WITTEN'S CONJECTURE, VIRASORO CONJECTURE, AND INVARIANCE OF TAUTOLOGICAL EQUATIONS

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Abstract. The main goal of this paper is to introduce a new technique, the *invariance of the tautological equations* under the loop group action, in the Gromov–Witten theory. Three applications are illustrated. The first two are the proofs of *Witten's conjecture* on the relations between higher spin curves and Gelfand–Dickey hierarchy and *Virasoro conjecture* for target manifolds with conformal semisimple quantum cohomology, both for genus up to two. This technique also provides some conjectural descriptions of the tautological equations of the moduli spaces of curves. In particular, it gives an effective algorithm to calculate, conjecturally, all tautological equations using only linear algebra.

0. Introduction

In this Introduction, we start with our motivation of introducing the concept of the *invariance of the tautological equations*: a proof of the Witten's and Virasoro conjectures.

0.1. **Two dimensional quantum gravity.** The famous conjecture by E. Witten [33] in 1990 predicted a striking relation between two seemingly unrelated objects: A generating functions of intersection numbers on moduli spaces of stable curves and a τ-function of the KdV hierarchy. The physical basis of this conjecture comes from the identification of two approaches to the two dimensional quantum gravity. Roughly, the correlators of the two dimensional quantum gravity are Feynman path integrals over the "space of metrics" on two dimensional topological real surfaces. One approach of evaluating this path integral involves a topological field theory technique which is expected to reduce to the integration over the moduli space of curves. The other approach considers an approximation of the space of the metrics by piecewise flat metrics and then take a suitable continuous limit.

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In the first approach, the free energy becomes the following geometrically defined function

$$\tau^{pt}(t_0, t_1, \ldots) = e^{\sum_{g=0}^{\infty} \hbar^{g-1} F_g^{pt}(t_0, t_1, \ldots)},$$

where $F_g^{pt}(t)$ is the generating function of (tautological) intersection numbers on the moduli space of stable curves of genus g

$$F_g^{pt}(t_0, t_1, \ldots) := \sum_n \frac{1}{n!} \int_{\overline{M}_{g,n}} \prod_{i=1}^n (\sum_k t_k \psi_i^k).$$

($\hbar$ is usually set to be 1 in the literature.) Moreover, from elementary geometry of moduli spaces, one easily deduces that τ^{pt} satisfies an additional equation, called the *string equation* (or puncture equation). It is a basic fact in the theory of KdV (or in general KP) hierarchies that the string equation uniquely determines one τ-function for the KdV hierarchy from all τ-functions parameterized by Sato's grassmannian.

In the second approach, the generating function in the double scaling limit yields the τ-function τ^{qg} of the KdV hierarchy whose initial value is $d^2/dx^2 + 2x$. From here Witten asserts that τ^{pt} must be equal to τ^{qg} since there should be only one quantum gravity.

0.2. **Witten's conjecture on spin curves and Gelfand–Dickey hierarchies.** In 1991 Witten formulated a remarkable generalization of the above conjecture. He argued that an analogous generating function $\tau^{r\text{-spin}}$ of the intersection numbers on moduli spaces of r-spin curves should be identified as a τ-function of Gelfand–Dickey (r-KdV) hierarchies [34]. When $r = 2$, this conjecture eventually reduces to the previous one as 2-KdV is the ordinary KdV.

The special case $\tau^{pt} = \tau^{qg}$ was soon proved by M. Kontsevich [22]. More recently a new proof was given by Okounkov–Pandharipande [30]. However, the generalized conjecture remains open up to this day.

Throughout the 12 years, there has been substantial progress in the foundational issues involved in the 1991 conjecture. In particular, Jarvis–Kimura–Vaintrob [20] established the genus zero case of the conjecture; T. Mochizuki and A. Polishchuk independently established the following property for $\tau^{r\text{-spin}}$:

Theorem 1. [29, 32] *All tautological equations hold for* $F_g^{r\text{-}spin}$. [1]

[1] Tautological relations in this article means both the relations of the tautological classes on moduli spaces of curves and the induced relations in the "cohomological field theories".

In fact, $F_g^{r\text{-spin}}$ satisfies all "expected functorial properties", similar to the axioms formulated by Kontsevich–Manin in the Gromov–Witten theory.

However, Riemann's trichotomy of Riemann surfaces has taught us that things are very different in genus one and in genus ≥ 2. Our Main Theorem therefore provides a solid confirmation for Witten's 1991 conjecture, covering one example ($g = 1$ and $g = 2$) for the other two cases of the trichotomy. *In fact, this work starts as a project trying to understand this conjecture in higher genus.*

For more background information about Witten's conjecture, the readers are referred to Witten's original article [34] and the paper [20] by Jarvis–Kimura–Vaintrob. In the remaining of this article, "Witten's conjecture" means the 1991 conjecture if not otherwise specified.

0.3. **Virasoro conjecture.** In 1997 another generalization of Witten's 1990 conjecture was proposed by T. Eguchi, K. Hori and C. Xiong. Witten's 1990 conjecture has an equivalent formulation [4] [10] [22]: τ^{pt} is annihilated by infinitely many differential operators $\{L_n^{pt}\}, n \geq -1$, satisfying the Virasoro relations

$$[L_m, L_n] = (m - n)L_{m+n}$$

such that $L_{-1}\tau^{pt} = 0$ is the string equation alluded above. To generalize Witten's 1990 conjecture to any projective smooth variety X, consider the moduli spaces of curves as the moduli spaces of maps to a point. It is clear that τ^{pt} should be replaced by

$$\tau_{GW}^X(t) := e^{\sum_{g=0}^{\infty} \hbar^{g-1} F_g^X(t)}, \tag{1}$$

where $F_g^X(t)$ is the generating function of genus g Gromov–Witten invariants with descendents for X. Based on that Eguchi–Hori–Xiong [9], and S. Katz, managed to to define $\{L_n^X\}$ for $n \geq -1$, satisfying the Virasoro relations. [2] They conjectured that

$$L_n^X \tau^X(t) = 0, \quad \text{for } n \geq -1.$$

This conjecture is commonly referred to as the *Virasoro conjecture.*

Eguchi–Hori–Xiong gave a partial proof for their conjecture in genus zero and a proof of $L_0^X \tau^X = 0$, among other things. Later X. Liu and G. Tian [28] proved the genus zero case in general. Using a very different method, Dubrovin–Zhang [7] established the genus one case of Virasoro conjecture for *conformal* semisimple Frobenius manifolds. [3]

[2] By the Virasoro relations, one only has to construct L_2^X, and the rest will follow.

[3] The definition of Frobenius manifolds in this article does not require existence of an Euler field, which is assumed in Dubrovin's definition. Dubrovin's definition will be referred to as *conformal Frobenius manifold* instead.

The recent progress by Givental [16] and by Okounkov–Pandharipande [30] have confirmed the conjecture for toric Fano manifolds and curves respectively at all genus. Some background information on Virasoro conjecture can be found in [13] and [25].

0.4. **Main results.**

Main Theorem. *Witten's conjecture and Virasoro conjecture for manifolds with conformal semisimple quantum cohomology hold up to genus two.*

A few words about the main idea in the proof. Givental in a series of papers [15] [16] [17] [18] introduces a definition of higher genus potentials for any semisimple Frobenius manifold which is not necessarily the quantum cohomology of a projective manifold. This definition is "formulaic" in the sense that the higher genus potentials are *defined* by a formula from the data of semisimple Frobenius manifolds (i.e. genus zero data). This enables him to prove that his theory satisfies Virasoro conjecture and, in the case of A_n singularities, Witten's conjecture. However, Givental's theory is conjecturally equivalent to the geometric theory, whether it is the Gromov–Witten theory or the theory of spin curves. Therefore, what is needed here is a proof that Givental's theory is equal to the geometric theory.

The bulk of this paper is devoted to this proof at genus two. Similar statement in genus one is proved in the conformal case in [6] and in the general case in [19].

Remark. There are other possible approaches to this problem. Our earlier approach in [24] reduces the checking of the Main Theorem to complicated, but finite-time checkable, identities. Nevertheless, it lacks the underlying simplicity of this approach.

After this result was announced, X. Liu [27] informed us that he was also able to reduce the genus two Virasoro conjecture for semisimple Gromov–Witten theory to some complicated identities and he was able to check these identities by hand and by a Mathematica program.

Acknowledgement. This idea of this work first comes in the form of [24] while working jointly with R. Pandharipande in the book project [25], and this current approach has its root in a recent joint work with A. Givental [19]. It is a great pleasure to thank both of them. Thanks are also due to T. Jarvis, T. Kimura, X. Liu, Y. Ruan, A. Vaintrob and especially E. Getzler for useful discussions and communications. Part of this work was done during a visit to NCTS, whose hospitality is greatly appreciated.

1. Review of geometric Gromov–Witten theory

Gromov–Witten theory studies the tautological intersection theory on $\overline{M}_{g,n}(X,\beta)$, the moduli spaces of stable maps from curves C of genus g with n marked points to a smooth projective variety X. The intersection numbers, or *Gromov–Witten invariants*, are integrals of tautological classes over the virtual fundamental classes of $\overline{M}_{g,n}(X,\beta)$

$$\int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\psi_i^{k_i}.$$

Here $\gamma_i \in H^*(X)$ and ψ_i are the *cotangent classes* (gravitational descendents).

For the sake of the later reference, let us fix some notations.

(i) $H := H^*(X,\mathbb{C})$, assumed of rank N.
(ii) Let $\{\phi_\mu\}_{\mu=1}^N$ be an *orthonormal* basis of H with respect to the Poincaré pairing.
(iii) Let $\mathcal{H}_t := \oplus_0^\infty H$ be the infinite dimensional complex vector space with basis $\{\phi_\mu\psi^k\}$.
(iv) Let $\{t_k^\mu\}$, $\mu = 1,\ldots,N$, $k = 0,\ldots,\infty$, be the dual coordinates of the basis $\{\phi_\mu\psi^k\}$.

We note that at each marked point, the insertion is $\mathcal{H}_t$-valued. Let $t := \sum_{k,\mu} t_k^\mu \phi_\mu \psi^k$ denote a general element in the vector space $\mathcal{H}_t$.

(v) Define $\langle \partial_{k_1}^{\mu_1} \ldots \partial_{k_n}^{\mu_n} \rangle_{g,n,\beta} := \displaystyle\int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\phi_{\mu_i})\psi_i^{k_i}$ and define $\langle t^n \rangle_{g,n,\beta} = \langle t \ldots t \rangle_{g,n,\beta}$ by multi-linearity.
(vi) Let $F_g^X(t) := \displaystyle\sum_{n,\beta} \frac{1}{n!} \langle t^n \rangle_{g,n,\beta}$ be the generating function of all genus g Gromov–Witten invariants.

The "τ-function of X" is the formal expression $\tau_{GW}^X := e^{\sum_{g=0}^\infty \hbar^{g-1} F_g^X}$ defined in (1).

1.1. Tautological equations.

Tautological equations. Let $E = 0$ be a *tautological equation*, i.e. a equation of the tautological classes in the moduli space of stable curves $\overline{M}_{g,n}$. Since there is a morphism

$$\overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$$

by forgetting the map, one can pull-back $E = 0$ to $\overline{M}_{g,n}(X,\beta)$. Due to the functorial properties of the virtual fundamental classes, the pull-backs of the *tautological equations hold for the Gromov–Witten theory of any target space*. The term *tautological equations* will also be used for the corresponding equations in the Gromov–Witten theory and in the theory of spin curves.

2. Genus zero axiomatic theory

Let H be a complex vector space of dimension N with a distinguished element $\mathbf{1}$. Let $(\cdot, \cdot)$ be a $\mathbb{C}$-bilinear metric on H, i.e. a nondegenerate symmetric $\mathbb{C}$-bilinear form. Let $\mathcal{H}$ denote the infinite dimensional complex vector space $H((z^{-1}))$ consisting of Laurent formal series in $1/z$ with vector coefficients. [4] Introduce the symplectic form Ω on $\mathcal{H}$:

$$\Omega(f, g) = \frac{1}{2\pi i} \oint (f(-z), g(z)) \, dz.$$

There is a natural polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ by the Lagrangian subspaces $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = z^{-1}H[[z^{-1}]]$ which provides a symplectic identification of $(\mathcal{H}, \Omega)$ with the cotangent bundle $T^*\mathcal{H}_+$.

Let $\{\phi_\mu\}$ be an *orthonormal* basis of H. An H-valued Laurent formal series can be written in this basis as

$$\ldots + (p_1^1, \ldots, p_1^N) \frac{1}{(-z)^2} + (p_0^1, \ldots, p_0^N) \frac{1}{(-z)} + (q_0^1, \ldots, q_0^N) + (q_1^1, \ldots, q_1^N) z + \ldots .$$

In fact, $\{p_k^\mu, q_k^\mu\}$ for $k = 0, 1, 2, \ldots$ and $\mu = 1, \ldots, N$ are the Darboux coordinates compatible with this polarization in the sense that

$$\Omega = \sum dp_k^\mu \wedge dq_k^\mu.$$

To simplify the notations, p_k will stand for the vector $(p_k^1, \ldots, p_k^N)$ and p^μ for $(p_0^\mu, p_1^\mu, \ldots)$.

The parallel between $\mathcal{H}_+$ and $\mathcal{H}_t$ is evident, and is in fact given by the affine coordinate transformation, the *dilaton shift*,

$$t_k^\mu = q_k^\mu + \delta^{\mu 1} \delta_{k1}.$$

Definition. Let $G_0(t)$ be a (formal) function on $\mathcal{H}_+$. The pair $(\mathcal{H}, G_0)$ is called a $g = 0$ *axiomatic theory* if G_0 satisfies three sets of genus zero tautological equations: the *Topological Recursion Relations* (TRR) (4), the *String Equation* (3) and the *Dilaton Equation* (2).

[4] Different completions of this spaces are used in different places, but this will be not be discussed in details in the present article as it is involving too much. See [25] for details.

$$\frac{\partial F_0(t)}{\partial t_1^1}(t) = \sum_{n=0}^{\infty} \sum_{\nu} t_n^{\nu} \frac{\partial F_0(t)}{\partial t_n^{\nu}} - 2F_0(t), \tag{2}$$

$$\frac{\partial F_0(t)}{\partial t_0^1} = \frac{1}{2}(t_0, t_0) + \sum_{n=0}^{\infty} \sum_{\nu} t_{n+1}^{\nu} \frac{\partial F_0(t)}{\partial t_n^{\nu}}, \tag{3}$$

$$\frac{\partial^3 F_0(t)}{\partial t_{k+1}^{\alpha} \partial t_l^{\beta} \partial t_m^{\gamma}} = \sum_{\mu} \frac{\partial^2 F_0(t)}{\partial t_k^{\alpha} \partial t_0^{\mu}} \frac{\partial^3 F_0(t)}{\partial t_0^{\mu} \partial t_l^{\beta} \partial t_m^{\gamma}} \tag{4}$$

It can be shown that this is equivalent to the definition of abstract formal Frobenius manifolds, not necessarily conformal. The coordinates on the corresponding Frobenius manifold is given by the following map [5]

$$s^{\mu} := \frac{\partial}{\partial t_0^{\mu}} \frac{\partial}{\partial t_0^1} G_0(t). \tag{5}$$

In the case of the geometric theory, one may elect to use either formulation. The main advantage, seems to us, is the expansion of the formulation from $\mathcal{H}_t$ to $\mathcal{H}$ where a symplectic structure is available. In the latter case, many properties can be reformulated in terms of the symplectic structure Ω and hence independent of the choice of the polarization. This suggests that the space of "genus zero axiomatic Gromov–Witten theories", i.e. the space of G_0 satisfying the string equation, dilaton equation, topological recursion relations (TRR), has a huge symmetry group.

Definition. Let $\mathcal{L}^{(2)}GL(H)$ denote the *twisted loop group* which consists of End(H)-valued formal Laurent formal series $M(z)$ in the indeterminate z^{-1} satisfying $M^*(-z)M(z) = \mathbf{1}$. Here * denotes the adjoint with respect to $(\cdot, \cdot)$.

The condition $M^*(-z)M(z) = \mathbf{1}$ means that $M(z)$ is a symplectic transformation on $\mathcal{H}$.

Theorem 2. [18] *The twisted loop group acts on the space of axiomatic genus zero theories. Furthermore, the action is transitive on the semisimple theories of a fixed rank N.*

3. Quantization and higher genus potentials

3.1. **Preliminaries on quantization.** To quantize an infinitesimal symplectic transformation, or its corresponding quadratic hamiltonians, we recall the standard Weyl quantization. A polarization $\mathcal{H} =$

$T^*\mathcal{H}_+$ on the symplectic vector space $\mathcal{H}$ (the phase space) defines a configuration space $\mathcal{H}_+$. The quantum "Fock space" will be a certain class of functions $f(\hbar, q)$ on $\mathcal{H}_+$ (containing at least polynomial functions), with additional formal variable $\hbar$ ("Planck's constant"). The classical observables are certain functions of p, q. The quantization process is to find for the classical mechanical system on $\mathcal{H}$ a "quantum mechanical" system on the Fock space such that the classical observables, like the hamiltonians $h(q,p)$ on $\mathcal{H}$, are quantized to become operators $\hat{h}(q, \frac{\partial}{\partial q})$ on the Fock space.

Let $A(z)$ be an $\mathrm{End}(H)$-valued Laurent formal series in z satisfying

$$(A(-z)f(-z), g(z)) + (f(-z), A(z)g(z)) = 0,$$

then $A(z)$ defines an infinitesimal symplectic transformation

$$\Omega(Af, g) + \Omega(f, Ag) = 0.$$

An infinitesimal symplectic transformation A of $\mathcal{H}$ corresponds to a quadratic polynomial $P(A)$ in p, q

$$P(A)(f) := \frac{1}{2}\Omega(Af, f).$$

For example, let $\dim H = 1$ and $A(z) = 1/z$. It is easy to see that $A(z)$ is infinitesimally symplectic and

$$P(z^{-1}) = -\frac{q_0^2}{2} - \sum_{m=0}^{\infty} q_{m+1}p_m. \tag{6}$$

In the above Darboux coordinates, the quantization $P \mapsto \hat{P}$ assigns

$$\begin{aligned} &\hat{1} = 1, \ \hat{p}_k^i = \sqrt{\hbar}\frac{\partial}{\partial q_k^i}, \hat{q}_k^i = q_k^i/\sqrt{\hbar}, \\ &(p_k^i p_l^j)\hat{} = \hat{p}_k^i \hat{p}_l^j = \hbar \frac{\partial}{\partial q_k^i}\frac{\partial}{\partial q_l^j}, \\ &(p_k^i q_l^j)\hat{} = q_l^j \frac{\partial}{\partial q_k^i}, \\ &(q_k^i q_l^j)\hat{} = \hat{q}_k^i \hat{q}_l^j/\hbar, \end{aligned} \tag{7}$$

Note that one often has to quantize the symplectic instead of the infinitesimal symplectic transformations. Following the common practice in physics, we define

$$(e^{A(z)})\hat{} := e^{(A(z))\hat{}}, \tag{8}$$

for $e^{A(z)}$ an element in the twisted loop group.

3.2. **τ-function for semisimple Frobenius manifolds.** Let H^{Npt} be the rank N Frobenius manifold corresponding to X being N points. In this case, the delta-functions at the N points form an orthonormal basis $\{\phi_\mu\}$ and the idempotents of the quantum product

$$\phi_\mu * \phi_\nu = \delta_{\mu\nu}\phi_\mu.$$

The genus zero potential is nothing but a sum of genus zero potentials of N points

$$F_0^{Npt}(t^1, \dots, t^N) = F_0^{pt}(t^1) + \dots + F_0^{pt}(t^N).$$

Note that $G_0^{H^{Npt}} = F_0^{Npt}$. By Theorem 2, the genus zero potential G_0^H of any semisimple formal Frobenius manifold H can be obtained from $G_0^{H^{Npt}}$ by the action of an element O_H in the twisted loop group. By Birkhoff factorization, $O_H = S_H(z^{-1})R_H(z)$, where $S(z^{-1})$ (resp. $R(z)$) is an matrix-valued functions in z^{-1} (resp. z). [5]

In order to define the higher genus potentials G_g^H, one first introduces the "τ-function of H"

$$\tau_G^H := \hat{S}_H \hat{R}_H \tau^{Npt}, \tag{9}$$

and define G_g^H via the formula (cf. (1))

$$\tau_G^H =: e^{\sum_{g=0}^{\infty} \hbar^{g-1} G_g^H}. \tag{10}$$

Strictly speaking, the multiplication, $\hat{S}_H \hat{R}_H$, is not well-defined. However, the function $\hat{S}_H(\hat{R}_H \tau^{Npt})$ is, thanks to the $(3g-2)$-jet properties. We will not discuss this subtle point here but refer the interested readers to [25].

What makes the above model especially attractive are the facts that

(a) It works for any semisimple Frobenius manifolds.
(b) It enjoys properties often complementary to the geometric theory.

Thanks to (a), one also has a definition for the Frobenius manifolds $H_{A_{r-1}}$ of the miniversal deformation space of A_{r-1} singularity. It turns out that this Frobenius manifold is isomorphic to the Frobenius manifold defined by the genus zero potential of r-spin curves. Furthermore, Givental has proved

Theorem 3. [17] $\tau_G^{H_{A_{r-1}}}$ *is a τ-function of r-KdV hierarchy.*

As in the case of the ordinary KdV, it is easy to show that both $\tau_G^{H_{A_{r-1}}}$ and $\tau^{r\text{-spin}}$ satisfy the additional string equation. Therefore, in

[5] In fact $R(z)$ is a series in z and therefore not really an element in the twisted loop group, but rather in its suitable completion. See [25].

order to prove Witten's conjecture, one only has to show $G_g^{H_{A_{r-1}}} = F_g^{r\text{-spin}}$.

As for (b), note for example that the Virasoro constraints for τ_G^H follow almost from the definition. As discussed in Section 1.1, τ_{GW}^X satisfies the tautological equations due to some functorial properties built in the definition of the Gromov–Witten theory. However, the Virasoro constraints for τ_{GW}^X and tautological equations for τ_G^H are highly nontrivial challenges. An obvious, and indeed very good, strategy to resolve all the above questions at once is to answer the following question: [6]

Question. *Is $G_g = F_g$ That is, does Givental's construction coincide with the geometric one when both are available?*

3.3. **Tautological equations and uniqueness theorems.** Our approach to the Question is to show that G_g satisfies enough geometric properties of F_g so that they have to be equal by some uniqueness theorems. More specifically, the geometric properties we will utilize are the tautological equations. For simplicity of language, let us call the *genus zero tautological equations* the following genus zero equations: topological recursion relations (TRR), string equation and dilaton equation; the *genus one tautological equations* the following two equations: genus one Getzler's equation [11] and genus one TRR; the *genus two tautological equations* the set of 3 equations by Mumford (16), Getzler [12], and Belorousski–Pandharipande (BP) [3].

In genus one, Dubrovin and Zhang [6] made the following important observation of the uniqueness property.

Lemma 1. [6] *Let $G_0(t)$ be the genus zero potential of a semisimple Frobenius manifold H. Suppose that both pairs $(G_0(t), F_1(t))$ and $(G_0(t), G_1(t)$ satisfy genus one Getzler's equation and topological recursion relations. Then $F_1 - G_1$ is a linear combination of canonical coordinates. Furthermore, if H is conformal and both pairs satisfy the conformal equation, then $F_1 - G_1$ is a constant.*

The proof of this fact goes as follows. First, genus one TRR guarantees that the descendent invariants are uniquely determined by primary invariants. Second, genus one Getzler's equation, when written in canonical coordinates u^i, is equal to $\frac{\partial^2 F_1}{\partial u^i \partial u^j} = B_{ij}$ where B_{ij} involves only genus zero invariants. Moreover, the conformal structure determined by a linear vector field (Euler field), uniquely determines the

[6]This is basically Givental's conjecture [16], although we have included r-spin curves on the geometric side, which is strictly speaking not a geometric Gromov–Witten theory.

linear term. We will refer to this fact casually as "The genus one potential for a conformal Frobenius manifold is uniquely determined by genus one tautological equations." In the same spirit, the genus two uniqueness theorem of X. Liu is formulated:

Theorem 4. [26] *The genus two descendent potentials for any conformal semisimple Frobenius manifolds are uniquely determined by genus two tautological equations.*

The proof of Liu's theorem rests on some very complicated calculation. We note that it is not known at this point whether this uniqueness theorem, or any weaker version, holds for non-conformal semisimple Frobenius manifolds.

Remark. There is another type of uniqueness theorem: Dubrovin and Zhang [8] have proved that Virasoro conjecture plus $(3g-2)$-jet property uniquely determines τ-function for any semisimple Frobenius manifold. The $(3g-2)$-jet property is proved by Getzler [14] in the geometric Gromov–Witten theory and by Givental [16] in the context of semisimple Frobenius manifolds. It is also expected to hold for the $\tau^{r\text{-spin}}$. Therefore, a proof of the Virasoro conjecture for τ^X should also answer the above Question positively.

Note that

- $\tau_G = \hat{S}\hat{R}\tau^{Npt}$ and $\tau^{Npt}(t^1, \ldots t^N) = \prod_{i=1}^N \tau^{pt}(t^i)$.
- $\tau^{pt}(t)$ satisfies all tautological equations.

It follows that

Main Lemma. *In order to show that a set of tautological equations holds for G_g, it suffices to show that it is invariant under arbitrary* $\mathrm{End}(H)$*-valued series $\hat{S}(z^{-1})$ and $\hat{R}(z)$.*

This lemma is our main technical tool to prove the Main Theorem. In fact, in order to prove the invariance of the tautological equations, it is enough to prove the *infinitesimal* invariance of the tautological equations. Before we proceed, let us study more carefully the quantized twisted loop groups.

4. Quantization of twisted loop groups

The twisted loop group is generated by "lower triangular subgroup" and the "upper triangular subgroup". The lower triangular subgroup consists of $\mathrm{End}(H)$-valued formal series $S(z^{-1}) = e^{s(z^{-1})}$ in z^{-1} satisfying $S^*(-z)S(z) = \mathbf{1}$ or equivalently

$$s^*(-z^{-1}) + s(z^{-1}) = 0.$$

The upper triangular subgroup consists of the regular part of the twisted loop groups $R(z) = e^{r(z)}$ satisfying $R^*(-z)R(z) = \mathbf{1}$ or equivalently

$$r^*(-z) + r(z) = 0. \tag{11}$$

In fact, we will use $R(z)$ to denote an $\mathrm{End}(H)$-valued series in z, and call it an element in the "upper triangular subgroup" by abusing the language.

4.1. **Quantization of lower triangular subgroups.** The quadratic hamiltonian of $s(z^{-1}) = \sum_{l=1}^{\infty} s_l z^{-l}$ is

$$\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{i,j}(s_l)_{ij} q^j_{l+n} p^i_n + \sum \frac{1}{2}(-1)^n (s_l)_{ij} q^i_n q^j_{l-n-1}.$$

The fact that $s(z^{-1})$ is a series in z^{-1} implies that the quadratic hamiltonian $P(s)$ of s is of the form q^2-term + qp-term where q in qp-term does not contain q_0. The quantization of the $P(s)$

$$\hat{s} = \sum (s_l)_{ij} q^j_{l+n} \partial_{q^i_n} + \frac{1}{2\hbar} \sum (-1)^n (s_l)_{ij} q^i_n q^j_{l-n-1}.$$

Here i, j are the indices of the orthonormal basis. (The indices μ, ν will be reserved for the "gluing indices" at the nodes.) For simplicity of the notation, we adopt the *summation convention* to *sum over all repeated indices.*

Let $\frac{d\tau_G}{d\epsilon_s} := \hat{s}(z)\tau_G$. Then

$$\frac{dG_0(\epsilon_s)}{d\epsilon_s} = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{i,j}(s_l)_{ij} q^j_{l+n} \partial_{q^i_n} G_0 + \frac{1}{2}(-1)^n (s_{l+n+1})_{ij} q^i_n q^j_l.$$

$$\frac{dG_g(\epsilon_s)}{d\epsilon_s} = \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{i,j}(s_l)_{ij} q^j_{l+n} \partial_{q^i_n} G_g, \quad \text{for } g \geq 1.$$

Define

$$\langle \partial^{i_1}_{k_1} \partial^{i_2}_{k_2} \dots \partial^{i_n}_{k_n} \rangle_g := \frac{\partial^n G_g}{\partial t^{i_1}_{k_1} \partial t^{i_2}_{k_2} \dots \partial t^{i_n}_{k_n}},$$

and denote $\langle\ldots\rangle := \langle\ldots\rangle_0$. These functions $\langle\ldots\rangle_g$ will be called *axiomatic Gromov–Witten invariants*. Then

$$\begin{aligned}&\frac{d}{d\epsilon_s}\langle\partial^{i_1}_{k_1}\partial^{i_2}_{k_2}\ldots\rangle\\ &=\sum(s_l)_{ij}q^j_{l+n}\langle\partial^i_n\partial^{i_1}_{k_1}\ldots\rangle+\sum_{l=1}^{\infty}\sum_{i,a}(s_l)_{ii_a}\langle\partial^i_{k_a-l}\partial^{i_1}_{k_1}\ldots\hat{\partial}^{i_a}_{k_a}\ldots\rangle\\ &+\frac{\delta}{2}\Big((-1)^{k_1}\sum(s_{k_1+k_2+1})_{i_1i_2}+(-1)^{k_2}\sum(s_{k_1+k_2+1})_{i_2i_1}\Big),\end{aligned}\tag{12}$$

where $\delta=0$ when there are more than 2 insertions and $\delta=1$ when there are two insertions. The notation $\hat{\partial}^i_k$ means that ∂^i_k is omitted from the summation. We *assume that there are at least two insertions*, as this is the case in our application.

For $g\geq 1$

$$\begin{aligned}&\frac{d}{d\epsilon_s}\langle\partial^{i_1}_{k_1}\partial^{i_2}_{k_2}\ldots\rangle_g\\ &=\sum(s_l)_{ij}q^j_{l+n}\langle\partial^i_n\partial^{i_1}_{k_1}\ldots\rangle_g+\sum\sum_{a}(s_l)_{ii_a}\langle\partial^i_{k_a-l}\partial^{i_1}_{k_1}\ldots\hat{\partial}^{i_a}_{k_a}\ldots\rangle_g\end{aligned}\tag{13}$$

4.2. **Quantization of upper triangular subgroups.** The quantization of $r(z)$ is

$$\begin{aligned}\hat{r}(z)=&\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{i,j}(r_l)_{ij}q^j_n\partial_{q^i_{n+l}}\\ &+\frac{\hbar}{2}\sum_{l=1}^{\infty}\sum_{m=0}^{l-1}(-1)^{m+1}\sum_{ij}(r_l)_{ij}\partial_{q^i_{l-1-m}}\partial_{q^j_m}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{d}{d\epsilon_r}\langle\partial^{i_1}_{k_1}\partial^{i_2}_{k_2}\ldots\rangle=\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\sum_{i,j}(r_l)_{ij}q^j_n\langle\partial^i_{n+l}\partial^{i_1}_{k_1}\ldots\rangle\\ &+\sum_{l=1}^{\infty}\sum_{i,a}(r_l)_{ii_a}\langle\partial^i_{k_a+l}\partial^{i_1}_{k_1}\ldots\hat{\partial}^{i_a}_{k_a}\ldots\rangle\\ &+\frac{1}{2}\sum_{l=1}^{\infty}\sum_{m=0}^{l-1}(-1)^{m+1}\sum_{ij}(r_l)_{ij}\partial^{i_1}_{k_1}\partial^{i_2}_{k_2}\ldots(\langle\partial^i_{l-1-m}\rangle\langle\partial^j_m\rangle).\end{aligned}\tag{14}$$

For $g \geq 1$

$$
\begin{aligned}
&\frac{d\langle \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \ldots \rangle_g}{d\epsilon_r} \\
&= \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \sum_{i,j} (r_l)_{ij} q_n^j \langle \partial_{n+l}^i \partial_{k_1}^{i_1} \ldots \rangle_g \\
&+ \sum_{l=1}^{\infty} \sum_{i,a} (r_l)_{ii_a} \langle \partial_{k_a+l}^{i} \partial_{k_1}^{i_1} \ldots \hat{\partial_{k_a}^{i_a}} \ldots \rangle_g \\
&+ \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m=0}^{l-1} (-1)^{m+1} \sum_{ij} (r_l)_{ij} \langle \partial_{l-1-m}^i \partial_m^j \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \ldots \rangle_{g-1} \\
&+ \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m=0}^{l-1} (-1)^{m+1} \sum_{ij} \sum_{g'=0}^{g} (r_l)_{ij} \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \ldots (\langle \partial_{l-1-m}^i \rangle_{g'} \langle \partial_m^j \rangle_{g-g'}).
\end{aligned}
\tag{15}
$$

5. Invariance of tautological equations

5.1. Invariance under lower triangular subgroups.

Theorem 5. (S-invariance theorem) *All tautological equations are invariant under action of lower triangular subgroups of the twisted loop groups.*

Proof. Let $E = 0$ be a tautological equation of axiomatic Gromov–Witten invariants. Suppose that this equation holds for a given semisimple Frobenius manifold, e.g. $H^{Npt} \cong \mathbb{C}^N$. We will show that $\hat{s}E = 0$. This will prove the theorem.

$\hat{s}E = 0$ follows from the following facts:

(a) The combined effect of the first term in (12) (for genus zero invariants) and in (13) (for $g \geq 1$ invariants) vanishes.

(b) The combined effect of the remaining terms in (12) and in (13) also vanishes.

(a) is due to the fact that the sum of the contributions from the first term is a derivative of the original equation $E = 0$ with respect to q variables. Therefore it vanishes.

It takes a little more work to show (b). Recall that all tautological equations are induced from moduli spaces of curves. Therefore, any relations of tautological classes on $\overline{M}_{g,n}$ contain no genus zero components of two or less marked points. However, when one writes down the induced equation for (axiomatic) Gromov–Witten invariants, the genus zero invariants with two insertions will appear. This is due to the difference between the cotangent classes on $\overline{M}_{g,n+m}(X, \beta)$ and the pull-back classes from $\overline{M}_{g,n}$. Therefore the only contribution from the third term of (12) comes from these terms. More precisely, let ψ_j (descendents) denote the j-th cotangent class on $\overline{M}_{g,n+m}(X,\beta)$ and $\bar{\psi}_j$ (ancestors) the pull-backs of cotangent classes from $\overline{M}_{g,n}$ by the combination of the stabilization and forgetful morphisms (forgetting the maps and extra marked points, and stabilizing if necessary). Let D_j be the divisor on $\overline{M}_{g,n+m}(X,\beta)$ defined by the image of the gluing morphism

$$\sum_{\beta'+\beta''=\beta}\sum_{m'+m''=m}\overline{M}^{(j)}_{0,2+m'}(X,\beta')\times_X\overline{M}_{g,n+m''}(X,\beta'')\to\overline{M}_{g,n+m}(X,\beta),$$

where $\overline{M}_{g,n+m''}(X,\beta'')$ carries all first n marked points except the j-th one, which is carried by $\overline{M}^{(j)}_{0,2+m'}(X,\beta')$. It is easy to see geometrically that $\psi_j-\bar{\psi}_j=D_j$. (See e.g. [23].) Let us denote $\langle\partial^{\mu}_{k,\bar{l}},\ldots\rangle$ the generalized (axiomatic) Gromov–Witten invariants with $\psi_1^k\bar{\psi}_1^l\mathrm{ev}_1^*(\phi_\mu)$ at the first marked point. The above relation can be rephrased in terms of invariants as

$$\langle\partial^i_{k,\bar{l}}\ldots\rangle_g=\langle\partial^i_{k+1,\overline{l-1}}\ldots\rangle_g-\langle\partial^i_k\partial^\mu\rangle\langle\partial^\mu_{\overline{l-1}}\ldots\rangle_g.$$

Repeat this process of reducing $\bar{l}$, one can show by induction that

$$\begin{aligned}&\langle\partial^i_{k,\bar{l}}\ldots\rangle_g=\langle\partial^i_{k+r,\overline{l-r}}\ldots\rangle_g-\langle\partial^i_{k+r-1}\partial^{\mu_1}\rangle\langle\partial^{\mu_1}_{\overline{l-r}}\ldots\rangle_g-\ldots\\&-\langle\partial^i_k\partial^{\mu_1}\rangle\left[\sum_{p=1}^{r}(-1)^{p+1}\sum_{k_1+\ldots+k_p=r-p}\langle\partial^{\mu_1}_{k_1}\partial^{\mu_2}\rangle\ldots\langle\partial^{\mu_{p-1}}_{k_{p-1}}\partial^{\mu_p}\rangle\langle\partial^{\mu_p}_{k_p,\overline{l-r}}\ldots\rangle_g\right].\end{aligned}$$

Now suppose that one has an equation of tautological classes of $\overline{M}_{g,n}$. Use the above equation (for $r=l$) one can translate the equation of tautological classes on $\overline{M}_{g,n}$ into an equation of the (axiomatic) Gromov–Witten invariants. The term-wise cancellation of the contributions from the second and the third terms of (12) and (13) can be seen easily by straightforward computation. □

If the above description is a bit abstract, the reader might want to try the following simple example. ψ_1^2 on $\overline{M}_{g,1}$ is translated into invariants:

$$\langle\partial^x_2\rangle_g-\langle\partial^x_1\partial^\mu\rangle\langle\partial^\mu\rangle_g-\langle\partial^x\partial^\mu\rangle\langle\partial^\mu_1\rangle_g+\langle\partial^x\partial^\mu\rangle\langle\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_g.$$

The above "translation" from tautological classes to Gromov–Witten invariants are worked out explicitly in some examples in Sections 6 and 7 of [12].

5.2. **Reduction to $q_0 = 0$.** The arguments in this section are mostly taken from [19].

Let $E = 0$ be a tautological equation of (axiomatic) Gromov–Witten invariants. Since we have already proved $\hat{s}(E) = 0$, our next goal would be to show $\hat{r}(E) = 0$. In this section, we will show that it suffices to check $\hat{r}(E) = 0$ on the subspace $q_0 = 0$.

Lemma 2. *It suffices to show $\hat{r}E = 0$ on each level set of the map $q \mapsto s$ in* (5)*.*

Proof. The union of the level sets is equal to $\mathcal{H}_+$. □

Lemma 3. *It suffices to check the relation for all $\hat{r}(z)E = 0$ along $z\mathcal{H}_+$ (i.e. $q_0 = 0$).*

Proof. It is proved in Section 4 of [13] that a particular lower triangular matrix S_s, which is called "calibration" of the Frobenius manifold, transforms the level set at s to $z\mathcal{H}_+$. S-invariance Theorem then concludes the proof. □

In fact, S_s is a fundamental solution of the horizontal sections of the Dubrovin (flat) connection, in z^{-1} formal series. It was discovered in [16], with preceding work in [23] and [14], that $\mathcal{A} := \hat{S}_s \tau^X$ is the corresponding generating function for "ancestors". Therefore the transformed equation $\hat{S}_s E \hat{S}_s^{-1} = 0$ is really an equation of *ancestors*.

5.3. **Invariance under the upper triangular subgroup.**

Theorem 6. (R-invariance theorem) *The union of the sets of genus g' equations for $g' \leq g$ is invariant under the action of upper triangular subgroup, for $g \leq 2$.* [7]

In fact, a "filtered" statement holds. We will state the genus two part:

(I) The combination of genus zero equations, genus one equations and Mumford's equation is R-invariant.
(II) The combination of genus zero equations, genus one equations and genus two Mumford's and Getzler's equations is R-invariant.
(III) The combination of genus zero equations, genus one equations and genus two equations by Mumford, Getzler and BP is R-invariant.

[7] In genus two, there are possibly other equations which have not been discovered. What is alluded in the Theorem is really about the known equations.

Remark. 1. R-invariance in $g = 1$ is proved in [19].

2. There are other genus two tautological equations, like the 6-point equation discovered by Faber–Pandharipande (private communication). However, its role in semisimple Gromov–Witten theory is not clear at this point.

3. R-invariance theorem is expected to hold for all g. In fact, under plausible assumptions, R-invariance technique can be used to "derive" all known tautological equations, including all tautological equations appeared in this paper, and other new equations in higher genus. This will be discussed in separate papers [1] [2].

The rest of the section is devoted to the proof of R-invariance theorem, and therefore the Main Theorem. In fact, the proof of (I), (II) and (III) follow the same line of arguments, so we will only treat (I) in details.

Recall that Mumford's genus two equation is of the form, *in the orthonormal basis* as usual, with summation convention,

$$\begin{aligned} M := & - \langle \partial_2^x \rangle_2 + \langle \partial_1^x \partial^\mu \rangle \langle \partial^\mu \rangle_2 + \langle \partial^x \partial^\mu \rangle \langle \partial_1^\mu \rangle_2 \\ & - \langle \partial^x \partial^\mu \rangle \langle \partial^\mu \partial^\nu \rangle \langle \partial^\nu \rangle_2 + \frac{7}{10} \langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^\mu \rangle_1 \langle \partial^\nu \rangle_1 \\ & + \frac{1}{10} \langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^\mu \partial^\nu \rangle_1 - \frac{1}{240} \langle \partial^\mu \partial^\nu \partial^\nu \rangle \langle \partial^x \partial^\mu \rangle_1 \\ & + \frac{13}{240} \langle \partial^x \partial^\mu \partial^\mu \partial^\nu \rangle \langle \partial^\nu \rangle_1 + \frac{1}{960} \langle \partial^x \partial^\mu \partial^\mu \partial^\nu \partial^\nu \rangle = 0. \end{aligned} \tag{16}$$

Lemma 4. *It suffices to check $(r(z))^\wedge M = 0$ for $l = 1$ and $l = 2$ (and on $q_0 = 0$).*

Proof. It is easy to see that when $l \geq 3$, all terms in $(r(z))^\wedge M$ (17) vanish by the $(3g-2)$-jet property, which is satisfied for the ancestor invariants ([14] and Section 5 of [16]).

In geometric terms, this is due to the fact that Mumford's equation (16) is a codimension 2 tautological relation in $\overline{M}_{2,1}$, whose dimension is equal to 4. Since $(r_l z^l)^\wedge$ carries codimension k strata to codimension $k + l$ ones, $(r_l z^l)^\wedge M = 0$ for $l \geq 3$. □

In fact, as we will see, the checking of invariance is really straightforward for $l = 1$, and almost trivial for $l = 2$. Note we will use the following conventions:

- If some term does not contain l, that means $l = 1$ and it vanishes for $l \geq 2$.
- ∂^x is any flat vector field, e.g. $\partial^x = \partial_k^\mu$. Therefore ∂_2^x means the descendent index is at least two, but could be greater.

$$
\begin{aligned}
&(r(z))\hat{}\, M \\
= &\sum_l (r_l)_{ij} \Big[-\frac{1}{2}\sum(-1)^{m+1}\langle\partial^i_{l-1+m}\partial^j_m\partial^x_2\rangle_1 \\
&-\sum(-1)^{m+1}\langle\partial^i_{l-1-m}\partial^x_2\rangle_1\langle\partial^j_m\rangle_1 \\
&+\frac{7}{5}\sum\langle\partial^x\partial^j\partial^\nu\rangle\langle\partial^i_1\rangle_1\langle\partial^\nu\rangle_1 \\
&-\frac{7}{10}\sum\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^j\partial^\mu\rangle\langle\partial^\nu\rangle_1 \\
&+\frac{1}{5}\sum\langle\partial^x\partial^j\partial^\nu\rangle\langle\partial^i_l\partial^\nu\rangle_1 \\
&+\frac{1}{20}\sum(-1)^{m+1}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i_{l-1-m}\partial^j_m\partial^\mu\partial^\nu\rangle \\
&+\frac{1}{10}\sum(-1)^{l}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j_{l-1}\rangle_1 \\
&-\frac{1}{240}\sum\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^i_l\rangle_1 \\
&-\frac{1}{480}\sum(-1)^{m+1}\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^i_{l-1-m}\partial^j_m\partial^x\partial^\mu\rangle \\
&-\frac{1}{240}\sum(-1)^{l}\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^x\partial^\mu\rangle\langle\partial^j_{l-1}\rangle_1 \\
&+\frac{13}{120}\sum\langle\partial^x\partial^i_l\partial^j\partial^\nu\rangle\langle\partial^\nu\rangle_1 \\
&+\frac{13}{240}\sum\langle\partial^x\partial^\mu\partial^\mu\partial^i_l\rangle\langle\partial^j\rangle_1 \\
&+\frac{13}{240}\sum\langle\partial^x\partial^\mu\partial^\mu\partial^j\rangle\langle\partial^i_l\rangle_1 \\
&-\frac{13}{480}\sum\langle\partial^x\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^j\partial^\nu\rangle \\
&-\frac{13}{120}\sum\langle\partial^i\partial^x\partial^\mu\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_1 \\
&-\frac{13}{240}\sum\langle\partial^i\partial^x\partial^\nu\rangle\langle\partial^j\partial^\mu\partial^\mu\rangle\langle\partial^\nu\rangle_1 \\
&+\frac{1}{240}\sum\langle\partial^x\partial^i_l\partial^j\partial^\nu\partial^\nu\rangle \\
&-\frac{1}{480}\sum\langle\partial^i_{l-1}\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle \\
&-\frac{1}{240}\sum\langle\partial^i_{l-1}\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle \\
&-\frac{1}{240}\sum\langle\partial^i_{l-1}\partial^x\partial^\mu\rangle\langle\partial^j\partial^\mu\partial^\nu\partial^\nu\rangle \Big]
\end{aligned}
\tag{17}
$$

In the above, the following observations simplify the calculation:

- The contributions from first terms of (14) (15) vanish.
- The contributions from the second terms of (14) (15), when acting on ∂^x_k, cancell with each other.

The first is already explained in the proof of Theorem 5. The second is due to the fact that the sum of the second term merely changes one flat vector to another: $\partial^\mu_k \mapsto \sum (r_l)_{i\mu} \partial^i_{k+l}$. Since ∂^x is an arbitrary flat vector field, this change sums to zero. The last case is due to the fact that it sums to a derivative of $M = 0$ with respect to q.

The two cases of $l = 1$ and $l = 2$ will be discussed separately.

5.3.1. *The case* $l = 1$. Since there are only invariants in genus zero and one, all descendents can be removed by TRR. (The flat vector field ∂^x might contain descendents, but we will only remove the "apparent descendents".)

After applying TRRs, all terms can be classified into two types:

- Type 1: terms involving genus one invariants and
- Type 2: terms with only genus zero invariants.

Type 1: After obvious vanishing (on the dimensional ground), the contribution is

$$\begin{aligned}
\sum (r_1)_{ij} \Big[& \frac{1}{2} \langle \partial^x_1 \partial^i \partial^j \partial^\mu \rangle \langle \partial^\mu \rangle_1 + \frac{1}{24} \langle \partial^x_1 \partial^i \partial^\mu \partial^\mu \rangle \langle \partial^j \rangle_1 \\
& + \frac{7}{120} \langle \partial^i \partial^\mu \partial^\mu \rangle \langle \partial^x \partial^j \partial^\nu \rangle \langle \partial^\nu \rangle_1 - \frac{7}{10} \langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^j \partial^\mu \rangle \langle \partial^\nu \rangle_1 \\
& + \frac{1}{5} \langle \partial^x \partial^j \partial^\nu \rangle \langle \partial^i \partial^\mu \partial^\nu \rangle \langle \partial^\mu \rangle_1 - \frac{1}{10} \langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^\mu \partial^\nu \rangle \langle \partial^j \rangle_1 \\
& - \frac{1}{240} \langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^j \partial^\nu \partial^\nu \rangle \langle \partial^\mu \rangle_1 + \frac{1}{240} \langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^\mu \partial^\nu \partial^\nu \rangle \langle \partial^j \rangle_1 \\
& + \frac{13}{240} \langle \partial^x \partial^i_1 \partial^j \partial^\mu \rangle \langle \partial^\mu \rangle_1 + \frac{13}{240} \langle \partial^x \partial^i_1 \partial^\mu \partial^\mu \rangle \langle \partial^j \rangle_1 \\
& - \frac{13}{120} \langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^j \partial^\mu \partial^\nu \rangle \langle \partial^\nu \rangle_1 - \frac{13}{120} \langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^j \partial^\nu \partial^\nu \rangle \langle \partial^\nu \rangle_1 \Big]
\end{aligned}$$

The above sum can be put into two groups. One with genus one invariants of the form $\langle \partial^j \rangle_1$, the other with $\langle \partial^\mu \rangle_1$. It is easy to see that each group gets exact cancellation by genus zero TRR (and WDVV equation, which is a consequence of TRR).

Type 2:

$$\sum(r_1)_{ij}\Big[\frac{1}{48}\langle\partial_1^x\partial^i\partial^j\partial^\mu\partial^\mu\rangle+\frac{1}{120}\langle\partial^x\partial^j\partial^\nu\rangle\langle\partial^i\partial^\nu\partial^\mu\partial^\mu\rangle$$
$$-\frac{1}{20}\langle\partial^i\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^\nu\partial^\mu\rangle-\frac{1}{5760}\langle\partial^i\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle$$
$$+\frac{1}{480}\langle\partial^i\partial^j\partial^x\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle+\frac{13}{5760}\langle\partial^x\partial^j\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^\nu\partial^\nu\rangle$$
$$-\frac{13}{480}\langle\partial^x\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^j\partial^\nu\rangle+\frac{1}{240}\langle\partial^x\partial^\mu\partial^\mu\partial_1^i\partial^j\rangle$$
$$+\frac{1}{240}\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^j\partial^\mu\partial^\nu\partial^\nu\rangle-\frac{1}{480}\langle\partial^x\partial^i\partial^\mu\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle$$
$$-\frac{1}{240}\langle\partial^x\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle\Big]$$

This can be seen to vanish by elementary calculation, utilizing genus zero TRR and WDVV.

5.3.2. *The case* $l = 2$. Due to (11), $(r_2)_{ij} = -(r_2)_{ji}$. However, all terms in (17) with nonvanishing contribution at $l = 2$ are all *symmetric* in i and j. For example the first term contributes

$$-\frac{1}{2}\sum_{ij}(r_2)_{ij}\langle\partial_1^i\partial^j\partial_2^x\rangle_1+\frac{1}{2}\sum_{ij}(r_2)_{ij}\langle\partial^i\partial_1^j\partial_2^x\rangle_1$$
$$=\sum_{ij}(r_2)_{ij}\langle\partial^i\partial_1^j\partial_2^x\rangle_1=\sum_{ij}(r_2)_{ij}[\langle\partial^i\partial^j\partial^\mu\rangle\langle\partial_2^x\partial^\mu\rangle+\frac{1}{24}\langle\partial^i\partial^j\partial^\mu\partial^\mu\partial_2^x\rangle],$$

which vanishes as $(r_2)_{ij}$ in anti-symmetric while the other factor is symmetric in i and j.

The terms do not cancell by itself after summation over i, j will contribute zero by cancellation. More precisely, the summation is

$$\sum(r_2)_{ij}(\frac{-1}{576}+\frac{1}{240}-\frac{1}{5760}-\frac{1}{5760}-\frac{1}{480})\langle\partial^i\partial^x\partial^\mu\rangle\langle\partial^\mu\partial^{nu}\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle,$$

which is easily seen to vanish.

5.3.3. *R-invariance of Getzler's and BP's equations.* The R-invariance of Getzler's and BP's equations are proved in the same way. The details are not recorded here due to the following two reasons. First, the proof follows the same line of arguments as in the Mumford's equation; second, a stronger version of the calculations has been done in [1]. Therefore, we will list only some main steps here. Recall that $q_0 = 0$ is always assumed.

For Getzler's genus two equation:

(1) $G' := (r(z))\hat{}$ (Getzler's equation) involves only up to $l = 3$, using the same argument in Lemma 4.
(2) G' contains only one term with genus two component: $3\langle \partial^x \partial^y \partial^\mu \rangle \langle \partial^\mu_{l+1} \rangle_2$. This can be written, via Mumford's equation as a summation of genus 1 and 0 invariants only.
(3) One may remove the "apparent descendents" in genus one invariants by TRR. Here the apparent descendents means the lower indices in our notation. Although ∂^x might already contain descendents implicitly, they won't be removed.
(4) One can group terms together according to the types of the factors of the genus one invariants in these terms. Since there is no additional relation in genus one, they will cancell within each group. We note that no 4-point genus one invariants appeared in the calculation. Therefore Getzler's genus one equation is actually never used.
(5) The cancellations of the genus zero part of (17) involves only (derivatives of) WDVV equation.

There are only two different points in the proof of R-invariance of BP's equation. First, the calculation will go up to $l = 4$. Second, Getzler's genus two equation will be used.

Appendix A. Conjectures on the Tautological Equations

The purpose of this appendix is to formulate a few conjectures on the relations of tautological classes in $A(\overline{M}_{g,n}, \mathbb{Q})$.

Some notations are needed. We assume that the readers are familiar with the presentation of the boundary strata of $\overline{M}_{g,n}$ by their dual graphs: Assign a vertex to each irreducible component of the generic curve; assign an edge between two vertices each time the two components intersect; assign a tail to each marked point. Consider an edge as gluing of two half-edges. Label each vertex with its genus and each tail with its number $1, \ldots, n$. Label each half-edge/tail with other classes (ψ, κ, λ, ...) it carries. Note that we are thinking of an edge as gluing of two half edges and therefore the automorphisms of graphs are different from the usual convention (and is close to the convention in Gromov–Witten theory).

Define the operations $\mathfrak{r}_l$ on the dual graphs Γ of tautological strata.

- Cut one edge and regard the two half edges as two new tails. Produce two graphs by assigning extra ψ^l to one of the two new tails. Produce more graphs by proceeding to the next edge. (Thus the number of the new graphs is the same as two times the number of edges in Γ.) Retain only the stable graphs. Take formal sum of these final graphs.

- For each vertex, produce l graphs. Reduce the genus of this given vertex by one, add two new tails and label them by ψ^m and ψ^{l-1-m} where $0 \leq m \leq l-1$. Do this to all vertices, and retain only the stable graphs. Take formal sum of these graphs with coefficient $\frac{1}{2}(-1)^{m+1}$.
- Split one vertex into two. Add two new tails to these two new vertices and label the two new tails by ψ^m and ψ^{l-1-m} where $0 \leq m \leq l-1$. Produce new graphs by separating the genus g between the two new vertices (g_1, g_2 such that $g_1 + g_2 = g$), and distributing to the two new vertices the tails and half-edges which belongs to the original chosen vertex, in all possible ways. Do this to all vertices, and retain only the stable graphs. Take formal sum of these graphs with coefficient $\frac{1}{2}(-1)^{m+1}$.
- When l is odd, symmetrize the two extra tails. When l is even, anti-symmetrize the two extra tails.

Some remarks are in order. First, the form of $\mathfrak{r}_l$ is dictated by the equation (15), where $\mathfrak{r}_l$ corresponds to $(r_l z^l)\hat{}$ in the Gromov–Witten theory. The three operations here correspond to the last three terms in (15). The first term and some contributions of the second term of (15) vanish by the arguments given in Section 5.3. Second, although the graph Γ we start with is connected, the graphs produced by $\mathfrak{r}_l$ might be disconnected. A disconnected graph is stable if each connected component is stable. In any case, the graphs produced from $\mathfrak{r}_l$ acting on Γ can be translated back to (disjoint unions of) tautological strata. Thus we will not distinguish between the strata and their corresponding graphs.

Conjecture 1. Given a tautological equation $\sum_i c_i \Gamma_i = 0$ of codimension k strata in $\overline{M}_{g,n}$, $\mathfrak{r}_l(\sum_i c_i \Gamma_i) = 0$, modulo the equations for $g' \leq g$ or $n' \leq n$.

Note that $\mathfrak{r}_l \Gamma = 0$ when $k+l > \dim \overline{M}_{g,n}$. This can be easily checked, and the Gromov–Witten version of that is explained in Lemma 4. Therefore, there are only finitely many (g', n') will be involved, depending on l. These observations can easily be verified from the definition of $\mathfrak{r}_l$.

Conjecture 1 is inspired by its numerical version in the Gromov–Witten theory.

Conjecture 1'. The induced equation on the Gromov–Witten invariants are invariant under the corresponding action by $(r_l z^l)\hat{}$ (15).

Conjecture 1 implies Conjecture 1' by the argument given in Section 5. The opposite implication is possible if the Poincaré duality conjecture for tautological rings holds.

Conjectures 1 and 1' will be referred to as *R-invariance Conjecture.* They have been established up to genus two in the main text. Note that the calculations given there are for the Gromov–Witten version, but the translation to graphical notations is completely straightforward. (The typesetting would be much more involved though.)

Conjecture 2. Let E be a given linear combination of codimension k tautological strata in $\overline{M}_{g,n}$ and $k \neq \dim \overline{M}_{g,n}$.

If $\mathfrak{r}_l(E) = 0$ for all l, modulo tautological equations in $\overline{M}_{g',n'}$ for $g' \leq g$ or $n' \leq n$, then $E = 0$ is a tautological equation.

There is also a version of Conjecture 2 in Gromov–Witten theory.

Conjecture 2'. Let $E = 0$ be an equation of Gromov–Witten invariants induced from $E = 0$ in Conjecture 2. If for every semisimple theory $(r_l z^l)\hat{}\,(E) = 0$, modulo $E = 0$ and other tautological equations in $g' \leq g$ or $n' \leq n$, then $E = 0$ holds for all semisimple theory.

Conjectures 2 and 2' have been used to "re-discover" all known tautological equations (not of top codimension): Getzler's genus one equation ([11]) in [19], Mumford's equation, Getzler's genus two equation ([12]) and Belorousski–Pandharipande's equation ([3]) in [1]. There is also a new tautological equation derived in codimension 3 of $\overline{M}_{3,1}$ [2] (see also [21]).

The basic strategy is to write a linear combination of all known strata Γ_i in a fixed codimension with *unknown* coefficients c_i, $E = \sum_i c_i \Gamma_i$. Apply the R-invariance condition $\mathfrak{r}_l(E) = 0$, one gets enough linear equations on c_i's to determine them completely. What please us most is the fact while the four equations were discovered in four different ways, we have been able to reproduce all four equations using the same method. Furthermore, this method can be used to discover more tautological equations.

We expect that one might be able to prove these conjectures via localization on moduli spaces of (relative/orbifold) stable maps to $\mathbb{P}^1$, especially Conjectures 1 and 1'.

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