How to find the Jordan canonical form of a matrix

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First of all, there is a systematic way to do this, but explaining it would take 20 pages! However, here are some examples to make you understand the general procedure! From now on, we'll only be working with 3×3 matrices to make things easier.

Warning: All of this is nonrigorous, and it's ok if at the end you're still confused about this topic!

Note: L.I. means 'linearly independent'

1 Introduction

How would we go about this? Suppose (v_1, v_2, v_3) is a basis of $\mathbb{R}^3$ and the matrix of T with respect to this basis is:

$$\mathcal{M}(T) = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

By definition of a matrix of a linear transformation, this means that:

$$\begin{aligned} T(v_1) &= 3v_1 \\ T(v_2) &= v_1 + 3v_2 \\ T(v_3) &= v_2 + 3v_3 \end{aligned}$$

Rewriting this, we get:

$$\begin{aligned} (T - 3I)v_1 &= 0 \\ (T - 3I)v_2 &= v_1 \\ (T - 3I)v_3 &= v_2 \end{aligned}$$

From this example, we can guess the following strategy, which we'll follow throughout this handout:

General strategy:

1) First find all the eigenvectors of T corresponding to a certain eigenvalue!

2) The number of L.I. eigenvectors you found gives you the number of Jordan blocks (here there was only 'one' L.I eigenvector, hence only one Jordan block)

3) Once you found that eigenvector, solve $(T - \lambda I)v =$ that eigenvector, and continue

Let's apply this strategy to a couple of examples to see how useful this is!

2 3 L.I. eigenvectors

This is really the best-case scenario, and there isn't much to do!

Example: Put $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$ into Jordan canonical form.

1) If you calculate the eigenvalues, you find that there are two: $\lambda = 5$ and $\lambda = 3$

2) Find the eigenspaces:

$$Nul(A - 5I) = Span \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$Nul(A - 3I) = Span \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Notice that here we have $3 = dim(\mathbb{R}^3)$ L.I. eigenvectors (more precisely, three linearly independent eigenvectors), so there's no need to go further and search for generalized eigenvectors!

If you put all three eigenvectors together, you find that a Jordan canonical basis for $\mathbb{R}^3$ is:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

And a Jordan canonical form for A is:

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Here there are precisely 3 Jordan blocks: $\begin{bmatrix} 5 \end{bmatrix}, \begin{bmatrix} 3 \end{bmatrix}, \begin{bmatrix} 3 \end{bmatrix}$.

3) Again, we already found our answer, so no need to go further!

3 2 L.I. eigenvectors

Example: Put $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ into Jordan canonical form.

1) Then you can check that $\lambda = 1$ is the only eigenvalue of A.

2) $Nul(A - I) = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

3) Notice that since there are only two L.I. eigenvectors, this implies that A can only have the following Jordan canonical forms:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The reason for this is that notice that every column that has only one nonzero entry corresponds to an eigenvector of T (so here, the first and second columns of the first matrix correspond to eigenvectors, whereas for the second matrix, it's the first and third columns).

Let's focus on the first matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. What this says is that:

$$\begin{aligned} Av_1 &= v_1 \\ Av_2 &= v_2 \\ Av_3 &= v_3 + v_2 \end{aligned}$$

That is:

$$
\begin{aligned}
(A-I)v_1 &= 0 \\
(A-I)v_2 &= 0 \\
(A-I)v_3 &= v_2
\end{aligned}
$$

Now here's the 'guessing-part'. We know that v_1 and v_2 need to be eigenvectors, so how about let $v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.[1]

To find v_3, all we nee to solve is the equation $(A-I)v_3 = v_2$, that is:

$$
\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

Where $v_3 = (x, y, z)$.

If you solve this, then you should get that the solution set is:

$$
x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

Since we're only looking for *one* vector, we can set $x = 0$ and $y = 0$, and we get:

$$
v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

So our answer is:

$$
(v_1, v_2, v_3) = \left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)
$$

And the corresponding Jordan canonical form is:

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
$$

[1] If this fails, then just try $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

which has two Jordan blocks, namely $\begin{bmatrix} 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Remark 1: If you chose the basis (v_2, v_3, v_1), then you would have gotten the other Jordan canonical form $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Remark 2: What if you chose $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and tried to solve $(A - I)v_3 = v_2$? Then you would find that there are no solutions, which is a sign that you didn't chose the right basis. In that case, do as I said and switch your v_1 and v_2 (see footnote). Of course this gets way more complicated for $n \times n$ matrices, but as I said, there is a systematic procedure, but it's a bit complicated!

4 1 L.I. eigenvector

That's in some sense the worst-case-scenario, but it's not as bad as the previous section because there's basically only one way of doing this!

Example: Put $A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$ into Jordan canonical form.

1) There is only one eigenvalue $\lambda = -1$

2)
$$Nul(A - (-I)) = Nul(A + I) = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

3) Here there is only one L.I. eigenvector, which means that there is only one Jordan canonical form of A, namely:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

And looking at this matrix, it follows that v_1 must be an eigenvector of A, and moreover:

$Av_2 = -v_2 + v_1$, that is $(A + I)v_2 = v_1$ and $Av_3 = -v_3 + v_2$, that is $(A + I)v_3 = v_2$.

This means that:

(a) We let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(b) We solve for v_2 in $(A+I)v_2 = v_1$

(c) And then solve for v_3 in $(A+I)v_3 = v_2$.

Now if you solve $(A+I)v_2 = v_1$ for $v_2 = (x, y, z)$, you need to solve:

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Whose solution set is:

$$x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Again, for simplicity, choose $x = 0$, and you get that $v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ [2]

Now if you solve $(A+I)v_3 = v_2$ for $v_3 = (x', y', z')$, you need to solve:

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Whose solution set is:

$$x' \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

Again, for simplicity, choose $x' = 0$, and you get that $v_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$

So our answer is:

$$(v_1, v_2, v_3) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} \right)$$

[2]Notice that it's **NOT** ok to rescale v_2 to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, that would give you the **wrong** answer!

And the corresponding Jordan canonical form is:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$