Invariance of tautological equations

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An observation: Tautological equations hold for any geometric Gromov–Witten theory.

Question 1. How about non-geometric GW theory? e.g. abstract Frobenius manifolds and its higher genus counterparts.

Question 2. Is this enough to determine the tautological equations (in cohomology theory)?

Conjecture. Yes to both questions!

Outline of the talk

1. Review of the geometric GW theory.
2. Givental's axiomatic GW theory.
3. Invariance of tautological equations – partial answer to Question 1.
4. Using invariance to derive tautological equations – partial answer to Question 2.

1. Geometric GW theory

GW theory studies the tautological intersection theory on $\overline{M}_{g,n}(X,\beta)$, the moduli spaces of stable maps from curves C of genus g with n marked points to a smooth projective variety X. The intersection numbers, or *Gromov–Witten invariants* (where $\gamma_i \in H^*(X)$)

$$GW_{g,n,\beta,\gamma,k} := \int_{[\overline{M}_{g,n}(X,\beta)]^{\text{vir}}} \prod_{i=1}^{n} \text{ev}_i^*(\gamma_i)\psi_i^{k_i}$$

ψ_i: Define the cotangent bundles L_i at a moduli point (i.e. a map) to be $T^*_{x_i}C$. $\psi_i := c_1(L_i)$.

Notations:
1. $H := H^*(X,\mathbb{C})$, assumed of rank N.
2. $\{\phi_\mu\}$ be an (*orthonormal*) basis of H w.r.t. the Poincaré pairing (such that $\phi_1 = 1$).
3. $\mathcal{H} = \oplus_0^\infty H =$ infinite dim complex v.s. with basis $\{\phi_\mu \psi^k\}$.
4. $\{t_k^\mu\}$, $\mu = 1,\ldots,N$, $k = 0,\ldots,\infty$, be the dual coordinates of the basis $\{\phi_\mu\psi^k\}$.
Observation: At each marked point, the insertion is $\mathcal{H}$-valued.

Let $t := \sum_{k,\mu} t_k^\mu \phi_\mu \psi^k$ denote a general element in the vector space $\mathcal{H}$.
$F_g(t) = \sum_{n,\beta} \frac{1}{n!} GW_{n,\beta}$ the generating function of all genus g GW invariants.

Tautological equations: Suppose that there is an equation $E = 0$ of the cohomology/Chow classes in $\overline{M}_{g,n}$, moduli space of stable curves. Since there is a morphism

$$\overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n}$$

by forgetting the map,

One can pull-back $E = 0$ to $\overline{M}_{g,n}(X, \beta)$.

If $E = 0$ is an equation of tautological classes, it is called an *tautological equation*. — Without going into the details, tautological classes "basically" includes ψ-classes and boundary classes coming from gluing:

$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}$$
$$\overline{M}_{g,n+2} \to \overline{M}_{g+1,n}.$$

Conclusion: Tautological equations hold for all geometric GW theory.

2. Givental–Gromov–Witten theory

Symplectic loop space. (abstract setting)

1. H = a cpx v.s. of dim N with a distinguished element $\phi_1 = \mathbf{1}$.

2. $(\cdot,\cdot)$ be a $\mathbb{C}$-bilinear metric on H.
$g_{\mu\nu} := (\phi_\mu, \phi_\nu)$, $g^{\mu\nu}$ is the inverse matrix.

3. $\mathbb{H} := H[z^{-1}, z]$, a $\mathbb{C}$-v.s. of H-valued Laurent polynomials in $1/z$. (Note: Various completions of this spaces will be used.)

4. $\mathbb{H}_+ := H[z]$, $\mathbb{H}_- := z^{-1}H[z^{-1}]$.

5. The symplectic form Ω on $\mathbb{H}$:

$$\Omega(f, g) = \frac{1}{2\pi i} \oint (f(-z), g(z))\, dz.$$

There is a natural polarization $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$ by the Lagrangian subspaces, which also provides a symplectic identification of $(\mathbb{H}, \Omega)$ with the cotangent bundle $T^*\mathbb{H}_+$.

Note: The parallel between $\mathbb{H}_+$ and $\mathcal{H}$ is evident, and is in fact given by the affine coordinate transformation

$$t_k^\mu = q_k^\mu + \delta^{\mu 1}\delta_{k1}.$$

An H-valued Laurent formal series can be written in this basis as

$$\ldots + p_1^\mu \phi_\mu \frac{1}{(-z)^2} + p_0^\mu \phi_\mu \frac{1}{(-z)} + q_0^\mu \phi_\mu + q_1^\mu \phi_\mu z + \ldots .$$

In fact, $\{p_k^\mu, q_k^\mu\}_{\mu=1,k=0}^{N,\infty}$ are the Darboux coordinates compatible with this polarization

$$\Omega = \sum dp_k^\mu \wedge dq_k^\mu .$$

Definition of $g = 0$ axiomatic theory. * Let $F_0(q)$ be a function on $\mathbb{H}_+$. The pair $(\mathbb{H}, F_0(q))$ is called a $g = 0$ axiomatic theory if F_0 satisfies 3 sets of tautological equations: *string equation, dilaton equation, topological recursion relations* (TRR).

Twisted loop groups.
$\mathcal{L}^{(2)}GL(H)$ = End(H)-valued formal Laurent polynomials/series $M(z)$ in the indeterminate z^{-1} preserving the symplectic structure

$$\Omega(Mf, Mg) = \Omega(f, g), \quad \forall f, g \in \mathbb{H}.$$

*It can be shown that this is equivalent to the definition of abstract formal Frobenius manifolds.

Semisimple theories.
A theory is called *semisimple* if the "quantum product" on the subspace H is diagonalizable. Roughly, it says that triple derivative of $F(q)$, when restricted to the finite dimensional subspace $q_k = 0, k \geq 1$ ($\cong H$), are diagonalizable in two of the indices. That is, the matrices $(A^\alpha)_{\mu\nu} := \frac{\partial^3 F_0(q)|_{q_{k>0}=0}}{\partial q_0^\alpha \partial q_0^\mu \partial q_0^\nu}$ are diagonalizable $\forall \alpha$.

Theorem 1 [Givental]
1. *The twisted loop group acts on the space of all $g = 0$ axiomatic theories.*
2. *The semisimple $g = 0$ theories form a "homogeneous space" of the twisted loop group. That is, the action is transitive.*

Question: What is the simplest semisimple $g = 0$ theory of rank N?
Answer: Let $H = \mathbb{C}^N$, with $g_{\mu\nu} = \delta_{\mu\nu}$ and $(A^\alpha)_{\mu\nu} = diag(1, 1, \ldots, 1)$. That is, the geometric GW theory of $X = N$ points. (Call this theory N_{pt}.)

Since any element in the loop group can be written as a product of the *upper triangular* part $R(z)$, a polynomial in z, and the *lower triangular* part $S(z)$, a polynomial in z^{-1} [Birkhoff decomposition]. One has

Corollary 1 *Any $g = 0$ semisimple theory can be obtained as $SR(N_{pt})$.*

Question: What about higher genus?

This will bring us to the **quantization**. Without going to the details, it is a process of associating a *quadratic differential operator* $\hat{A}$ to an element A of the Lie algebra of the twisted loop group. A *group* element e^A can be quantized as $e^{\hat{A}}$. More precisely,

$$A(z) \in \text{Lie algebra(loop group)}$$
$$\to P(A)(p,q) := \frac{1}{2}\Omega\Big(A(p.q),(p,q)\Big)$$
$$= \text{a quadratic polynomial on } (p,q) \in \mathbb{H}$$
$$\to \widehat{A} := P(A)(f) \text{ with } p \mapsto \sqrt{\hbar}\frac{\partial}{\partial q},\ q \mapsto \frac{1}{\sqrt{\hbar}}q;$$
$$pq \mapsto q\frac{\partial}{\partial q}$$

Example. $A(z) = z^{-1}$ and $N = 1$.

$$P(z^{-1}) = -\frac{q_0^2}{2} - \sum_{m=0}^{\infty} q_{m+1}p_m.$$
$$(z^{-1})\widehat{} = -\frac{q_0^2}{2\hbar} - \sum_{m=0}^{\infty} q_{m+1}\frac{\partial}{\partial q_m},$$

Definition

$$\tau^{N_{pt}}(q^1,\ldots,q^N) := e^{\sum_{g=0}^{\infty}\sum_{i=1}^{N} F_g^{pt}(q^i)}.$$

Let $T = SR(N_{pt})$ be a semisimple $g = 0$ theory.

$$\tau^T(q) := \widehat{S}\widehat{R}\tau^{N_{pt}} =: e^{\sum_{g=0}^{\infty} G_g(q)}.$$

Conjecture. [Givental] If abstract $T = GW_{g=0}(X)$ for a smooth projective variety X, then

$$\tau^T(q)\left(= e^{\sum_{g=0}^{\infty} G_g(q)}\right) = e^{\sum_{g=0}^{\infty} F_g^X(q)}$$

Comments. Givental's formulaic model usually enjoys properties *complementary* to the geometric model. For example, almost immediately

geometric model $\Rightarrow$ tautological equations

Givental's model $\Rightarrow$ Virasoro constraints.

However, it is in no way obvious when they are exchanged.

Question 1. (revisited) Does G_g satisfies tautological equations?

3. Invariance of tautological equations

Theorem 2 **1.** [Givental–L.] *Yes for $g = 1$ (TRR and Getzler's equation).*
2. [L.] *G_2 satisfies $g = 2$ tautological equations of Mumford, Getzler and BP.*

Corollary 2 *Virasoro conjecture for semisimple GW theories and Witten's (generalized) conjecture for spin curves hold up to genus 2.*

Remarks. **1.** The uniqueness theorems proved by Dubrovin–Zhang and Liu: G_g and F_g are uniquely determined by tautological equations for *semisimple* theories.
2. The non-geometric theories for spin curves $\sim$ miniversal deformations of A_N singualrity.
3. Witten's (generalized) conjecture for spin curves: the τ-functions of GW-type invariants constructed from spin curves satisfy the Gelfand–Dickey hierarchies. This is a generalization of the Kontsevich's theorem.

4. Using invariance to derive tautological equations

Example Getzler's $g = 1$ equation.

Theorem 3 [Givental-L.] *Getzler's equation is the only codimension 2 equation in $\overline{M}_{1,4}$ which is invariant under quantized twisted loop group action (assuming genus one TRR and genus zero equations).*

Furthermore, invariance, i.e. requiring equation to hold for any axiomatic GW theory, determines the coefficients of Getzler's equation up to scaling.

1. In $g = 1$, TRR $\Rightarrow$ ψ-classes can be written as boundary classes.

2. There are 9 codimension 2 boundary strata in $\overline{M}_{1,4}$.

Pictures

3. Write the boundary classes in terms of GW invariants.

$$
\begin{aligned}
G(q) := \sum_{\mu,\nu,S_4 \text{ permutation}} \quad & c_1\langle v_1, v_2, \partial^\mu\rangle_0 \langle v_3, v_4, \partial^\nu\rangle_0 \langle \partial^\mu, \partial^\nu\rangle_1 \\
&+ c_2\langle v_1, v_2, \partial^\mu\rangle_0 \langle v_3, \partial^\mu, \partial^\nu\rangle_0 \langle v_4, \partial^\nu\rangle_1 \\
&+ c_3\langle v_1, v_2, \partial^\mu\rangle_0 \langle v_3, v_4, \partial^\mu, \partial^\nu\rangle_0 \langle \partial^\nu\rangle_1 \\
&+ c_4\langle v_1, v_2, v_3, \partial^\mu\rangle_0 \langle v_4, \partial^\mu, \partial^\nu\rangle_0 \langle \partial^\nu\rangle_1 \\
&+ c_5\langle v_1, v_2, v_3, \partial^\mu\rangle_0 \langle v_4, \partial^\mu, \partial^\nu, \partial^\nu\rangle_0 \\
&+ c_6\langle v_1, v_2, v_3, v_4, \partial^\mu\rangle_0 \langle \partial^\mu, \partial^\nu, \partial^\nu\rangle_0 \\
&+ c_7\langle v_1, v_2, \partial^\mu, \partial^\nu\rangle_0 \langle v_3, v_4, \partial^\mu, \partial^\nu\rangle_0 \\
&+ c_8\langle v_1, v_2, \partial^\mu\rangle_0 \langle v_3, v_4, \partial^\mu, \partial^\nu, \partial^\nu\rangle_0 \\
&+ c_9\langle v_1, \partial^\mu, \partial^\nu\rangle_0 \langle v_2, v_3, v_4, \partial^\mu, \partial^\nu\rangle_0 = 0.
\end{aligned}
$$

Convention: $\{\phi_\mu\}$ orthonormal basis. Summation over repeated indices.

4. Invariance under $e^{(r_1 z)\widehat{}} \Rightarrow$

$$0 = \frac{dG}{d\epsilon} = \sum_{S_4, i, j, \mu, \nu, \ldots} (r_1)_{ij} \Big(\\
2c_1 \langle v_1, v_2, \partial^j \rangle \langle v_3, v_4, \partial^\mu \rangle \langle \partial^i, \partial^\mu, \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
- c_1 \langle v_1, v_2, \partial^\mu \rangle \langle v_3, v_4, \partial^\nu \rangle \langle \partial^i, \partial^\nu, \partial^\mu \rangle \langle \partial^j \rangle_1 \\
+ c_2 \langle v_1, v_2, \partial^\mu \rangle \langle v_3, \partial^j, \partial^\mu \rangle \langle \partial^i, v_4, \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
- c_2 \langle v_1, v_2, \partial^\mu \rangle \langle v_3, \partial^\mu, \partial^\nu \rangle \langle \partial^i, v_4, \partial^\nu \rangle \langle \partial^j \rangle_1 \\
+ c_3 \langle v_1, v_2, \partial^j \rangle \langle v_3, v_4, \partial^i_1, \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
+ c_3 \langle v_1, v_2, \partial^\mu \rangle \langle v_3, v_4, \partial^i_1, \partial^\nu \rangle \langle \partial^j \rangle_1 \\
- c_3 \langle v_1, v_2, \partial^\mu \rangle \langle v_3, v_4, \partial^i \rangle \langle \partial^j, \partial^\mu, \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
- 2c_3 \langle v_1, v_2, \partial^\mu \rangle \langle v_3, \partial^i, \partial^\mu \rangle \langle v_4, \partial^j, \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
+ c_4 \langle v_1, v_2, v_3, \partial^i_1 \rangle \langle v_4, \partial^j, \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
- 3c_4 \langle v_1, v_2, \partial^i \rangle \langle v_3, \partial^j, \partial^\mu \rangle \langle v_4, \partial^\mu, \partial^\nu \rangle \langle \partial^\nu \rangle_1 \Big) \\
+ \text{genus-zero-only terms.}$$

($(r_1)_{ij}$ arbitrary)

5. At $g = 1$: Collect terms of the same type and set the coefficients to 0.

It is easy to see that the terms containing $\langle \partial^j \rangle_1$ gives the condition (after applying genus zero TRR)

$$-c_1 - c_2 + c_3 = 0.$$

The terms containing $\langle v_*, v_{**}, \partial^j \rangle \langle \partial^\nu \rangle_1$ gives the equation

$$2c_1 - 3c_4 = 0.$$

The terms containing $\langle v_*, v_{**}, \partial^\nu \rangle \langle \partial^\nu \rangle_1$ gives the equation

$$c_2 - 2c_3 + c_4 = 0.$$

6. For the terms involving genus zero invariants only, the only relations are WDVV, after stripping off all descendents by genus zero TRR.

(a) Those terms containing a factor $\langle v_*, v_{**}, v_{***}, \partial^i \rangle$ give the equation

$$\sum (r_1)_{ij} \langle v_1, v_2, v_3, \partial^i \rangle \Big[c_5 \langle v_4, \partial_1^j, \partial^\nu, \partial^\nu \rangle \\ - 4c_6 \langle v_4, \partial^j, \partial^\mu \rangle \langle \partial^\mu, \partial^\nu, \partial^\nu \rangle \\ - c_9 \langle v_4, \partial^\mu, \partial^\nu \rangle \langle \partial^j, \partial^\mu, \partial^\nu \rangle \Big] = 0,$$

which gives condition

$$c_5 - 4c_6 - c_9 = 0.$$

(b) Those terms containing a factor $\langle v_*, v_{**}, \partial^i\rangle$ give the equation

$$\sum(r_1)_{ij}\langle v_1, v_2, \partial^i\rangle\Big[\frac{1}{12}c_1\langle v_3, v_4, \partial^\nu\rangle\langle\partial^j, \partial^\mu, \partial^\mu, \partial^\nu\rangle$$
$$-3c_5\langle v_3, \partial^j, \partial^\nu\rangle\langle v_4, \partial^\mu, \partial^\mu, \partial^\nu\rangle$$
$$-6c_6\langle v_3, v_4, \partial^j, \partial^\nu\rangle\langle\partial^\mu, \partial^\mu, \partial^\nu\rangle$$
$$-2c_7\langle v_3, v_4, \partial^\mu, \partial^\nu\rangle\langle\partial^j, \partial^\mu, \partial^\nu\rangle$$
$$+c_8\langle v_3, v_4, \partial_1^j, \partial^\nu, \partial^\nu\rangle$$
$$+c_8\langle v_3, v_4, \partial^\mu\rangle\langle\partial^j, \partial^\mu, \partial^\nu, \partial^\nu\rangle$$
$$-3c_9\langle v_3, \partial^\mu, \partial^\nu\rangle\langle v_4, \partial^j, \partial^\mu, \partial^\nu\rangle\Big]$$

$$=\sum(r_1)_{ij}\langle v_1, v_2, \partial^i\rangle\Big[$$
$$\Big(\frac{3}{2}c_5-6c_6-\frac{3}{2}c_9\Big)\partial^j\Big(\langle v_3, \partial^\mu, \partial^\nu\rangle\langle v_4, \partial^\mu, \partial^\nu\rangle\Big)$$
$$\Big(-3c_5-2c_7-c_8\Big)\langle v_3, v_4, \partial^\mu, \partial^\nu\rangle\langle\partial^j, \partial^\mu, \partial^\nu\rangle\Big)$$
$$\Big(\frac{1}{12}c_1-3c_5+6c_6=0\Big)\langle v_3, v_4, \partial^\mu, \partial^\nu\rangle\langle\partial^j, \partial^\mu, \partial^\nu\rangle\Big]$$
$$=0.$$

The above equation gives two conditions

$$-3c_5 - 2c_7 + 2c_8 = 0.$$

$$\frac{1}{12}c_1 - 3c_5 + 6c_6 = 0.$$

(c) The remaining terms, after genus zero TRR, contain no descendents. Therefore the only relations are WDVV's and their derivatives. However, WDVV's and their derivatives don't change the summation of the coefficients, therefore the summation has to vanish. This gives another equation

$$-\frac{1}{2}c_1 - \frac{11}{24}c_2 - \frac{11}{24}c_3 - \frac{11}{24}c_4 + 3c_6 - 3c_8 = 0.$$

(d) Combining the above equations, one can express all coefficients in terms of c_3 and c_9:

$$c_1 = -3c_3,\ c_2 = 4c_3,\ c_4 = -2c_3,\ c_5 = -\frac{1}{6}c_3 - c_9,$$

$$c_6 = -\frac{1}{24}c_3 - \frac{1}{2}c_9,\ c_7 = \frac{1}{4}c_3 + c_9,\ c_8 = -\frac{1}{2}c_9$$

(e) Summarizing, we have

$$-c_3(\text{Getzler's equation}) + c_9(T).$$

where T is a sum of rational GW invariants. It is not very difficult to see that WDVV implies that $T = 0$. This completes the proof.

Remark. 1. The same technique can be applied to $g = 2$ Mumford, Getzler and BP. (Partially checked.)
2. The set of all (axiomatic) invariants satisfying Graber–Vakil's is also invariant under twisted loop group action.

Conjecture. One can find many (all?) tautological equations by the invariance technique.

One should start with codimension 3 equation in $\overline{M}_{3,1}$. (Instead of 9 terms, there are about 100 terms....)