right)$$

$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}$

If $f(x)$ is sm. then $= f(x)$

• Q: How to find a_n, b_n ?

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(n\frac{2\pi}{T}x\right)dx$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\left(n\frac{2\pi}{T}x\right)dx$$

Thm2: If $f(x) =$ ~~odd~~ even then only $\cos(nx)$-terms

odd — $\sin(nx)$

Rmk: If $f(x)$ is defined on $[-\frac{T}{2}, \frac{T}{2}]$ only,

We can make is a T-periodic fn by defining $f_{new}(x+T) = f(x)$

f_{new} is defined on $\mathbb{R}$ ← whole real line

2. Complex Form of Fourier series

$z = x + iy$
$\bar{z} = x - iy$
$|z|^2 = z\bar{z} = x^2 + y^2$

Recall Euler's formula $e^{ix} = \cos x + i \sin x$

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

$\rightsquigarrow$ cos & sin can be written in term of e^{inx}, e^{-inx}

Thm 1' + 2'

$$\underset{\downarrow}{\frac{f(x^+) + f(x^-)}{2}}$$
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\frac{2\pi}{T}x} \qquad f(x) = T\text{-periodic}.$$

c_n can be cpx even if $f(x)$ is real

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\, e^{-in\frac{2\pi}{T}x}\, dx, \quad n = \text{any integer}$$

Orthogonality:

$$\int_0^T e^{in\frac{2\pi}{T}x} e^{im\frac{2\pi}{T}x} dx = \begin{cases} 0 & m \neq -n \\ T. & m = -n \end{cases}$$

$$= \int_0^T e^{\boxed{i(m+n)\frac{2\pi}{T}x}} dx = \frac{\int_0^{i(m+n)2\pi} e^u du}{i(m+n)\left(\frac{2\pi}{T}\right)}$$

$$= \frac{1}{*}\left(\underset{1}{e^{i(m+n)2\pi}} - \underset{1}{e^0}\right)$$

Parseval's identity

ⓐ $$\frac{1}{T}\int_0^T f(x)^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

ⓑ $$\frac{1}{T}\int_0^T f(x)^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

"pf": ⓐ $$LHS = \frac{1}{T}\int^T \left(\sum^{\infty} c_m e^{im\frac{2\pi}{T}x}\right)\left(\sum^{\infty} c_n e^{in\frac{2\pi}{T}x}\right) dx$$

"pf": ⓐ $LHS = \frac{1}{T}\int_0^T \left(\sum_{m=-\infty}^{\infty} c_m e^{im\frac{2\pi}{T}x}\right)\left(\sum_{n=-\infty}^{\infty} c_n e^{in\frac{2\pi}{T}x}\right)dx$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_m c_n \int_0^T e^{im\frac{2\pi}{T}}e^{in\frac{2\pi}{T}x}\,dx$$

$$= \cancel{T}\sum_{m=-\infty}^{\infty}(c_m c_{-m}) = \sum_{m=-\infty}^{\infty} c_m \overline{c_m} = \sum_{m=-\infty}^{\infty}|c_m|^2$$

(Recall $c_n = \overline{c_{-n}}$ as $f(x)$ is real)

ⓑ "proven similarly".

Half-range expansion

- Sometimes you have a fn defined on $[0,p]$. You want to make it an even & periodic fcns.

$\rightsquigarrow$ · define $f_{new}(-x) = f(x)$ $\forall x \geq 0$ $f_{new}(x)$ def on $[-p,p]$

· extend it periodically. $f_{newer}(x+2p) = f(x)$

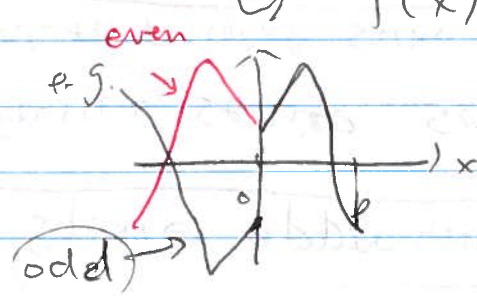
$\rightsquigarrow f_{newer}(x)$ defined on $\underline{\underline{\mathbb{R}}}$

In this case $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

- Similarly, odd & periodic

$\rightsquigarrow f(x) = \sum b_n \sin nx$

e.g.



Mean Square approximation

~~Fourier series~~ sums to ∞ !

N-th partial sum $S_N(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos n\frac{2\pi}{T}x + b_n \sin n\frac{2\pi}{T}x \right)$

$f(x) = S_N + $ (error term)

f = "square integrable" on $[0, T]$ $\quad \int_0^T f(x)^2\,dx = $ const

Thm: ① $\lim_{N\to\infty} \left(E_N := \frac{1}{T}\int_0^T (f(x) - S_N(x))^2\,dx \right) = 0$

② $E_N = \frac{1}{T}\int_0^T f(x)^2\,dx - a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2)$

$N \to \infty$ "Parseval equality"

H.W. §2.2 # 1, 5(a) 7(a)?

§2.3 # 1, 5.

§2.4 # 1(a) 5(a)

§2.5 # 1,

§2.6 # 1, 5,

§3.1 – §3.2