MT1

PROBLEM 1

(a) Write $\mathbf{v} = (1, 1)$ and $\mathbf{w} = (1, 5)$. Choose a number c so that $\mathbf{v}$ and $\mathbf{w} - c\mathbf{v}$ are perpendicular.
(b) More formally, let $\mathbf{v}$ and $\mathbf{w}$ be two nonzero vectors. Find a formula of c such that $\mathbf{v}$ and $\mathbf{w} - c\mathbf{v}$ are perpendicular.

PROBLEM 2

Let A be an $l \times m$ matrix, B and $m \times n$ matrix and C an $n \times p$ matrix.
(a) Prove that $(AB)C = A(BC)$ (associative law).
(b) Let $l = m = n = p$ (so all matrices are square). If $AB = I$ and $BC = I$, prove $A = C$.

PROBLEM 3

Use Gauss–Jordan elimination to find the U^{-1} for

$$U = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1. \end{bmatrix}$$

PROBLEM 4

If A is a 3×3 matrix.
(a) If A has row 1 + row 2 = row 3, prove that A is not invertible.
(b) If A has column 1 + column 2 = column 3, prove that A is not invertible.

PROBLEM 5

Explain the following facts (as best as you can!)

(a) If a system of linear equations is nonsingular, it has a unique solution.

(b) If the system is singular, it either has no solution, or has infinitely many solutions.

MT2

PROBLEM 6

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

(a) Find a basis for the null space of A.

(b) Find a basis for the column space of A.

PROBLEM 7

Let A be an $m \times n$ matrix. Consider A as a map from a domain vector space to a target vector space.

(a) What are the dimensions of these two vector spaces?

(b) Show that the null space of A is a subspace of the domain vector space.

PROBLEM 8

What conditions on b_1, b_2, b_3, b_4 make the following system solvable? Find $\mathbf{x}$ in that case:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

PROBLEM 9

This problem consists of 4 parts. You may use the conclusion of Part (a) to solve Part (b), and (b) to solve (c) etc.. You do not need to solve (a) in order to get the full credit for (b), etc.

Let A, B, M, N be two matrices such that AB and MN make sense.

(a) Show that $\text{rank}(MN) \leq \text{rank}(N)$.

(b) Suppose that A is invertible. Show that $\text{rank}(AB) = \text{rank}(B)$. (Hint: May use (a) twice for $M = A, N = B$ and $M = A^{-1}, N = AB$ to get 2 inequalities.)

(Parts (c) and (d) on the next page.)

(c) Explain why the dimension of column space $C(B)$ is the same as the dimension of $C(R)$, where R is the reduced form. (Hint: May use (b) for A as the elimination and permutation matrices.)

(d) Suppose that B is an $m \times n$ matrix. Show that the dimension of $\text{Null}(B)$ plus the rank of B is equal to n.

MT3

PROBLEM 10

Let A be an arbitrary $m \times n$ matrix. Prove that every $\mathbf{y}$ in $N(A^T)$ is perpendicular to every $A\mathbf{x}$ in the column space.

PROBLEM 11

Project $\mathbf{b}$ onto the column space of A for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

PROBLEM 12

Find orthogonal vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ by Gram-Schmidt from $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

$$\mathbf{a} = (1, -1, 0, 0),\ \mathbf{b} = (0, 1, -1, 0),\ \mathbf{c} = (0, 0, 1, -1).$$

PROBLEM 13

Find the determinant of the matrix A in the following two ways.
(a) By row operations to produce an upper triangular U.
(b) By the cofactor formula.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}.$$

PROBLEM 14

Let A be a 3×3 matrix. Prove Cramer's rule

$$A^{-1} = \frac{1}{\det(A)} C^T,$$

where C is the cofactor matrix.

MT4

PROBLEM 15

The transformation $T : V \to W$ takes the *second derivative*, where V and W are both the vector space of polynomials of degree less than or equal to 3. Let $\{\mathbf{v}_1 = 1, \mathbf{v}_2 = x, \mathbf{v}_3 = x^2, \mathbf{v}_4 = x^3\}$ be a basis of V and let $\{\mathbf{w}_1 = 1 - x, \mathbf{w}_2 = 1 + x, \mathbf{w}_3 = x^2, \mathbf{w}_4 = x^3\}$ be a basis of W.
(a) What are the dimensions of V and W? (4 points)
Pur your answers here: $\dim(V) =$, $\dim(W) =$. (**No justification needed.**
(b) Find the matrix A for T. (20 points)

PROBLEM 16

Diagonalize B and compute $S\Lambda S^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \qquad \text{has} \qquad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$$

PROBLEM 17

Solve the differential equation

$$\frac{d}{dt}\mathbf{u}(t) = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t)$$

with the initial condition $\mathbf{u}(0) = (5, -2)$.

PROBLEM 18

Let A be an $n \times n$ matrix. If λ is an eigenvalue of A, prove that there is always a *nonzero* eigenvector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

PROBLEM 19

Let

$$V = \{\mathbf{u}(t) \mid \mathbf{u}(t) \text{ is an } \mathbb{R}^2\text{-valued smooth function in } t\}.$$

In other words, V is the vector space of $\mathbb{R}^2$-valued smooth functions in one variable t (differentiable infinitely many times). Let A be a 2×2 constant matrix.

Prove that the set of solutions to

$$\left(\frac{d}{dt} - A\right) \mathbf{u}(t) = \vec{0}$$

is a vector space.