Thompson’s group

Yael Algom Kfir

February 5, 2006


1 Definition

Definition: Let \( F \) be the set of functions \( f \) on the unit interval \([0, 1]\) such that:

1. \( f \) is piecewise linear and continuous.
2. \( f(0) = 0 \) and \( f(1) = 1 \)
3. If \( 0 < x_1 < x_2 < \ldots < x_m < 1 \) are the points of non-differentiability the \( x_i = \frac{k}{l} \) for some \( k, l \in \mathbb{N} \).
4. If \( x \neq x_i \forall i \) then \( f \) is differentiable at \( x \) and \( f'(x) = 2^j \) for some \( j \in \mathbb{Z} \).

It is straightforward to check that \( F \) forms a subgroup of \( \text{Homeo}[0, 1] \).

Consider:

\[
A(x) = \begin{cases} 
\frac{x}{2} & 0 \leq x \leq \frac{1}{2} \\
x - \frac{1}{4} & \frac{1}{2} < x < \frac{3}{4} \\
2x - 1 & \frac{3}{4} \leq x \leq 1 
\end{cases}
\]

\[
B(x) = \begin{cases} 
x & 0 \leq x \leq \frac{1}{2} \\
\frac{x}{2} + \frac{1}{4} & \frac{1}{2} < x < \frac{3}{4} \\
x - \frac{1}{8} & \frac{3}{4} < x < \frac{7}{8} \\
2x - 1 & \frac{7}{8} \leq x \leq 1 
\end{cases}
\]

\( A, B \in F \). We will show that \( A, B \) generate \( F \).

2 Tree diagrams

We can completely describe any \( f \in F \) by specifying \( x_1 < x_2 < \ldots < x_m \) and \( f(x_1) < f(x_2) < \ldots < f(x_m) \). For example: \( A : (0, \frac{1}{2}, \frac{3}{4}, 1) \rightarrow (0, \frac{1}{4}, \frac{1}{2}, 1) \) and
$B : (0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1) \rightarrow (0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1)$. We can code this information using binary tree diagrams $(D, R)$ where $D$ is the domain tree and $R$ is the range tree of the diagram. For example, for $A$ we have:

![Diagram of A](image)

Omitting the labelling, the tree diagram for $B$ is:

![Diagram of B](image)

Observe that $f \in \mathcal{F}$ corresponds to infinitely many pairs of dihedral sequences. Indeed, if $f : (0, y_1, \ldots, y_n, 1) \rightarrow (0, f(y_1), \ldots, f(y_n), 1)$ fully determines $f$ and $y_i < a < y_{i+1}$ then $f : (\ldots y_i, a, y_{i+1} \ldots) \rightarrow (\ldots f(y_i), f(a), f(y_{i+1}) \ldots)$ also fully determines the same $f$. The tree diagram corresponding to the latter sequences will differ from the first tree diagram $(D, R)$ by attaching a wedge to the $i^{th}$ leaf of the $D$ and one to the $i^{th}$ leaf of $R$.

A tree diagram $(D, R)$ is reduced if the following condition holds: whenever the $i^{th}$ and $i + 1^{st}$ leaves in $D$ are brothers the $i^{th}$ and $i + 1^{st}$ leaves in $R$ are not.

### 3 Composition of tree diagrams

We can compose two tree diagrams $(D_1, R_1)$ and $(D_2, R_2)$ whenever $D_2 = R_1$ to obtain $(D_1, R_2)$. But what do we do if $R_1 \neq D_2$? What if we want to compose the tree diagram corresponding to $A$ with the one corresponding to $B$? We can always find tree diagrams, $(D_1, R_1)$ for $A$ and $(D_2, R_2)$ such that $D_2 = R_1$. But these diagrams will not in general be reduced. The procedure is as follows: Let
(D_A, R_A) (resp. (D_B, R_B)) be the reduced tree diagram for A (resp. B). Now, add subtrees of \(D_B\) to both \(D_A\) and \(R_A\) and add subtrees of \(R_A\) to \(D_B\) and \(R_B\) until we get \(R_1 = D_2\).

![Tree Diagram](image)

Computing \(A^{-1}BA\) and \(A^{-2}BA^2\) we get:

![Tree Diagram](image)

We can now guess what the tree diagram for \(X_n := A^{-n}BA^n\) will look like. For simplicity of notation define \(X_0 := A\) and \(X_1 := B\). We will also denote the domain tree of \(X_n\) by \(T_n\).

### 4 Normal Forms

**Definition:** Suppose T is a binary tree with \(n\) leaves. Its \(i^{th}\) exponent is the length of the longest path of left edges that doesn’t touch the right side of the tree.

**Lemma:** If \((D, T_n)\) is a reduced tree diagram. And \(a_1, a_2, \ldots, a_n\) are the exponents of \(D\) then \((D, T_n)\) corresponds to \(X_n^{-a_n} \cdots X_1^{-a_1} X_0^{-a_0} \in \mathcal{F}\).

The lemma is proved by induction but we can follow the argument by working out the example below:

![Tree Diagram](image)

**Definition:** Every non-trivial element of \(\mathcal{F}\) can be expressed in a unique normal
form:
\[ X_0^{b_0} X_1^{b_1} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_1^{-a_1} X_0^{-a_0} \quad (1) \]

where \( n, a_0, \ldots, a_n, b_n, \ldots, b_n \) are nonnegative integers such that: exactly one of \( a_n, b_n \) is nonzero, and if \( a_k, b_k > 0 \) then \( a_{k+1} > 0 \) or \( b_{k+1} > 0 \). Furthermore, every such normal form function is non-trivial.

Indeed, if \((D, R)\) is the reduced tree diagram corresponding to \( f \) and \( a_1, \ldots, a_n, b_1, \ldots, b_n \) are the exponents of \( D, R \) respectively then \( f \)'s normal form is identical to the one in expression (1). The restrictions on the \( a_i \)s and \( b_i \)s are equivalent to \((D, R)\) being reduced.

**Corollary:** \( F \) is generated by \( A, B \).

**Corollary:** There is a linear-time solution to the word problem in \( F \).

5 Some properties of \( F \)

1. \( F \) is torsion free of type \( FP_\infty \), i.e. \( F \) has no finite projective resolution but has an infinite resolution \( P \) such that \( P_n \) is finitely generated for every \( n \).

2. \( F \) acts freely properly discontinuously by isometries on a CAT(0) cube complex

3. The group \( T \) is the set of orientation preserving piecewise linear homeomorphisms of the unit circle, which are non-differentiable on finitely many diadic rationals and whose derivatives are powers of 2. This is an infinite simple group, finitely presented, and of type \( FP_\infty \).

6 Amenability

**Definition:** A group \( G \) is called amenable if there is a function \( \mu : P(G) \to [0, 1] \) such that:

1. \( \mu(gA) = \mu(A) \) for all \( g \in G \) and subsets \( A \) of \( G \)

2. \( \mu(G) = 1 \)

3. \( \mu(A \cup B) = \mu(A) + \mu(B) \) if \( A \) and \( B \) are disjoint subsets of \( G \).
The class of amenable groups is denoted $AG$. Clearly, finite groups are amenable.

In 1929, Von Neumann made the connection between Banach-Tarski paradoxes and non-amenability of isometry groups. He proved that all abelian groups are amenable and that the class of amenable groups is closed under quotients, subgroups, group extensions and direct unions. Define $EG$ to be the smallest class of groups containing all finite groups and all abelian groups which is closed under the operations described above. $EG$ is called the class of elementary amenable groups.

**Lemma**: $F_2 = \langle a, b \rangle$ is not amenable.

**Proof**: Suppose $\mu$ is a measure on $G$. Since $G$ is infinite and $\mu(G) = 1$ then $\mu\{a\} = 0$ for every $a \in G$. In particular $\mu\{1\} = 0$. For each $g \in \{a, b, a^{-1}, b^{-1}\}$ define $A_g := \{h \in G | h \text{ has a reduced representative beginning with } g\}$. $\mu(A_a) = \mu(a^{-1}A_a) = \mu(\{1\} \cup A_a \cup A_b \cup A_{b^{-1}}) = \mu(\{1\}) + \mu(\{1\}) + \mu(A_a) = \mu(A_a) + \mu(A_b) + \mu(A_{b^{-1}}) = \mu(A_a) + \mu(A_b) + \mu(A_{b^{-1}})$. Hence $\mu(A_a) + \mu(A_b) + \mu(A_{b^{-1}}) = 0$ which implies $\mu(A_a) = \mu(A_b) = \mu(A_{b^{-1}}) = 0$. Similarly, $\mu(A_a) = \mu(A_{a^{-1}}) = 0$. But this contradicts:

$$G = \{1\} \cup A_a \cup A_{a^{-1}} \cup A_b \cup A_{b^{-1}}$$

Let $NF$ denote the class of groups that do not contain $F_2$ as a subgroup. We get:

$$EG \subseteq AG \subseteq NF$$

Olshanskii and later Gromov gave examples of inequality of the latter inclusion. Grigorchuk gave an example of inequality for the first inclusion. All examples are not finitely generated. $\mathcal{F} \in NF \setminus EG$. If $\mathcal{F} \in AG$ it is the first finitely presented example of inequality of the second inclusion and if $\mathcal{F} \notin AG$ it is the first finitely presented counterexample of the first inclusion.

**Open Question**: Is $\mathcal{F}$ amenable?