MATH 3210 - SUMMER 2008 - ASSIGNMENT #6

Limits of Functions

Let us begin by showing another example of how to prove by definition that $\lim_{x\to a} f(x) = L$.

Exercise. *Prove by definition that* $\lim_{x\to 2} \frac{x+5}{x-1} = 7$

Proof. <u>NTS</u>: $\forall \varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that for all x such that $0 < |x-2| < \delta(\varepsilon)$ we have $|\frac{x+5}{x-1} - 7| < \varepsilon$
<u>Calc</u>:

$$(1) \quad \left|\frac{x+5}{x-1} - 7\right| = \left|\frac{x+5-7(x-1)}{x-1}\right| = \left|\frac{-6x+12}{x-1}\right| = \left|\frac{6(2-x)}{x-1}\right| = \frac{6|2-x|}{|x-1|} = \frac{6|x-2|}{|x-1|}$$

Notice that the numerator is a quantity that we control - it is the quantity which is smaller than 6δ. To prove that the fraction is small we must show that denomenator not too small (so it won't cancell out the numerator). Notice that if $|x-2|$ is very small then x is very close to 2. Therefore x cannot be very close to 1 so the numerator isn't small. Therefore:
$|x-1| = |x-2+1| \geq^1 ||x-2| - |1|| =^2 1 - |x-2| \geq^3 1 - \frac{1}{2} = \frac{1}{2}$
Inequality 1: is one of the triangle inequalities: $|x-y| \geq ||x| - |y||$
Equality 2: If we choose $\delta < \frac{1}{2}$ then $|x-2| - 1 < -\frac{1}{2}$ so the absolute value of this number is negative the number.
Inequality 3: If $\delta < \frac{1}{2}$ then $|x-1| < \frac{1}{2}$ and $1 - |x-2| \geq \frac{1}{2}$.
All in all $|x-1| \geq \frac{1}{2}$.
Getting back to equation 1 we have:

$$\frac{6|x-2|}{|x-1|} \geq \frac{6|x-2|}{\frac{1}{2}} = 12|x-2| < 12\delta = \varepsilon$$

<u>Proof</u>: Given an $\varepsilon > 0$ take $\delta(\varepsilon) = \min\{\frac{1}{2}, \frac{\varepsilon}{12}\}$ then by the calculation above we get that $|\frac{x+5}{x-1}| < \varepsilon$ for x s.t. $0 < |x-a| < \delta$ $\square$

(1) Using the definition of convergence of a sequence, prove the following:

1) $\lim_{x\to 4} 2x - 1 = 7$

2) $\lim_{x\to -1} 3x + 5 = 2$

3) $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$

4) $\lim_{x\to 3} \frac{x-3}{x+5} = 0$

5) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = 0$

6) $\lim_{x\to 1} \frac{x^2+2x-3}{x^2-5x+4} = -\frac{4}{3}$

7) $\lim_{x\to a} \frac{x^4-a^4}{x^2+ax-2a^2} = \frac{4}{3}a^2$

8) $\lim_{x\to \infty} \frac{x-5}{x-3} = 1$

(2) Prove that $\lim_{x\to 3} f(x) \neq 0$ for $f(x) = 2x - 8$ by producing a sequence x_n such that:

(a) $x_n \neq 3$ for all $n \in \mathbb{N}$

(b) $\lim_{n\to\infty} x_n = 3$

such that $\lim_{n\to\infty} f(x_n) \neq 0$

(3) Consider the function $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \notin \mathbb{Q} \end{cases}$

Prove that:

(a) $\lim_{x\to 3} f(x)$ doesn't exist (Hint: find two sequences x_n, y_n both converging to 3 so that $f(x_n) = -x_n$ for all n and $f(y_n) = y_n$ for all n)

(b) $\lim_{x\to 0} f(x) = 0$

Notice that this is an example of a function which has a limit only at one point.

(4) Which of the following conditions is equivalent to the definition of $\lim_{x\to a} f(x) = L$? If it is equivalent prove it. If the condition is not equivalent find an example of a sequence which satisfies the condition but doesn't converge, or a sequence which converges but doesn't satisfy the condition.

(a) For all $\varepsilon > 0$ there is an $\delta(\varepsilon) > 0$ such that for all x such that: $x \in (a-\delta, a+\delta) \setminus \{a\}$ we have: $|f(x) - L| < \varepsilon$

(b) For all $\varepsilon > 0$ there is an $\delta(\varepsilon) > 0$ such that for all x such that: $0 < |x-a| < \delta$ we have: $|f(x) - L| < 2\varepsilon$

(c) There is an ε for which there is an δ such that for all x such that: $0 < |x-a| < \delta$ we have: $|f(x) - L| < \varepsilon$

(d) (bonus) For all $n \in \mathbb{N}$ there is a $\delta(n)$ such that for all x such that: $0 < |x - a| < \delta$ we have: $|f(x) - L| < \frac{1}{n}$