

A RECONSTRUCTION THEOREM IN QUANTUM COHOMOLOGY AND QUANTUM K -THEORY

Y.-P. LEE AND R. PANDHARIPANDE

ABSTRACT. A reconstruction theorem for genus 0 gravitational quantum cohomology and quantum K -theory is proved. A new linear equivalence in the Picard group of the moduli space of genus 0 stable maps relating the pull-backs of line bundles from the target via different markings is used for the reconstruction result. Examples of calculations in quantum cohomology and quantum K -theory are given.

0. INTRODUCTION

0.1. Divisor relations. Let X be a nonsingular, projective, complex algebraic variety. Let L be a line bundle on X . The goal of the present article is to study the relationship between the different evaluation pull-backs $\text{ev}_i^*(L)$ and $\text{ev}_j^*(L)$ on the space of stable maps $\overline{M}_{0,n}(X, \beta)$. We will also examine the relationship between the cotangent line classes ψ_i and ψ_j at distinct markings.

Our method is to study the relationship first in the case of projective space. The moduli of stable maps $\overline{M}_{0,n}(\mathbb{P}^r, \beta)$ is a nonsingular Deligne–Mumford stack. The Picard group $\mathcal{P}ic(\overline{M}_{0,n}(\mathbb{P}^r, \beta))$ with \mathbb{Q} -coefficients has been analyzed in [12]. In case $n \geq 1$, the Picard group is generated by the evaluation pull-backs $\text{ev}_i^*(L)$ together with the boundary divisors. The cotangent line classes ψ_i also determine elements of the Picard group.

Theorem 1. *The following relations hold in $\mathcal{P}ic(\overline{M}_{0,n}(\mathbb{P}^r, \beta))$ for all $L \in \mathcal{P}ic(\mathbb{P}^r)$ and markings $i \neq j$:*

$$(1) \quad \text{ev}_i^*(L) = \text{ev}_j^*(L) + \langle \beta, L \rangle \psi_j - \sum_{\beta_1 + \beta_2 = \beta} \langle \beta_1, L \rangle D_{i, \beta_1 | j, \beta_2},$$

$$(2) \quad \psi_i + \psi_j = D_{i|j},$$

Y.L. was partially supported by DMS-0072547. R.P. was partially supported by DMS-0071473 and fellowships from the Sloan and Packard foundations.

where $\langle \beta, L \rangle$ denotes the intersection pairing

$$\langle \beta, L \rangle = \int_{\beta} c_1(L).$$

The boundary notation used in the Theorem is defined by the following conventions. Let $D_{S_1, \beta_1 | S_2, \beta_2}$ denote the boundary divisor in $\overline{M}_{0,n}(\mathbb{P}^r, \beta)$ parameterizing maps with reducible domains and splitting types

$$S_1 \cup S_2 = \{1, \dots, n\}, \quad \beta_1 + \beta_2 = \beta$$

of the marking set and the degree respectively (see [3, 12]). We then define:

$$D_{i, \beta_1 | j, \beta_2} = \sum_{i \in S_1, j \in S_2} D_{S_1, \beta_1 | S_2, \beta_2},$$

$$D_{i|j} = \sum_{i \in S_1, j \in S_2, \beta_1 + \beta_2 = \beta} D_{S_1, \beta_1 | S_2, \beta_2}.$$

Let $L \in \mathcal{P}ic(X)$ be a very ample line bundle on X . Let

$$\iota : X \rightarrow \mathbb{P}^r$$

be the embedding determined by L . There is a canonically induced embedding:

$$\bar{\iota} : \overline{M}_{0,n}(X, \beta) \rightarrow \overline{M}_{0,n}(\mathbb{P}^r, \iota_*[\beta]).$$

The $\bar{\iota}$ pull-backs of the relations of Theorem 1 together with the splitting axiom of Gromov–Witten theory yield relations in the rational Chow group of $\overline{M}_{0,n}(X, \beta)$:

$$\begin{aligned} \text{ev}_i^*(L) \cap [\overline{M}_{0,n}(X, \beta)]^{\text{vir}} &= (\text{ev}_j^*(L) + \langle \beta, L \rangle \psi_j) \cap [\overline{M}_{0,n}(X, \beta)]^{\text{vir}} \\ &\quad - \sum_{\beta_1 + \beta_2 = \beta} \langle \beta_1, L \rangle [D_{i, \beta_1 | j, \beta_2}]^{\text{vir}}. \end{aligned}$$

$$(\psi_i + \psi_j) \cap [\overline{M}_{0,n}(X, \beta)]^{\text{vir}} = [D_{i|j}]^{\text{vir}}.$$

Here, $[D_{i, \beta_1 | j, \beta_2}]^{\text{vir}}$ and $[D_{i|j}]^{\text{vir}}$ denote the push-forward to $\overline{M}_{0,n}(X, \beta)$ of the virtual classes of their constituent boundary divisors.

We note the classes of very ample divisors span $\mathcal{P}ic(X)$ with \mathbb{Q} -coefficients for projective X . As the above Chow relation is *linear* in L , we conclude:

Corollary 1. *The following relations hold in $A_*(\overline{M}_{0,n}(X, \beta))$ for all $L \in \mathcal{P}ic(X)$ and markings $i \neq j$:*

$$\begin{aligned} \text{ev}_i^*(L) \cap [\overline{M}_{0,n}(X, \beta)]^{\text{vir}} &= (\text{ev}_j^*(L) + \langle \beta, L \rangle \psi_j) \cap [\overline{M}_{0,n}(X, \beta)]^{\text{vir}} \\ &\quad - \sum_{\beta_1 + \beta_2 = \beta} \langle \beta_1, L \rangle [D_{i, \beta_1 | j, \beta_2}]^{\text{vir}}, \end{aligned}$$

$$(\psi_i + \psi_j) \cap [\overline{M}_{0,n}(X, \beta)]^{\text{vir}} = [D_{i|j}]^{\text{vir}}.$$

0.2. Reconstruction. We use the following standard notation for the Gromov–Witten invariants:

$$(\tau_{k_1}(\gamma_1), \dots, \tau_{k_n}(\gamma_n))_{0,n,\beta} = \int_{[\overline{M}_{0,n}(X,\beta)]^{\text{vir}}} \prod_i \psi_i^{k_i} \text{ev}_i^*(\gamma_i)$$

where $\gamma_i \in H^*(X)$. The *quantum K-invariants* [11] are:

$$(\tau_{k_1}(\gamma_1), \dots, \tau_{k_n}(\gamma_n))_{0,n,\beta}^K = \chi(\overline{M}_{0,n}(X, \beta), [\mathcal{O}_{\overline{M}_{0,n}(X,\beta)}^{\text{vir}}] \prod_i [\mathcal{L}_i]^{k_i} \text{ev}_i^*(\gamma_i)),$$

where $\gamma_i \in K^*(X)$, $[\mathcal{O}_{\overline{M}_{0,n}(X,\beta)}^{\text{vir}}] \in K_0(\overline{M}_{0,n}(X, \beta))$ is the virtual structure sheaf, \mathcal{L}_i is the i^{th} cotangent line bundle, and χ is the K -theoretic push-forward to $\text{Spec}(\mathbb{C})$. In algebraic K -theory,

$$\chi(M, [F]) = \sum_i (-1)^i R^i \pi_* [F]$$

where M is a variety and $\pi : M \rightarrow \text{Spec}(\mathbb{C})$ is the canonical map.

A subring $R \subset H^*(X)$ is *self-dual* if the restriction of the cohomological Poincaré pairing to R is nondegenerate. The K -theoretic Poincaré pairing on a nonsingular variety X is:

$$\langle u, v \rangle = \chi(X, u \otimes v).$$

A subring $R \subset K^*(X)$ is *self-dual* if the restriction of the K -theoretic Poincaré pairing to R is nondegenerate.

Theorem 2. *A reconstruction result from 1-point invariants holds in both quantum cohomology and quantum K-theory:*

- (i) *Let $R \subset H^*(X)$ be a self-dual subring generated by Chern classes of elements of $\mathcal{P}ic(X)$. Let R^\perp be the orthogonal complement (with respect to the cohomological Poincaré pairing). Suppose*

$$(\tau_{k_1}(\gamma_1), \dots, \tau_{k_{n-1}}(\gamma_{n-1}), \tau_{k_n}(\xi))_{0,n,\beta} = 0$$

for all n -point descendent invariants satisfying $\gamma_i \in R$ and $\xi \in R^\perp$. Then, all n -point descendent invariants of classes of R can be reconstructed from 1-point descendent invariants of R .

- (ii) Let $R \subset K^*(X)$ be a self-dual subring generated by elements of $\mathcal{P}ic(X)$. Let R^\perp be the orthogonal complement (with respect to the Poincaré pairing in K -theory). Suppose

$$(\tau_{k_1}(\gamma_1), \dots, \tau_{k_n}(\gamma_n), \tau_{k_{n+1}}(\xi))_{0,n,\beta}^K = 0$$

for all n -point descendent invariants satisfying $\gamma_i \in R$ and $\xi \in R^\perp$. Then, all n -point descendent invariants of classes from R can be reconstructed from 1-point descendent invariants of R .

Part (i) of Theorem 2 is a direct consequence of Corollary 1, the string equation, and the splitting axiom of Gromov–Witten theory. The self-dual and vanishing conditions on R are required to control the Künneth components of the diagonal arising in the splitting axiom. Part (ii) is proven by a parallel argument in quantum K -theory [11]. We note the subring R need *not* be generated by the entire Picard group for either part of Theorem 2.

In the Gromov–Witten case (i), a similar reconstruction result was proven independently by A. Bertram and H. Kley in [1] using a very different technique: recursive relations are found via a residue analysis of the virtual localization formula of [7] applied to the graph space of X . Our recursive equations differ from [1]. We point out the self-dual condition on R was omitted in the Bertram–Kley result [1] in error.

0.3. Applications.

0.3.1. The 1-point descendents are the most accessible integrals in quantum cohomology. Their generating function, the J -function, has been explicitly computed for many important target varieties (for example, toric varieties and homogeneous spaces). The n -point descendent invariants, however, remain largely unknown. Theorem 2 yields a reconstruction of all gravitational Gromov–Witten invariants from the J -function in the case $H^*(X)$ is generated by $\mathcal{P}ic(X)$.

For example, the 1-point invariants of all flag spaces X (associated to simple Lie algebras) have been computed by B. Kim [8]. As a result, a presentation of the quantum cohomology ring $QH^*(X)$ can be found. However, the 3-point invariants, or structure constants of $QH^*(X)$, remain mostly unknown. The 3-point invariants for the flag space of A_n -type have been determined by Fomin–Gelfand–Postnikov [4]. It is hoped that Theorem 1 may help to find a solution for other flag spaces. Computations in this direction have been done by H. Chang and the first author (in agreement with results of Fomin–Gelfand–Postnikov). The principal difficulty is to understand the combinatorics associated to Theorem 2.

0.3.2. The *Quantum Lefschetz Hyperplane Theorem* [10] determines the 1-point descendents of the restricted classes $i_Y^* H^*(X)$ of a nonsingular very ample divisor

$$i_Y : Y \hookrightarrow X$$

from the 1-point descendents of X . The following Lemma shows Theorem 2 may be applied to the subring $i_Y^* H^*(X) \subset H^*(Y)$.

Lemma 1. *Let Y be a nonsingular very ample divisor in X determined by the zero locus of a line bundle E . Assume:*

- (i) $H^*(X)$ is generated by $\text{Pic}(X)$,
- (ii) $i_{Y*} : H_2(Y) \xrightarrow{\sim} H_2(X)$.

Consider the ring $R = i_Y^ H^*(X) \subset H^*(Y)$. Then, R is self-dual, the vanishing condition for quantum cohomology in part (i) of Theorem 2 is satisfied, and the reconstruction result holds.*

Proof. We first prove $R \subset H^*(Y)$ is self-dual for the Poincaré pairing on Y . Equivalently, we will prove, for all non-zero $\epsilon_Y \in R$, there exists a element $\delta_Y \in R$ such that

$$\int_Y \epsilon_Y \cup \delta_Y \neq 0.$$

Let $\epsilon \in H^*(X)$ pull-back to ϵ_Y : $i_Y^*(\epsilon) = \epsilon_Y$. If

$$\epsilon \cup c_1(E) \neq 0 \in H^*(X),$$

then there exists $\delta \in H^*(X)$ satisfying:

$$\int_X \epsilon \cup \delta \cup c_1(E) \neq 0$$

as the Poincaré pairing on X is nondegenerate. Let $\delta_Y = i_Y^*(\delta)$. Then,

$$\int_Y \epsilon_Y \cup \delta_Y = \int_X \epsilon \cup \delta \cup c_1(E) \neq 0.$$

If $\epsilon \cup c_1(E) = 0 \in H^*(X)$, then we will apply the Hard Lefschetz Theorem (HLT) to prove $\epsilon_Y = 0 \in H^*(Y)$. Let n be the complex dimension of X . By HLT applied to $(X, c_1(E))$, we may assume

$$\epsilon \in H^{n-1+k}(X)$$

for $k > 0$. Thus, $\epsilon_Y \in H^{n-1+k}(Y)$. By HLT applied to $(Y, c_1(E_Y))$, there exists an element $\epsilon'_Y \in H^{n-1-k}(Y)$ satisfying:

$$\epsilon'_Y \cup c_1(E_Y)^k = \epsilon_Y \in H^*(Y).$$

By the Lefschetz Hyperplane Theorem applied to $Y \subset X$,

$$i_Y^* : H^{n-1-k}(X) \xrightarrow{\sim} H^{n-1-k}(Y).$$

Let $\epsilon' \in H^{n-1-k}(X)$ satisfy $i_Y^*(\epsilon') = \epsilon'_Y$. We find,

$$i_Y^*(\epsilon' \cup c_1(E)^k) = \epsilon_Y.$$

As $i_Y^*(\epsilon) = \epsilon_Y$ and $\epsilon \cup c_1(E) = 0 \in H^*(X)$, we find:

$$\epsilon' \cup c_1(E)^{k+1} = 0 \in H^*(X).$$

By HLT applied to $(X, c_1(E))$, we conclude $\epsilon' = 0$. The vanishing $\epsilon_Y = 0$ then follows.

Next, we prove the vanishing of Gromov–Witten invariants required for Theorem 2. Let $\gamma_i \in H^*(X)$ and $\xi \in R^\perp$.

$$\begin{aligned} & (\tau_{k_1}(i_Y^*\gamma_1), \dots, \tau_{k_{n-1}}(i_Y^*\gamma_{n-1}), \tau_{k_n}(\xi))_{0,n,\beta}^Y \\ &= \int_{[\overline{M}_{0,n}(Y,\beta)]^{\text{vir}}} \prod_{i=1}^{n-1} \psi_i^{k_i} \text{ev}_i^*(i_Y^*\gamma_i) \psi_n^{k_n} \text{ev}_n^*(\xi). \end{aligned}$$

Since $i_{Y*}[\overline{M}_{0,n}(Y,\beta)]^{\text{vir}} = c_{\text{top}}(E_\beta) \cap [\overline{M}_{0,n}(X,\beta)]^{\text{vir}}$ (see [2]) and

$$\prod_{i=1}^{n-1} \psi_i^{k_i} \text{ev}_i^*(\gamma_i) \psi_n^{k_n}$$

is a cohomology class pulled-back from $\overline{M}_{0,n}(X,\beta)$, the above integral may be rewritten as:

$$\int_Y i_Y^*(\text{ev}_{n*} \prod_{i=1}^{n-1} \psi_i^{k_i} \text{ev}_i^*(\gamma_i) \psi_n^{k_n} \cap [\overline{M}_{0,n}(X,\beta)]^{\text{vir}}) \cup \xi = 0.$$

□

The Quantum Lefschetz Hyperplane Theorem and Theorem 2 allow the determination of n -point descendants of $i_Y^*H^*(X)$ from the 1-point descendants of X in this case (see [10], Corollary 1).

The Quantum Lefschetz Hyperplane Theorem and Lemma 1 also hold when $Y \subset X$ is the nonsingular complete intersection of very ample divisors. The proof of Lemma 1 for complete intersections is the same.

0.3.3. Quantum K -theory is more difficult than Gromov–Witten theory. For example, there are no dimension restrictions for the quantum K -theoretic invariants. The genus 0 reconstruction results of Kontsevich–Manin via the WDVV-equations are less effective in quantum K -theory: the 3-point invariants needed for reconstruction are non-trivial even in the case of projective space. Theorem 2 provides a new tool for the study of quantum K -theory.

In case $X = \mathbb{P}^r$, the 1-point quantum K -invariants have been determined in [11] (see Section 2). Theorem 2 then allows a recursive

computation of all the genus 0 K -theoretic invariants. The set of invariants includes (and is essentially equivalent to) the holomorphic Euler characteristics of the *Gromov–Witten subvarieties* of $\overline{M}_{0,n}(\mathbb{P}^r, \beta)$. The Gromov–Witten subvarieties are defined by:

$$(3) \quad \text{ev}_1^{-1}(P_1) \cap \text{ev}_2^{-1}(P_2) \cap \cdots \cap \text{ev}_n^{-1}(P_n) \subset \overline{M}_{0,n}(\mathbb{P}^r, \beta),$$

where $P_1, P_2, \dots, P_n \subset \mathbb{P}^r$ are general linear subspaces. By Bertini’s Theorem, the intersection (3) is a nonsingular substack. The holomorphic Euler characteristics of the Gromov–Witten subvarieties specialize to enumerative invariants for \mathbb{P}^r when the intersection (3) is 0 dimensional.

The K -theoretic application was our primary motivation for the study of the linear relations on the moduli space of maps appearing in Theorems 1 and 2.

0.3.4. A. Givental has informed us that our equation (1) has a natural interpretation in *symplectic field theory*.

1. PROOFS

1.1. **Proof of Theorem 1.** Consider the moduli space $\overline{M}_{0,n}(\mathbb{P}^r, \beta)$. The curve class is a multiple of the class of a line: $\beta = d[\textit{line}]$. If $d = 0$, then $n \geq 3$ by the definition of stability. Equation (1) is trivial in the $d = 0$ case. Equation (2) is easily verified for $\overline{M}_{0,3}(\mathbb{P}^r, 0)$. For $d = 0$ and $n > 3$, the second equation is obtained by pull-back from the 3-pointed case. We may therefore assume $d > 0$.

It is sufficient to prove equations (1-2) on the 2-pointed moduli space $\overline{M}_{0,2}(\mathbb{P}^r, \beta)$ with marking set $\{i, j\}$. The equations on the n -pointed space $\overline{M}_{0,n}(\mathbb{P}^r, \beta)$ are then obtained by pull-back.

Let $B \hookrightarrow \overline{M}_{0,2}(\mathbb{P}^r, \beta)$ be a nonsingular curve intersecting the boundary divisors transversely at their interior points. By the main results of [12], equations (1-2) may be established in the Picard group of $\overline{M}_{0,2}(\mathbb{P}^r, \beta)$ by proving the equalities hold after intersecting with all such curves B (actually much less is needed).

Consider the following fiber square:

$$\begin{array}{ccccc} S & \longrightarrow & C & \xrightarrow{f} & \mathbb{P}^r \\ \downarrow \pi & & \downarrow \pi_C & & \\ B & \longrightarrow & \overline{M}_{0,2}(\mathbb{P}^r, \beta) & & \end{array}$$

where C is the universal curve and S is a nonsingular surface. The morphism π has sections s_i and s_j induced from the marked points of

π_C . We find:

$$\begin{aligned}\langle B, \text{ev}_i^* L \rangle &= \langle s_i, f^* L \rangle, \\ \langle B, \text{ev}_j^* L \rangle &= \langle s_j, f^* L \rangle. \\ \langle B, \psi_i \rangle &= -\langle s_i, s_i \rangle, \\ \langle B, \psi_j \rangle &= -\langle s_j, s_j \rangle,\end{aligned}$$

where the right sides are all intersection products in S .

S is a \mathbb{P}^1 -bundle P over B blown-up over points where B meets the boundary divisors. More precisely, each reducible fiber of S is a union of two (-1) -curves. After a blow-down of one (-1) -curve in each reducible fiber, a \mathbb{P}^1 -bundle P is obtained. Let $P = \mathbb{P}(V)$ where $V \rightarrow B$ is a rank two bundle. Therefore,

$$0 \rightarrow \mathcal{P}ic(P) \rightarrow \mathcal{P}ic(S) \rightarrow \bigoplus_{b \in \text{Sing}} \mathbb{Z}E_b \rightarrow 0,$$

where $\text{Sing} \subset B$ is the set of points $b \in B$ where S_b is singular. E_b is the corresponding exceptional divisor of S . A line bundle H on S is uniquely determined by three sets of invariants $(J, d, \{e_b\})$, where H is (the pull-back of) an element of $\mathcal{P}ic(B)$, d is the fiber degree of the \mathbb{P}^1 -bundle P over B , and $\{e_b\}$ is the set of degrees on the exceptional divisors E_b :

$$H = \pi^*(J) \otimes \mathcal{O}_P(d) \left(- \sum_{b \in \text{Sing}} e_b E_b\right).$$

We may assume $L = \mathcal{O}_{\mathbb{P}^r}(1)$. Then, f^*L is a line bundle on S of type $(J, d, \{d_b\})$ where d_b is the degree of the map f on the exceptional divisors. Similarly, the sections s_i and s_j are divisors on S of type $(J_i, 1, \{\delta_b^i\})$ and $(J_j, 1, \{\delta_b^j\})$ respectively. Here, $\delta_b^i = 1$ or 0 if s_i does or does not intersect E_b (and similarly for δ_b^j).

By intersection calculations in S , we find:

$$\begin{aligned}\langle s_i, f^* L \rangle &= \text{deg}(J) + d \cdot \text{deg}(J_i) + d \cdot c_1(V) - \sum_{b \in \text{Sing}} d_b \delta_b^i, \\ \langle s_j, f^* L \rangle &= \text{deg}(J) + d \cdot \text{deg}(J_j) + d \cdot c_1(V) - \sum_{b \in \text{Sing}} d_b \delta_b^j, \\ -\langle s_i, s_i \rangle &= -2\text{deg}(J_i) - c_1(V) + \sum_{b \in \text{Sing}} \delta_b^i, \\ -\langle s_j, s_j \rangle &= -2\text{deg}(J_j) - c_1(V) + \sum_{b \in \text{Sing}} \delta_b^j,\end{aligned}$$

As $(\pi : S \rightarrow B, f : S \rightarrow \mathbb{P}^r, s_i, s_j)$ is a family of stable maps, the relation

$$(4) \quad \langle s_i, s_j \rangle = \deg(J_i) + \deg(J_j) + c_1(V) - \sum_{b \in \text{Sing}} \delta_b^i \delta_b^j = 0$$

is obtained.

Let $\text{Sing}(i) \subset \text{Sing}$ denote the points b such that s_i intersects E_b and s_j does not. Similarly, let $\text{Sing}(j) \subset \text{Sing}$ denote the subset where s_j intersects E_b and s_i does not.

The intersection of equations (1) and (2) with B are easily proven by the above intersection calculations:

$$\begin{aligned} \langle f^*L, s_i \rangle &= \langle f^*L, s_j \rangle - d \langle s_j, s_j \rangle - \sum_{b \in \text{Sing}(i)} d_b - \sum_{b \in \text{Sing}(j)} (d - d_b), \\ -\langle s_i, s_i \rangle - \langle s_j, s_j \rangle &= \sum_{b \in \text{Sing}(i)} 1 + \sum_{b \in \text{Sing}(j)} 1. \end{aligned}$$

The proof of Theorem 1 is complete.

1.2. Proof of the Theorem 2. The result is obtained by an easy induction on the number of marked points n and the degree β . For simplicity, we assume $\text{Pic}(X) = \mathbb{Z}H$, $H^*(X)$ is generated by elements of $\text{Pic}(X)$, and $R = H^*(X)$. The general argument is identical.

An n -point invariant with classes in $H^*(X)$ may be written as

$$(5) \quad \langle \psi^{l_1} H^{k_1}, \dots, \psi^{l_n} H^{k_n} \rangle_{0,n,\beta}.$$

Suppose that all $(n-1)$ -point invariants and n -point invariants with degree strictly less than β are known.

An application of the first equation of Corollary 1 in case $i = n, j = 1$ together with the splitting axiom of Gromov–Witten theory relates the invariant (5) to the invariant

$$\langle \psi^{l_1} H^{k_1+1}, \dots, \psi^{l_n} H^{k_n-1} \rangle_{0,n,\beta}.$$

modulo products of invariants with classes in $H^*(X)$ with either fewer points or lesser degree. The self-dual and vanishing conditions on R in part (i) of Theorem 2 are required to kill the diagonal splittings not consisting of classes of R — of course these conditions are trivial in case $R = H^*(X)$. After repeating the procedure, we may assume $k_n = 0$.

Similarly, applications of the second equation of Corollary 1 allow a reduction of l_n to 0 (modulo known invariants).

Once $l_n = 0$ and $k_n = 0$, then the n -point invariant may be reduced to $(n-1)$ -point invariants by the string equation. This completes the induction step.

The proof of Theorem 1 in quantum cohomology is complete. The argument for quantum K -theory is identical. One simply replaces the divisor classes (and their products) by the K -products of the corresponding line bundles. The splitting axiom of quantum K -theory is slightly more complicated (see [11]).

2. EXAMPLES

2.1. Gromov–Witten invariants of \mathbb{P}^2 . Kontsevich’s formula for the genus 0 Gromov–Witten invariants of \mathbb{P}^2 is derived here from Theorem 2.

The cohomology ring $H^*(\mathbb{P}^2) = \mathbb{Q}[H]/(H^3)$ has a linear basis

$$1, H, H^2.$$

By the fundamental class and divisor axioms, only invariants of the form

$$N_d = (H^2, \dots, H^2)_{0,3d-1,d}$$

need be computed. As there is a unique line through two distinct points in \mathbb{P}^2 , we see $N_1 = 1$.

We may reformulate equation (1) in the following form, which is better suited for computations without descendants.

Proposition 1. *Let $n \geq 3$. Let i, j, k be distinct markings. Then*

$$(6) \quad \text{ev}_i^*(H) = \text{ev}_j^*(H) + \sum_{d_1+d_2=d} (d_2 D_{ik,d_1|j,d_2} - d_1 D_{i,d_1|jk,d_2}).$$

Proof. When $n \geq 3$, $\psi_j = D_{j,ik}$. Then, equation (1) easily implies the Proposition. \square

Consider the following cohomology class in $H^*(\overline{M}_{0,3d-1}(\mathbb{P}^2, d))$:

$$(7) \quad \text{ev}_1^*(H^2) \cdots \text{ev}_{3d-2}^*(H^2) \text{ev}_{3d-1}^*(H).$$

Let $i = 3d - 1$ and $j = 1, k = 2$. After intersecting (6) with (7) and applying the splitting axioms, we obtain:

$$N_d = \sum_{d_1+d_2=d, d_i>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

2.2. Quantum K -invariants of \mathbb{P}^1 . We explain here the computation of quantum K -invariants of \mathbb{P}^1 in genus 0 using Theorem 1.

Define the K -theoretic J -function of \mathbb{P}^r to be

$$J_{\mathbb{P}^r}^K(Q, q) := \sum_{d=0}^{\infty} Q^d \text{ev}_{d*} \left(\frac{1}{1 - q\mathcal{L}} \right)$$

where $\text{ev}_{d*} : K(\overline{M}_{0,1}(\mathbb{P}^r, d)) \rightarrow K(\mathbb{P}^r)$ is the K -theoretic push-forward. The virtual structure sheaf in this case is just the ordinary structure sheaf as $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is a smooth stack. The following result is proven in [11].

Proposition 2.

$$J_{\mathbb{P}^r}^K(Q, q) = \sum_d \frac{Q^d}{\prod_{m=1}^d (1 - q^m H)^{r+1}},$$

where $H = \mathcal{O}(1)$ is the hyperplane bundle in \mathbb{P}^r .

We will specialize now to \mathbb{P}^1 . By the classical result

$$K^*(\mathbb{P}^1) = \frac{\mathbb{Q}[H]}{(H-1)^2},$$

$K^*(\mathbb{P}^1)$ is a two dimensional \mathbb{Q} -vector space with basis

$$e_0 = \mathcal{O}, \quad e_1 = H - \mathcal{O}.$$

The K -theoretic Poincaré metric is

$$(g_{ij}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

with inverse matrix

$$(g^{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

From Proposition 2, we obtain the 1-point quantum K -invariants,

$$\begin{aligned} \left(\frac{\gamma}{1 - q\mathcal{L}}\right)_{g,n,d}^K &= \sum_k q^k (\tau_k(\gamma))_{g,n,d}^K \\ \left(\frac{e_1}{1 - q\mathcal{L}}\right)_{0,1,1}^K &= 1 + 2q + 3q^2 + \dots, \\ \left(\frac{e_1}{1 - q\mathcal{L}}\right)_{0,1,2}^K &= 1 + 2q + 5q^2 + \dots. \end{aligned}$$

Let $\gamma_i \in K^*(\mathbb{P}^1)$. The K -theoretic fundamental class equation is:

$$(8) \quad (\gamma_1 \cdots, \gamma_{n-1}, e_0)_{0,n,d}^K = (\gamma_1 \cdots, \gamma_{n-1})_{0,n-1,d}^K.$$

This equation holds as the fiber of $\overline{M}_{0,n}(\mathbb{P}^1, d) \rightarrow \overline{M}_{0,n-1}(\mathbb{P}^1, d)$ is rational.

The string equation is obtained from the geometry of the morphism $\pi : \overline{M}_{0,n}(X, \beta) \rightarrow \overline{M}_{0,n-1}(X, \beta)$ which forgets the last point. The K -theoretic string equation [11] takes the following form:

$$(9) \quad R^0 \pi_* (\otimes_{i=1}^{n-1} \mathcal{L}_i^{\otimes k_i}) = \otimes_{i=1}^{n-1} \mathcal{L}^{\otimes k_i} \otimes \left(\mathcal{O} + \sum_{i=1}^{n-1} \sum_{k=1}^{k_i} \mathcal{L}_i^{-k} \right),$$

$$R^1 \pi_* (\otimes_{i=1}^{n-1} \mathcal{L}^{\otimes k_i}) = 0.$$

We will illustrate the computational scheme of n -point quantum K -invariants for \mathbb{P}^1 by calculating $(e_1, e_1)_{0,2,2}^K$. We start with by rewriting the invariant:

$$(e_1, e_1)_{0,2,2}^K = (e_1, H)_{0,2,2}^K - (e_1, e_0)_{0,2,2}^K.$$

By equations (1) and (8) we find:

$$(e_1, e_1)_{0,2,2}^K = (\mathcal{L}_1^2 e_1 H, e_0)_{0,2,2}^K - \chi(D_{1,d=1|2,d=1}, \mathcal{L}_1^2 \text{ev}_1^*(e_1 H)) - (e_1)_{0,1,2}^K.$$

We may use the relation $e_1 H = e_1$ and the 1-point evaluations. Together with the string equation and splitting axiom, we find:

$$(10) \quad (e_1, e_1)_{0,2,2}^K = 7 - (\mathcal{L}_1^2 e_1, e_a)_{0,2,1}^K g^{ab} (e_b, e_0)_{0,2,1}^K.$$

The following invariants are easy to compute by equation (1) and the string equation:

$$(e_1, e_1)_{0,2,1}^K = 1, \quad (\mathcal{L}_1^2 e_1, e_1)_{0,2,1}^K = 4.$$

Substitution in equation (10) yields:

$$(e_1, e_1)_{0,2,2}^K = 1.$$

A similar, but much longer, computation shows $(e_1, e_1, e_1)_{0,3,2}^K = 1$.

We conclude with two remarks about the quantum K -theory of \mathbb{P}^1 and the rationality of the Gromov–Witten subvarieties of $\overline{M}_{0,n}(\mathbb{P}^1, d)$.

(i) By Proposition 2 and the fundamental class equation, we see:

$$(e_0, e_0, \dots, e_0)_{0,n,d}^K = 1.$$

This result may also be deduced from the rationality of the moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ proven in [9].

(ii) By the exact sequence,

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathcal{O}_p \rightarrow 0,$$

where p is a point in \mathbb{P}^1 , we find $e_1 = [\mathcal{O}_p]$. Hence,

$$(e_1, \dots, e_1)_{0,n,d}^K = \chi(\cap_{i=1}^n \text{ev}_i^{-1}(p_i))$$

where $\text{ev}_i^{-1}(p_i)$ and their intersections are Gromov–Witten subvarieties (see § 0.3.3). For small pairs (n, d) , the space $\cap_{i=1}^n \text{ev}_{i,d}^{-1}(p_i)$ is also rational. For example, rationality certainly holds, from explicit geometric considerations, for the cases $(2, 2)$ and $(3, 2)$ discussed in the above computations.

It is interesting to ask which Gromov–Witten subvarieties of $\overline{M}_{0,n}(\mathbb{P}^1, d)$ are rational. For \mathbb{P}^2 , irrational Gromov–Witten subvarieties have been found in [13].¹

REFERENCES

- [1] A. Bertram and H. Kley, *New recursions for genus-zero Gromov–Witten invariants*, math.AG/0007082.
- [2] D. A. Cox, S. Katz and Y.-P. Lee, *Virtual Fundamental Classes of Zero Loci*, Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), 157–166, Contemp. Math., 276, Amer. Math. Soc., Providence, RI, 2001.
- [3] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Symp. Pure. Math. 62, Part 2, (1997) 45–96.
- [4] S. Fomin, S. Gelfand, and A. Postnikov, *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10** (1997), no. 3, 565–596.
- [5] A. Givental, *On the WDVV equation in quantum K-theory*, Michigan Math. J. **48** (2000), 295–304.
- [6] A. Givental, Y.-P. Lee, *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups*, Invent. Math. **151** (2003), no. 1, 193–219.
- [7] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518.
- [8] B. Kim, *Quantum cohomology of flag manifolds G/B and quantum Toda lattices*, Ann. of Math. (2) **149** (1999), no. 1, 129–148.
- [9] B. Kim and R. Pandharipande, *The connectedness of the moduli space of maps to homogeneous spaces*, Symplectic geometry and mirror symmetry (Seoul, 2000), 187–201, World Sci. Publishing, River Edge, NJ, 2001.
- [10] Y.-P. Lee, *Quantum Lefschetz hyperplane theorem*, Invent. Math. 145 (2001), no. 1, 121–149.
- [11] Y.-P. Lee, *Quantum K-theory I: foundation*, to appear in Duke Math. J. *Quantum K-theory II*, in preparation.
- [12] R. Pandharipande, *Intersections of \mathbf{Q} -divisors on Kontsevich’s moduli space $\overline{M}_{0,n}(\mathbf{P}^r, d)$ and enumerative geometry*. Trans. Amer. Math. Soc. **351** (1999), no. 4, 1481–1505
- [13] R. Pandharipande, *The canonical class of $\overline{M}_{0,n}(\mathbf{P}^r, d)$ and enumerative geometry*, Internat. Math. Res. Notices (1997), no. 4, 173–186.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY,
UT 84112-0090

E-mail address: yplee@math.utah.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ
08544-1000

E-mail address: rahulp@math.princeton.edu

¹Added in proof: Using Riemann–Roch theorem, we were able to show that there are at most finitely many rational Gromov–Witten loci for a fixed d .