process until you run out of variables or equations. sider the example discussed on page 2:

$$\begin{vmatrix} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{vmatrix}.$$

can solve the first equation for x:

$$x = 39 - 2y - 3z.$$

n we substitute this equation into the other equations:

$$\begin{vmatrix} (39 - 2y - 3z) + 3y + 2z = 34 \\ 3(39 - 2y - 3z) + 2y + z = 26 \end{vmatrix}.$$

can simplify:

$$\begin{vmatrix} y - z = -5 \\ -4y - 8z = -91 \end{vmatrix}.$$

, $y = z - 5$, so that $-4(z-5) - 8z = -91$, or

$$-12z = -111.$$

ind that $z = \frac{111}{12} = 9.25$. Then

$$y = z - 5 = 4.25,$$

$$x = 39 - 2y - 3z = 2.75.$$

Explain why this method is essentially the same as the method discussed in this section; only the bookkeeping is different.

46. A hermit eats only two kinds of food: brown rice and yogurt. The rice contains 3 grams of protein and 30 grams of carbohydrates per serving, while the yogurt contains 12 grams of protein and 20 grams of carbohydrates.

a. If the hermit wants to take in 60 grams of protein and 300 grams of carbohydrates per day, how many servings of each item should he consume?

b. If the hermit wants to take in P grams of protein and C grams of carbohydrates per day, how many servings of each item should he consume?

47. I have 32 bills in my wallet, in the denominations of US$ 1, 5, and 10, worth \$100 in total. How many do I have of each denomination?

48. Some parking meters in Milan, Italy, accept coins in the denominations of 20¢, 50¢, and € 2. As an incentive program, the city administrators offer a big reward (a brand new Ferrari Testarossa) to any meter maid who brings back exactly 1,000 coins worth exactly € 1,000 from the daily rounds. What are the odds of this reward being claimed anytime soon?

Matrices, Vectors, and Gauss–Jordan Elimination

When mathematicians in ancient China had to solve a system of simultaneous linear equations such as[4]

$$\begin{vmatrix} 3x + 21y - 3z = 0 \\ -6x - 2y - z = 62 \\ 2x - 3y + 8z = 32 \end{vmatrix},$$

they took all the numbers involved in this system and arranged them in a rectangular pattern (*Fang Cheng* in Chinese), as follows:[5]

3	21	−3	0
−6	−2	−1	62
2	−3	8	32

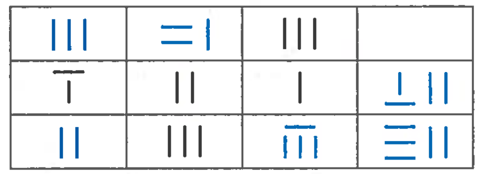
All the information about this system is conveniently stored in this array of numbers.

The entries were represented by bamboo rods, as shown below; red and black rods stand for positive and negative numbers, respectively. (Can you detect how this

[4]This example is taken from Chapter 8 of the *Nine Chapters on the Mathematical Art*; see page 1. Our source is George Gheverghese Joseph, *The Crest of the Peacock, Non-European Roots of Mathematics*, 2nd ed., Princeton University Press, 2000.

[5]Actually, the roles of rows and columns were reversed in the Chinese representation.

number system works?) The equations were then solved in a hands-on fashion, by manipulating the rods. We leave it to the reader to find the solution.



Today, such a rectangular array of numbers,

$$\begin{bmatrix} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{bmatrix},$$

is called a *matrix*.[6] Since this particular matrix has three rows and four columns, it is called a 3×4 matrix ("three by four").

The four columns of the matrix

The three rows of the matrix

$$\begin{bmatrix} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{bmatrix}$$

Note that the first column of this matrix corresponds to the first variable of the system, while the first row corresponds to the first equation.

It is customary to label the entries of a 3×4 matrix A with double subscripts as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

The first subscript refers to the row, and the second to the column: The entry a_{ij} is located in the ith row and the jth column.

Two matrices A and B are equal if they are the same size and if corresponding entries are equal: $a_{ij} = b_{ij}$.

If the number of rows of a matrix A equals the number of columns (A is $n \times n$), then A is called a *square matrix*, and the entries $a_{11}, a_{22}, \ldots, a_{nn}$ form the (main) *diagonal* of A. A square matrix A is called *diagonal* if all its entries above and below the main diagonal are zero; that is, $a_{ij} = 0$ whenever $i \neq j$. A square matrix A is called *upper triangular* if all its entries below the main diagonal are zero; that is, $a_{ij} = 0$ whenever i exceeds j. *Lower triangular* matrices are defined analogously. A matrix whose entries are all zero is called a *zero matrix* and is denoted by 0 (regardless of its size). Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & 0 & 0 \\ 4 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

[6]It appears that the term *matrix* was first used in this sense by the English mathematician J. J. Sylvester, in 1850.

The matrices B, C, D, and E are square, C is diagonal, C and D are upper triangular, and C and E are lower triangular.

Matrices with only one column or row are of particular interest.

> **Vectors and vector spaces**
> A matrix with only one column is called a column vector, or simply a vector. The entries of a vector are called its components. The set of all column vectors with n components is denoted by $\mathbb{R}^n$; we will refer to $\mathbb{R}^n$ as a *vector space*.
> A matrix with only one row is called a row vector.
> In this text, the term *vector* refers to column vectors, unless otherwise stated. The reason for our preference for column vectors will become apparent in the next section.

Examples of vectors are

$$\begin{bmatrix} 1 \\ 2 \\ 9 \\ 1 \end{bmatrix},$$

a (column) vector in $\mathbb{R}^4$, and

$$\begin{bmatrix} 1 & 5 & 5 & 3 & 7 \end{bmatrix},$$

a row vector with five components. Note that the m columns of an $n \times m$ matrix are vectors in $\mathbb{R}^n$.

In previous courses in mathematics or physics, you may have thought about vectors from a more geometric point of view. (See the Appendix for a summary of basic facts on vectors.) Let's establish some conventions regarding the geometric representation of vectors.

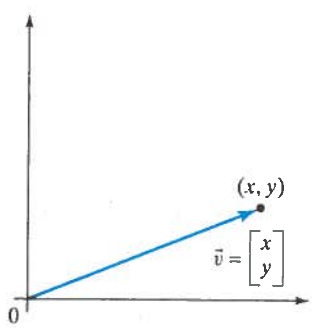


Figure 1

> **Standard representation of vectors**
> The standard representation of a vector
>
> $$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$
>
> in the Cartesian coordinate plane is as an *arrow* (a directed line segment) from the origin to the point (x, y), as shown in Figure 1.
> The standard representation of a vector in $\mathbb{R}^3$ is defined analogously.
> In this text, we will consider the standard representation of vectors, unless stated otherwise.

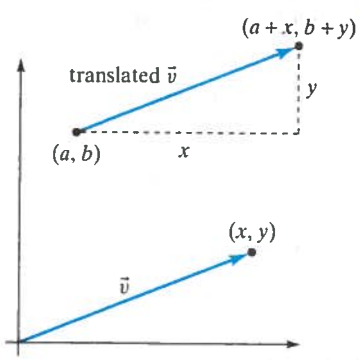


Figure 2

Occasionally, it is helpful to translate (or shift) the vector in the plane (preserving its direction and length), so that it will connect some point (a, b) to the point $(a + x, b + y)$, as shown in Figure 2.

When considering an infinite set of vectors, the arrow representation becomes impractical. In this case, it is sensible to represent the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ simply by the point (x, y), the head of the standard arrow representation of $\vec{v}$.

For example, the set of all vectors $\vec{v} = \begin{bmatrix} x \\ x+1 \end{bmatrix}$ (where x is arbitrary) can be represented as the *line* $y = x + 1$. For a few special values of x we may still use the arrow representation, as illustrated in Figure 3.

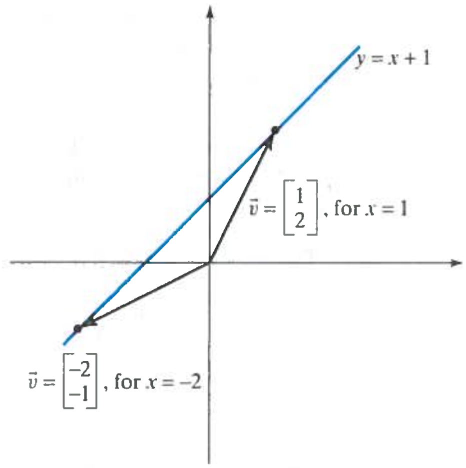


Figure 3

In this course, it will often be helpful to think about a vector numerically, as a list of numbers, which we will usually write in a column.

In our digital age, information is often transmitted and stored as a string of numbers (i.e., as a vector). A section of 10 seconds of music on a CD is stored as a vector with 440,000 components. A weather photograph taken by a satellite is transmitted to Earth as a string of numbers.

Consider the system

$$\begin{vmatrix} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{vmatrix}.$$

Sometimes we are interested in the matrix

$$\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix},$$

which contains the coefficients of the system, called its *coefficient matrix*.

By contrast, the matrix

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix},$$

which displays all the numerical information contained in the system, is called its *augmented matrix*. For the sake of clarity, we will often indicate the position of the equal signs in the equations by a dotted line:

$$\left[\begin{array}{ccc:c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right].$$

To solve the system, it is more efficient to perform the elimination on the augmented matrix rather than on the equations themselves. Conceptually, the two approaches are equivalent, but working with the augmented matrix requires less writing yet is easier to read, with some practice. Instead of dividing an *equation* by a scalar,[7] you can divide a *row* by a scalar. Instead of adding a multiple of an equation to another equation, you can add a multiple of a row to another row.

As you perform elimination on the augmented matrix, you should always remember the linear system lurking behind the matrix. To illustrate this method, we perform the elimination both on the augmented matrix and on the linear system it represents:

$$\left[\begin{array}{rrr|r}2&8&4&2\\2&5&1&5\\4&10&-1&1\end{array}\right]\begin{array}{l}\div 2\\ \\ \\ \end{array}\qquad\left|\begin{array}{rrrr}2x+&8y+&4z=&2\\2x+&5y+&z=&5\\4x+&10y-&z=&1\end{array}\right|\begin{array}{l}\div 2\\ \\ \\ \end{array}$$

$$\downarrow\qquad\qquad\qquad\qquad\downarrow$$

$$\left[\begin{array}{rrr|r}1&4&2&1\\2&5&1&5\\4&10&-1&1\end{array}\right]\begin{array}{l} \\ -2\text{ (I)}\\ -4\text{ (I)}\end{array}\qquad\left|\begin{array}{rrrr}x+&4y+&2z=&1\\2x+&5y+&z=&5\\4x+&10y-&z=&1\end{array}\right|\begin{array}{l} \\ -2\text{ (I)}\\ -4\text{ (I)}\end{array}$$

$$\downarrow\qquad\qquad\qquad\qquad\downarrow$$

$$\left[\begin{array}{rrr|r}1&4&2&1\\0&-3&-3&3\\0&-6&-9&-3\end{array}\right]\begin{array}{l} \\ \div(-3)\\ \\ \end{array}\qquad\left|\begin{array}{rrrr}x+&4y+&2z=&1\\&-3y-&3z=&3\\&-6y-&9z=&-3\end{array}\right|\begin{array}{l} \\ \div(-3)\\ \\ \end{array}$$

$$\downarrow\qquad\qquad\qquad\qquad\downarrow$$

$$\left[\begin{array}{rrr|r}1&4&2&1\\0&1&1&-1\\0&-6&-9&-3\end{array}\right]\begin{array}{l}-4\text{ (II)}\\ \\ +6\text{ (II)}\end{array}\qquad\left|\begin{array}{rrrr}x+&4y+&2z=&1\\&y+&z=&-1\\&-6y-&9z=&-3\end{array}\right|\begin{array}{l}-4\text{ (II)}\\ \\ +6\text{ (II)}\end{array}$$

$$\downarrow\qquad\qquad\qquad\qquad\downarrow$$

$$\left[\begin{array}{rrr|r}1&0&-2&5\\0&1&1&-1\\0&0&-3&-9\end{array}\right]\begin{array}{l} \\ \\ \div(-3)\end{array}\qquad\left|\begin{array}{rrrr}x&-&2z=&5\\&y+&z=&-1\\&&-3z=&-9\end{array}\right|\begin{array}{l} \\ \\ \div(-3)\end{array}$$

$$\downarrow\qquad\qquad\qquad\qquad\downarrow$$

$$\left[\begin{array}{rrr|r}1&0&-2&5\\0&1&1&-1\\0&0&1&3\end{array}\right]\begin{array}{l}+2\text{ (III)}\\ -\text{ (III)}\\ \\ \end{array}\qquad\left|\begin{array}{rrrr}x&-&2z=&5\\&y+&z=&-1\\&&z=&3\end{array}\right|\begin{array}{l}+2\text{ (III)}\\ -\text{ (III)}\\ \\ \end{array}$$

$$\downarrow\qquad\qquad\qquad\qquad\downarrow$$

$$\left[\begin{array}{rrr|r}1&0&0&11\\0&1&0&-4\\0&0&1&3\end{array}\right]\qquad\left|\begin{array}{rrrr}x&&=&11\\&y&=&-4\\&&z=&3\end{array}\right|.$$

The solution is often represented as a vector:

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}11\\-4\\3\end{bmatrix}.$$

Thus far we have been focusing on systems of 3 linear equations with 3 unknowns. Next we will develop a technique for solving systems of linear equations of arbitrary size.

---

[7]In vector and matrix algebra, the term *scalar* is synonymous with (real) number.

Here is an example of a system of three linear equations with five unknowns:

$$\begin{vmatrix} x_1 - x_2 \quad\quad + 4x_5 = 2 \\ x_3 \quad - \quad x_5 = 2 \\ x_4 - \quad x_5 = 3 \end{vmatrix}$$

We can proceed as in the example on page 4. We solve each equation for the leading variable:

$$\begin{vmatrix} x_1 = 2 + x_2 - 4x_5 \\ x_3 = 2 \quad\quad + \; x_5 \\ x_4 = 3 \quad\quad + \; x_5 \end{vmatrix}.$$

Now we can freely choose values for the nonleading variables, $x_2 = t$ and $x_5 = r$, for example. The leading variables are then determined by these choices:

$$x_1 = 2 + t - 4r, \qquad x_3 = 2 + r, \qquad x_4 = 3 + r.$$

This system has infinitely many solutions; we can write the solutions in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 & +t & -4r \\ & t & \\ 2 & & +r \\ 3 & & +r \\ & & r \end{bmatrix}.$$

Again, you can check this answer by substituting the solutions into the original equations, for example, $x_3 - x_5 = (2 + r) - r = 2$.

What makes this system so easy to solve? The following three properties are responsible for the simplicity of the solution, with the second property playing a key role:

- P1: The leading coefficient in each equation is 1. (The leading coefficient is the coefficient of the leading variable.)
- P2: The leading variable in each equation does not appear in any of the other equations. (For example, the leading variable x_3 of the second equation appears neither in the first nor in the third equation.)
- P3: The leading variables appear in the "natural order," with increasing indices as we go down the system (x_1, x_3, x_4 as opposed to x_3, x_1, x_4, for example).

Whenever we encounter a linear system with these three properties, we can solve for the leading variables and then choose arbitrary values for the other, nonleading variables, as we did above and on page 4.

Now we are ready to tackle the case of an arbitrary system of linear equations. We will illustrate our approach by means of an example:

$$\begin{vmatrix} 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 = 2 \\ x_1 + 2x_2 - x_3 + 2x_4 \quad\quad = 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 = 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 = 9 \end{vmatrix}.$$

We wish to reduce this system to a system satisfying the three properties (P1, P2, and P3); this reduced system will then be easy to solve.

We will proceed from equation to equation, from top to bottom. The leading variable in the first equation is x_1, with leading coefficient 2. To satisfy property P1, we will divide this equation by 2. To satisfy property P2 for the variable x_1, we will then subtract suitable multiples of the first equation from the other three equations to eliminate the variable x_1 from those equations. We will perform these operations both on the system and on the augmented matrix.

$$\left|\begin{array}{rcl} 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 &=& 2 \\ x_1 + 2x_2 - x_3 + 2x_4 \phantom{{}+ 4x_5} &=& 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 &=& 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 &=& 9 \end{array}\right| \begin{array}{l} \div 2 \\ \\ \\ \\ \end{array} \qquad \left[\begin{array}{rrrrr|r} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \begin{array}{l} \div 2 \\ \\ \\ \\ \end{array}$$

$$\downarrow$$

$$\left|\begin{array}{rcl} x_1 + 2x_2 - x_3 + x_4 + 2x_5 &=& 1 \\ x_1 + 2x_2 - x_3 + 2x_4 \phantom{{}+ 2x_5} &=& 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 &=& 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 &=& 9 \end{array}\right| \begin{array}{l} \\ -(I) \\ -3(I) \\ -5(I) \end{array} \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \begin{array}{l} \\ -(I) \\ -3(I) \\ -5(I) \end{array}$$

$$\downarrow$$

$$\left|\begin{array}{rcl} x_1 + 2x_2 - x_3 + x_4 + 2x_5 &=& 1 \\ x_4 - 2x_5 &=& 3 \\ x_3 - 2x_4 + 3x_5 &=& -2 \\ x_3 \phantom{{}- 2x_4} - x_5 &=& 4 \end{array}\right| \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right]$$

Now on to the second equation, with leading variable x_4 and leading coefficient 1. We could eliminate x_4 from the first and third equations and then proceed to the third equation, with leading variable x_3. However, this approach would violate our requirement P3 that the variables must be listed in the natural order, with increasing indices as we go down the system. To satisfy this requirement, we will swap the second equation with the third equation. (In the following summary, we will specify when such a swap is indicated and how it is to be performed.)

Then we can eliminate x_3 from the first and fourth equations.

$$\left|\begin{array}{rcl} x_1 + 2x_2 - x_3 + x_4 + 2x_5 &=& 1 \\ x_3 - 2x_4 + 3x_5 &=& -2 \\ x_4 - 2x_5 &=& 3 \\ x_3 \phantom{{}- 2x_4} - x_5 &=& 4 \end{array}\right| \begin{array}{l} +(II) \\ \\ \\ -(II) \end{array} \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right] \begin{array}{l} +(II) \\ \\ \\ -(II) \end{array}$$

$$\downarrow$$

$$\left|\begin{array}{rcl} x_1 + 2x_2 \phantom{{}- x_3} - x_4 + 5x_5 &=& -1 \\ x_3 - 2x_4 + 3x_5 &=& -2 \\ x_4 - 2x_5 &=& 3 \\ 2x_4 - 4x_5 &=& 6 \end{array}\right| \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 2 & -4 & 6 \end{array}\right]$$

Now we turn our attention to the third equation, with leading variable x_4. We need to eliminate x_4 from the other three equations.

$$\left|\begin{array}{rcl} x_1 + 2x_2 \phantom{{}- x_3} - x_4 + 5x_5 &=& -1 \\ x_3 - 2x_4 \quad 3x_5 &=& -2 \\ x_4 - 2x_5 &=& 3 \\ 2x_4 - 4x_5 &=& 6 \end{array}\right| \begin{array}{l} +(III) \\ +2(III) \\ \\ -2(III) \end{array} \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 2 & -4 & 6 \end{array}\right] \begin{array}{l} +(III) \\ +2(III) \\ \\ -2(III) \end{array}$$

$$\downarrow$$

$$\left|\begin{array}{rcl} x_1 + 2x_2 \phantom{{}- x_3 - x_4} + 3x_5 &=& 2 \\ x_3 \phantom{{}- 2x_4} - x_5 &=& 4 \\ x_4 - 2x_5 &=& 3 \\ 0 &=& 0 \end{array}\right| \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since there are no variables left in the fourth equation, we are done. Our system now satisfies properties P1, P2, and P3. We can solve the equations for the leading variables:

$$\begin{vmatrix} x_1 = 2 - 2x_2 - 3x_5 \\ x_3 = 4 \qquad\quad + \ x_5 \\ x_4 = 3 \qquad\quad + 2x_5 \end{vmatrix}$$

If we let $x_2 = t$ and $x_5 = r$, then the infinitely many solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 & -2t & -3r \\ & t & \\ 4 & & +r \\ 3 & & +2r \\ & & r \end{bmatrix}.$$

Let us summarize.

Solving a system of linear equations
We proceed from equation to equation, from top to bottom.

Suppose we get to the ith equation. Let x_j be the leading variable of the system consisting of the ith and all the subsequent equations. (If no variables are left in this system, then the process comes to an end.)

- If x_j does not appear in the ith equation, swap the ith equation with the first equation below that does contain x_j.
- Suppose the coefficient of x_j in the ith equation is c; thus this equation is of the form $cx_j + \cdots = \cdots$. Divide the ith equation by c.
- Eliminate x_j from all the other equations, above and below the ith, by subtracting suitable multiples of the ith equation from the others.

Now proceed to the next equation.

If an equation *zero* = *nonzero* emerges in this process, then the system fails to have solutions; the system is *inconsistent*.

When you are through without encountering an inconsistency, solve each equation for its leading variable. You may choose the nonleading variables freely; the leading variables are then determined by these choices.

This process can be performed on the augmented matrix. As you do so, just imagine the linear system lurking behind it.

In the preceding example, we reduced the augmented matrix

$$M = \left[\begin{array}{ccccc|c} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \quad \text{to} \quad E = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We say that the final matrix E is in reduced row-echelon form (rref).

Reduced row-echelon form

A matrix is in reduced row-echelon form if it satisfies all of the following conditions:

a. If a row has nonzero entries, then the first nonzero entry is a 1, called the *leading* 1 (or *pivot*) in this row.

b. If a column contains a leading 1, then all the other entries in that column are 0.

c. If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

Condition c implies that rows of 0's, if any, appear at the bottom of the matrix.

Conditions a, b, and c defining the reduced row-echelon form correspond to the conditions P1, P2, and P3 that we imposed on the system.

Note that the leading 1's in the matrix

$$E = \left[\begin{array}{ccccc|c} \textcircled{1} & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & \textcircled{1} & 0 & -1 & 4 \\ 0 & 0 & 0 & \textcircled{1} & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

correspond to the leading variables in the reduced system,

$$\left| \begin{array}{rcrcrcrcl} \boxed{x_1} & + & 2x_2 & & & & + & 3x_5 & = 2 \\ & & & \boxed{x_3} & & & - & x_5 & = 4 \\ & & & & \boxed{x_4} & & - & 2x_5 & = 3 \end{array} \right| .$$

Here we draw the staircase formed by the leading variables. This is where the name *echelon form* comes from. According to Webster, an echelon is a formation "like a series of steps."

The operations we perform when bringing a matrix into reduced row-echelon form are referred to as elementary row operations. Let's review the three types of such operations.

Types of elementary row operations

- Divide a row by a nonzero scalar.
- Subtract a multiple of a row from another row.
- Swap two rows.

Consider the following system:

$$\left| \begin{array}{rcrcrcrcr} x_1 & - & 3x_2 & & & - & 5x_4 & = & -7 \\ 3x_1 & - & 12x_2 & - & 2x_3 & - & 27x_4 & = & -33 \\ -2x_1 & + & 10x_2 & + & 2x_3 & + & 24x_4 & = & 29 \\ -x_1 & + & 6x_2 & + & x_3 & + & 14x_4 & = & 17 \end{array} \right| .$$

The augmented matrix is

$$\left[\begin{array}{rrrr|r} 1 & -3 & 0 & -5 & -7 \\ 3 & -12 & -2 & -27 & -33 \\ -2 & 10 & 2 & 24 & 29 \\ -1 & 6 & 1 & 14 & 17 \end{array}\right].$$

The reduced row-echelon form for this matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

(We leave it to you to perform the elimination.)

Since the last row of the echelon form represents the equation $0 = 1$, the system is inconsistent.

This method of solving linear systems is sometimes referred to as *Gauss–Jordan elimination*, after the German mathematician Carl Friedrich Gauss (1777–1855; see Figure 4), perhaps the greatest mathematician of modern times, and the German engineer Wilhelm Jordan (1844–1899). Gauss himself called the method *eliminatio vulgaris*. Recall that the Chinese were using this method 2,000 years ago.

Figure 4 Carl Friedrich Gauss appears on an old German 10-mark note. (In fact, this is the *mirror image* of a well-known portrait of Gauss.[8])

How Gauss developed this method is noteworthy. On January 1, 1801, the Sicilian astronomer Giuseppe Piazzi (1746–1826) discovered a planet, which he named Ceres, in honor of the patron goddess of Sicily. Today, Ceres is called a dwarf planet, because it is only about 1,000 kilometers in diameter. Piazzi was able to observe Ceres for 40 nights, but then he lost track of it. Gauss, however, at the age of 24, succeeded in calculating the orbit of Ceres, even though the task seemed hopeless on the basis of a few observations. His computations were so accurate that the German astronomer W. Olbers (1758–1840) located the asteroid on December 31, 1801. In the course of his computations, Gauss had to solve systems of 17 linear equations.[9] In dealing with this problem, Gauss also used the method of least

[8] Reproduced by permission of the German Bundesbank.

[9] For the mathematical details, see D. Teets and K. Whitehead, "The Discovery of Ceres: How Gauss Became Famous," *Mathematics Magazine*, 72, 2 (April 1999): 83–93.

12. $\left| \begin{array}{l} 2x_1 - 3x_3 + 7x_5 + 7x_6 = 0 \\ -2x_1 + x_2 + 6x_3 - 6x_5 - 12x_6 = 0 \\ x_2 - 3x_3 + x_5 + 5x_6 = 0 \\ -2x_2 + x_4 + x_5 + x_6 = 0 \\ 2x_1 + x_2 - 3x_3 + 8x_5 + 7x_6 = 0 \end{array} \right|$

Solve the linear systems in Exercises 13 through 17. You may use technology.

13. $\left| \begin{array}{l} 3x + 11y + 19z = -2 \\ 7x + 23y + 39z = 10 \\ -4x - 3y - 2z = 6 \end{array} \right|$

14. $\left| \begin{array}{l} 3x + 6y + 14z = 22 \\ 7x + 14y + 30z = 46 \\ 4x + 8y + 7z = 6 \end{array} \right|$

15. $\left| \begin{array}{l} 3x + 5y + 3z = 25 \\ 7x + 9y + 19z = 65 \\ -4x + 5y + 11z = 5 \end{array} \right|$

16. $\left| \begin{array}{l} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 = 11 \end{array} \right|$

17. $\left| \begin{array}{l} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 = 24 \end{array} \right|$

18. Determine which of the matrices below are in reduced row-echelon form:

a. $\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}$

19. Find all 4×1 matrices in reduced row-echelon form.

20. We say that two $n \times m$ matrices in reduced row-echelon form are of the same type if they contain the same number of leading 1's in the same positions. For example,

$$\begin{bmatrix} \textcircled{1} & 2 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \textcircled{1} & 3 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there?

21. How many types of 3×2 matrices in reduced row-echelon form are there? (See Exercise 20.)

22. How many types of 2×3 matrices in reduced row-echelon form are there? (See Exercise 20.)

23. Suppose you apply Gauss–Jordan elimination to a matrix. Explain how you can be sure that the resulting matrix is in reduced row-echelon form.

24. Suppose matrix A is transformed into matrix B by means of an elementary row operation. Is there an elementary row operation that transforms B into A? Explain.

25. Suppose matrix A is transformed into matrix B by a sequence of elementary row operations. Is there a sequence of elementary row operations that transforms B into A? Explain your answer. (See Exercise 24.)

26. Consider an $n \times m$ matrix A. Can you transform rref(A) into A by a sequence of elementary row operations? (See Exercise 25.)

27. Is there a sequence of elementary row operations that transforms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

Explain.

28. Suppose you subtract a multiple of an equation in a system from another equation in the system. Explain why the two systems (before and after this operation) have the same solutions.

29. *Balancing a chemical reaction.* Consider the chemical reaction

$$a\, NO_2 + b\, H_2O \rightarrow c\, HNO_2 + d\, HNO_3,$$

where a, b, c, and d are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must be the same before and after the reaction. For example, because the number of oxygen atoms must remain the same,

$$2a + b = 2c + 3d.$$

While there are many possible values for a, b, c, and d that balance the reaction, it is customary to use the smallest possible positive integers. Balance this reaction.

30. Find the polynomial of degree 3 [a polynomial of the form $f(t) = a + bt + ct^2 + dt^3$] whose graph goes through the points (0, 1), (1, 0), (−1, 0), and (2, −15). Sketch the graph of this cubic.

31. Find the polynomial of degree 4 whose graph goes through the points (1, 1), (2, −1), (3, −59), (−1, 5), and (−2, −29). Graph this polynomial.

32. *Cubic splines.* Suppose you are in charge of the design of a roller coaster ride. This simple ride will not make any left or right turns; that is, the track lies in a vertical plane. The accompanying figure shows the ride as viewed from the side. The points (a_i, b_i) are given to you, and your job is to connect the dots in a reasonably smooth way. Let $a_{i+1} > a_i$.