ROLLE'S MEAN VALUE THEOREM

**Theorem** (Rolle). Let $f$ be a continuous function on $[a, b]$ which is differentiable on $(a, b)$. Suppose $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$

**Proof.** Since $f$ is continuous on the closed interval $[a, b]$, by Weierstrauss’ extreme value theorem, $f$ assumes a minimum $m$ and a maximum $M$ on this interval. That is, there are points $c, d \in [a, b]$ such that:

i) $f(x) \leq M$ for all $x \in [a, b]$, and $f(c) = M$.

ii) $f(x) \geq m$ for all $x \in [a, b]$, and $f(d) = m$.

If $m = M$ then $f(x) = m$ for all $x \in [a, b]$ and we’ve computed that $f'(x) = 0$ for all $x \in (a, b)$ so in this case we are done.

Now assume that $m < M$. $f(a) \neq m$ or $f(a) \neq M$ assume without loss of generality that $f(a) \neq M$. Since $f(b) = f(a)$ then $f(b) \neq M$. Therefore $c \neq a$ and $c \neq b$ hence $a < c < b$ and so $f$ is differentiable at $c$. $c$ is the source of the maximum so it is a local maximum (a maximum which is not at an endpoint). We appeal to Fermat’s theorem which says:

*If $f$ is defined on $(c-t_0, c+t_0)$ for some $t_0$, $f$ differentiable at $c$, and $f$ has a local maximum or minimum at $c$ then $f'(c) = 0$*

Appealing to this theorem, we have: $f'(c) = 0$. The case where $f(a) \neq m$ is similar. In this case it follows that $f'(d) = 0$. $\square$