MATH 1100-2 Spring 2007

First Midterm Exam

Instructor: Y.-P. Lee

LAST NAME ______________________________

First NAME ______________________________

UID NO. ______________________________

INSTRUCTION: Show all of your work. Make sure your answers are clear and legible. Use *specified* method to solve the question. It is not necessary to simplify your final answers.

Problem 1	10	______
Problem 2	20	______
Problem 3	20	______
Problem 4	15	______
Problem 5	20	______
Problem 6	15	______
Total	100	______

PROBLEM 1

Use the limit definition to find the derivative of the function $f(x) = x^2 + 4$.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 + 4] - [x^2 + 4]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\Delta x)(2x + \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x) \\
&= 2x
\end{aligned}$$

PROBLEM 2

$$x^2 + y^2 = 25.$$

One can solve y in terms of x (two branches)

$$y = \sqrt{25 - x^2}, \qquad \text{or} \qquad y = -\sqrt{25 - x^2}.$$

Find $\frac{dy}{dx}$ at $(-4, -3)$ implicitly and explicitly and show that the results are equivalent. *Note:* $(-4, -3)$ *lies on one of the branch.*

Implicitly. Implicit differentiation of $x^2 + y^2 = 25$ with respect to x:

$$2x + 2y\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

At $(-4, -3)$, $\frac{dy}{dx} = -\frac{4}{3}$.

Explicitly. $(-4, -3)$ lies on the branch $\qquad y = -\sqrt{25 - x^2}$.

$$\frac{dy}{dx} = -\frac{1}{2}(25 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{25 - x^2}}.$$

At $(-4, -3)$, $\frac{dy}{dx} = -\frac{4}{3}$.

Therefore, they are equivalent.

PROBLEM 3

Let $f(x) = x\sqrt{x+1}$. Find
(a) the domain of the function, (b) critical numbers, (c) the open intervals on which $f(x)$ is increasing or decreasing.

(a) Domain: $x + 1 \geq 0$. Thus domain $= [-1, \infty)$.
(b)

$$f'(x) = \sqrt{x+1} + x\left(\frac{1}{2\sqrt{x+1}}\right) = \frac{3x+2}{2\sqrt{x+1}}.$$

$$f'(x) = 0 \Leftrightarrow 3x + 2 = 0 \Leftrightarrow x = -\frac{2}{3},$$

$$f'(x) \text{ DNE } \Leftrightarrow x = -1.$$

The critical numbers are $x = -1, -\frac{2}{3}$.

(c) The critical numbers $x = -1, -\frac{2}{3}$ divide the domain $[-1, \infty)$ into two open intervals $(-1, -\frac{2}{3})$ and $(-\frac{2}{3}, \infty)$. Pick a point from each open interval, say $-3/4$ from the first and 0 from the second.

$$f'(-\frac{3}{4}) = -\frac{1}{4} < 0 \Rightarrow \text{increasing.}$$

$$f'(0) = 1 > 0. \Rightarrow \text{decreasing.}$$

Increasing on $(-1, -\frac{2}{3})$ and decreasing on $(-\frac{2}{3}, \infty)$.

PROBLEM 4

Find the intervals on which the graph of the function

$$y = f(x) = \frac{x^2 - 1}{2x + 1}$$

is concave upward and those on which it is concave downward.

Straightforward calculations via the quotient rule

$$f'(x) = \frac{(2x+1)(2x) - (x^2-1)(2)}{(2x+1)^2} = \frac{2(x^2+x+1)}{(2x+1)^2}$$

and via the quotient rule and general power rule

$$\begin{aligned} f''(x) &= 2\frac{(2x+1)^2(2x+1) - (x^2+x+1)\frac{d}{dx}(2x+1)^2}{(2x+1)^4} \\ &= 2\frac{(2x+1)^2(2x+1) - (x^2+x+1)(2(2x+1)2)}{(2x+1)^4} \\ &= \frac{-6}{(2x+1)^3}. \end{aligned}$$

$f''(x) > 0 \Leftrightarrow (2x+1)^3 < 0 \Leftrightarrow (2x+1) < 0 \Leftrightarrow x < -\dfrac{1}{2} \Leftrightarrow$ concave upward.
Similarly,

$$f''(x) < 0 \Leftrightarrow x > -\frac{1}{2} \Leftrightarrow \text{concave downward.}$$

Conclusion: concave upward on $(-\infty, -\frac{1}{2})$. concave downward on $(-\frac{1}{2}, -\infty)$.

PROBLEM 5

A right triangle is formed in the first quadrant by the x- and y-axes and a line through $(1, 2)$. The three vertices of the triangles are $(0, 0), (x, 0), (0, y)$. Find x and y such that the area of the triangle is minimum.

Two similar triangles: First triangle has vertices $(0, 0), (x, 0), (0, y)$, and the second $(1, 0), (x, 0), (1, 2)$.

Using the fact that similar triangles have proportional sides, one has

$$\frac{x-1}{x} = \frac{2}{y} \Rightarrow y = \frac{2x}{x-1}.$$

The area function is

$$f(x) = \frac{1}{2}xy = \frac{2x^2}{2(x-1)},$$

with domain (x variable) equal to $(1, \infty)$. Note that if $x \leq 1$, the line through $(1, 2)$ will not form a triangle in the first quadrant.

$$f'(x) = \frac{x^2 - 2x}{(x-1)^2}$$

In the domain $(1, \infty)$ $f'(x)$ is always well-defined. Therefore the only critical points come from

$$f'(x) = 0 \Leftrightarrow x^2 - 2x = x(x-2) = 0 \Leftrightarrow x = 2 \text{ or } x = 0.$$

However, $x = 0$ does not lie in the domain, therefore the only critical number is $x = 2$.

Checking if $x = 2$ produces an absolute minimum. Note that $x = 2$ cuts the domain into two open intervals $(1, 2)$ and $(2, \infty)$. Observe that

$$f'(x) < 0 \text{ for } 1 < x < 2 \Rightarrow f(x) \text{ is decreasing in } (1, 2)$$
$$f'(x) > 0 \text{ for } 2 < x \Rightarrow f(x) \text{ is increasing in } (2, \infty)$$

Therefore, $x = 2$ produce the minimum.

When $x = 2$, $y = \frac{2x}{x-1} = 4$.

PROBLEM 6

Compare the values of dy and Δy, where

$$y = f(x) = x^4 + 1, \qquad x = -1, \qquad \Delta x = dx = 1.$$

$$\Delta y = f(x + \Delta x) - f(x) = f(-1 + 1) - f(-1) = 1 - 2 = -1.$$

$$dy = 4x^3 dx = 4(-1)^3(1) = -4.$$