MATH 3210 - SUMMER 2008 - ASSIGNMENT #5

Theorems about converging sequences

(1) Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$ prove that $\lim_{n\to\infty} a_n \cdot b_n = a \cdot b$ (Hint: when you do the calculation step first add and subtract $a_n \cdot b$)

(2) Prove that if $\lim_{n\to\infty} a_n = 0$ and b_n is bounded then $\lim_{n\to\infty} a_n \cdot b_n = 0$

(3) If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ and $a < b$ then there is an $N \in \mathbb{N}$ such that for all $n > N$: $a_n < b_n$.

(4)
a. Find an example of sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} a_n = a$ and $\{b_n\}_{n=1}^{\infty}$ diverges but $a_n \cdot b_n$ converges.

b. Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences such that $\{a_n\}_{n=1}^{\infty}$ converges and $\{a_n + b_n\}_{n=1}^{\infty}$ converges. Does $\{b_n\}_{n=1}^{\infty}$ necessarily converge? If so prove it, otherwise give an example of such sequences where $\{b_n\}_{n=1}^{\infty}$ diverges.

(5) Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ (Hint: Use the inequality of the means for $\sqrt{n}, \sqrt{n}, 1, \ldots, 1$)

(6) Prove that the following sequence converges, and find its limit: $a_1 = 0$ and $a_{n+1} = \frac{a_n+1}{2}$.

(7) Prove that the following sequence converges, and find its limit: $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 2}$.

(8) Prove that $\lim_{n\to\infty} \frac{n^3}{4n^2+3n+15} = \infty$

(9) a. Prove that if $\lim_{n\to\infty} a_n = \infty$ then $\lim_{n\to\infty} \frac{1}{a_n} = 0$

b. Find a sequence a_n such that $\lim_{n\to\infty} a_n = 0$ but $\lim_{n\to\infty} \frac{1}{a_n} \neq 0$