MATH 3210 - SUMMER 2008 - ASSIGNMENT #10

The Mean Value theorem, Rolle's theorem, L'Hopital's theorem

(1) Prove that $p(x) = x^7 - 3x + 15$ has at most three real roots (hint: if it had four roots or more, how many times would the derivative have to vanish?)

(2) Prove that for $0 < x < y < \frac{\pi}{2}$: $\frac{1}{\cos^2(x)} < \frac{\tan(y)-\tan(x)}{y-x} < \frac{1}{\cos^2(y)}$. (hint: start by proving that in this domain the derivative of tan is monotonic)

(3) Prove that for all $x > 0$: $\frac{x}{1+x^2} < \arctan(x) < x$ (You can use the fact that $\arctan'(x) = \frac{1}{1+x^2}$).

(4) Prove that for all $x > 0$: $1 + x < e^x < 1 + xe^x$

(5) Let $f(x) = x + \frac{1}{x+e^x}$ then for all $0 \le a \le b$ prove that: $f(b) - f(a) \le b - a$

(6) Use L'Hopital's theorem to compute the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{2x}$

(b) $\lim\limits_{x\to\infty} \dfrac{x^3}{e^x}$

(c) $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{1 - \cos(x)}$

(d) $\lim\limits_{x\to 0} \dfrac{1}{\sin(x)} - \dfrac{1}{x}$

(7) Suppose f is differentiable at 0 and $f(0) = 0$. Define $h(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}$. Prove that h is differentiable at 0 and find its derivative (hint: use the fact that f has a linear approximation.)