Rational Homotopy of the Homotopy Fixed Point Sets of Lie Group Actions

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1. Background and Our main results
   - Background
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An action of a group $G$ on a space $M$ gives rise to two natural spaces:

- the fixed point set $M^G$,
- the homotopy fixed point set $M^{hG}$.

Recall that the homotopy fixed point set of a given $G$-action on $X$ is defined as the space $\text{map}^G(EG, X)$ of equivariant maps from the universal $G$-space $EG$ into $X$. In fact, the space

$$\text{map}^G(EG, X) = \text{map}(EG, M)^G,$$

the fixed point set of the group $G$-action on $\text{map}(EG, M)$, where the action is given by $(g \cdot f)(x) = gf(g^{-1}x)$, for $\forall g \in G$, $f \in \text{map}(EG, M)$, $x \in EG$. 
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Some known results on the generalized Sullivan conjecture

The trivial map $\eta : EG \to \{\ast\}$ induces a natural map

$$k : M^G = \text{map}^G(\ast, M) \longrightarrow M^{hG} = \text{map}^G(EG, M).$$

The generalized Sullivan conjecture

When $G$ is a finite $p$-group, and $M$ is a $G$-CW-complex, then the $p$-completion of the natural mapping $k$,

$$k_p^\wedge : (M^G)^{\wedge}_p \to (M^{hG})^{\wedge}_p$$

is a homotopy equivalence.

- Miller (Ann. of Math., 1984) achieved the first major breakthrough and is given credit for solving the Sullivan conjecture. This was published in 1984 and one version reads: $k : M^G \to M^{hG}$ is a weak homotopy equivalence, where $G$ is a finite group and trivially acts on a finite CW-complex $M$. Note in this case

$$M^{hG} = \text{map}(BG, M).$$
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$$M^{hG} = \text{map}(BG, M).$$
For a finite group $G$, the homotopy fixed point set has been considered by Goyo (Thesis, University of Toronto, 1989). In this case, the action of $G$ can be extended to an action on the rationalization $M_\mathbb{Q}$. Goyo proved that the graded rational homotopy group of the fixed point set, $\pi_\ast (M^{hG})_\mathbb{Q}$, is isomorphic to the invariant subgroup $(\pi_\ast (M_\mathbb{Q}))^G$ of the induced action of $G$ on $\pi_\ast (M)$. Moreover,

$$\text{cat}(M^{hG})_\mathbb{Q} \leq \text{cat}(M_\mathbb{Q}),$$

where $\text{cat}X$ denotes the Lusternik-Schnirelmann category of the space $X$.

Urtzi Buijs et al (Proc. Lond. Math. Soc., 2015) studied the rational homotopy of $M^{hG}$ for a compact Lie group with particular emphasis when $G$ is the circle. They also show that if $M$ is elliptic, that is, it has finite dimensional rational homotopy and cohomology, then each path component of $M^{hG}$ is also elliptic.
Our main results

Recently, we determined the rational homotopy types of the homotopy fixed point sets of Lie groups $S^3$ and $S^1 \times S^3$ actions on $S^n_Q$ and $\mathbb{C}P^n_Q$. Moreover, we obtained some useful results on the inclusion $k$.

Theorem (Hao, Liu, Sun; 2016)

Given an $S^3$-action on the rational $n$-sphere $S^n_Q$.

When $n$ is odd, $S^n_Q hS^3$ has the rational homotopy type of products of odd dimensional spheres, precisely, we have

$$S^n_Q hS^3 \simeq Q S^a \times S^{a+4} \times \cdots \times S^n,$$

where

$$a = \begin{cases} 
1, & n = 4k + 1, \\
3, & n = 4k + 3. 
\end{cases}$$
Our main results

Theorem (continued)

- When $n = 4k$, $S^n_Q^{hS^3}$ is either path connected, and of the rational homotopy type of $S^3 \times K_k$, where $K_k$ has the minimal Sullivan model $(\Lambda((x_s)_{1 \leq s \leq k}, (y_r)_{2 \leq r \leq 2k}), d)$ with $|x_s| = 4s$, $|y_r| = 4r - 1$, $dx_s = 0$ ($1 \leq s \leq k$), $dy_r = \sum_{s+t=r} x_s x_t$ ($2 \leq r \leq 2k$), or else, it has 2 components, each of them has the rational homotopy type of $S^{4k+3} \times S^{4k+7} \times \ldots \times S^{8k-1}$.

- When $n = 4k + 2$, $S^n_Q^{hS^3}$ is path connected, and of the rational homotopy type of $S^3 \times S^7 \times T_k$, where $T_k$ has the minimal Sullivan model $(\Lambda((x_s)_{1 \leq s \leq k}, (y_r)_{3 \leq r \leq 2k+1}), d)$ with $|x_s| = 4s + 2$, $|y_r| = 4r - 1$, $dx_s = 0$ ($1 \leq s \leq k$), $dy_r = \sum_{s+t=r-1} x_s x_t$ ($3 \leq r \leq 2k + 1$).
Our main results

Theorem (Hao, Liu, Sun; 2016)

Given an $S^3$-action in the rational complex projective space $\mathbb{C}P^n_{\mathbb{Q}}$.

When $n$ is odd, $\mathbb{C}P^n_{\mathbb{Q}}^{hS^3}$ is path connected, and has the rational homotopy type of one of the following spaces:

- $\mathbb{C}P^1 \times S^7 \times S^{11} \times \ldots \times S^{2n+1}$,
- $S^3 \times \mathbb{C}P^3 \times S^{11} \times \ldots \times S^{2n+1}$,
- $S^3 \times S^7 \times \mathbb{C}P^5 \times \ldots \times S^{2n+1}$,
- $\ldots$,
- $S^3 \times S^7 \times \ldots \times S^{2n-3} \times \mathbb{C}P^n$. 
Our main results

Theorem (continued)

Given an $S^3$-action in the rational complex projective space $\mathbb{C}P^n_Q$.

- When $n$ is even, $\mathbb{C}P^n_Q^{hS^3}$ is path connected, and has the rational homotopy type of one of the following spaces:

$$\ast \times S^5 \times S^9 \times \cdots \times S^{2n+1},$$

$$S^1 \times \mathbb{C}P^2 \times S^9 \times \cdots \times S^{2n+1},$$

$$S^1 \times S^5 \times \mathbb{C}P^4 \times \cdots \times S^{2n+1},$$

$$\cdots,$$

$$S^1 \times S^5 \times \cdots \times S^{2n-3} \times \mathbb{C}P^n.$$
Our main results

Xie and Liu considered the case that $G = S^1 \times S^3$ and showed the rational homotopy types of $S^1 \times S^3$ actions on $S^n_Q$ and $\mathbb{C}P^n_Q$.

**Theorem (Xie, Liu; 2018)**

Given an $S^1 \times S^3$-action on the rational $n$-sphere $S^n_Q$.

- When $n$ is odd, $S^n_Q^{nh} S^1 \times S^3$ is path connected and has the rational homotopy type of
  \[ S^1 \times S^3 \times \cdots \times S^n \times S^1 \times S^5 \times \cdots \times S^{n-4}, \]
  or
  \[ S^1 \times S^3 \times \cdots \times S^n \times S^3 \times S^7 \times \cdots \times S^{n-4}. \]
When $n$ is even, $S^{n+1}_Q \times S^3$ is either path connected and has the rational homotopy type of

$$S^1 \times SO(n+2)/U\left(\frac{n+2}{2}\right) \times |\Lambda((x_t)_{1 \leq t \leq k}, (y_l)_{1 \leq l \leq 2k}, d)|,$$

where $|x_t| = 4t$, $|y_l| = 4l - 1$; $dx_t = 0$, $dy_1 = 0$, $l > 1$,

$dy_l = \sum_{s+t=l} x_s x_t$, or

$$S^1 \times SO(n+2)/U\left(\frac{n+2}{2}\right) \times |\Lambda((x_t)_{1 \leq t \leq k}, (y_l)_{1 \leq l \leq 2k}, d)|,$$

where $dx_t = 0$, $dy_1 = y_2 = 0$, $l > 2$, $dy_l = \sum_{s+t=l-1} (x_s x_t)$, or else,

it has two components, each of them has the same rational homotopy type of $S^1 \times SO(n+2)/U\left(\frac{n+2}{2}\right) \times S^{4k+3} \times S^{4k+7} \times \ldots \times S^{8k-1}$, or $S^{n+1} \times S^{n+3} \times \ldots \times S^{2n-1} \times |\Lambda((x_t)_{1 \leq t \leq k}, (y_l)_{1 \leq l \leq 2k}, d)|$, where

$|x_t| = 4t$, $|y_l| = 4l - 1$; $dx_t = 0$, $dy_1 = 0$, $l > 1$, $dy_l = \sum_{s+t=l} x_s x_t$. 
Our main results

Theorem (X. Sang and X. Liu, 2018)

Given an $S^1 \times S^3$-action on the complex projective space $CP^n_{\mathbb{Q}}^{h}(S^1 \times S^3)$. When $n$ is odd, $CP^n_{\mathbb{Q}}^{h}(S^1 \times S^3)$ either path connected and has the rational homotopy type of one of the following spaces:

\[ S^1 \times S^3 \times \ldots \times S^{2n-1} \times \mathbb{C}P^1 \times S^7 \times \ldots \times S^{4k+3}, \]

\[ \ldots, \]

\[ S^1 \times S^3 \times \ldots \times S^{2n-1} \times S^3 \times \ldots \times S^{4k-4i-1} \times \mathbb{C}P^{2k+1-2i} \times \]

\[ S^{4k-4i+3} \times \ldots \times S^{4k+3}, \]

\[ \ldots, \]

\[ S^1 \times S^3 \times \ldots \times S^{2n-1} \times S^3 \times S^7 \times \ldots \times S^{4k-1} \times \mathbb{C}P^n, \]
or else, it has at most \( n + 1 \) components, each of them has the same rational homotopy type of one of the following spaces:

\[
\mathbb{C}P^0 \times S^3 \times S^5 \times \ldots \times S^{2n+1} \times S^3 \times S^7 \times \ldots S^{4k-1},
\]

\[
\ldots ,
\]

\[
S^1 \times S^3 \times \ldots \times S^{2i-3} \times \mathbb{C}P^{i-1} \times S^{2i+1} \times \ldots \times S^{2n+1} \times S^3 \times S^7 \times \ldots S^{4k-1},
\]

\[
\ldots ,
\]

\[
S^1 \times S^3 \times \ldots \times S^{2n-1} \times \mathbb{C}P^n \times S^3 \times S^7 \times \ldots S^{4k-1}.
\]
Our main results

Theorem (continued)

- When $n$ is even, $\mathbb{C}P^n S^1 \times S^3$ is either path connected and has the same rational homotopy type of one of the following spaces:

\[
S^1 \times S^3 \times \cdots \times S^{2n-1} \times \mathbb{C}P^0 \times S^5 \times S^9 \times \cdots \times S^{2n+1}, \\
S^1 \times S^3 \times \cdots \times S^{2n-1} \times S^1 \times \mathbb{C}P^2 \times S^9 \times \cdots \times S^{2n+1}, \\
\vdots, \\
S^1 \times S^3 \times \cdots \times S^{2n-1} \times S^1 \times S^5 \times \cdots \times S^{2n-3} \times \mathbb{C}P^n,
\]
Theorem (S. Xie and X. Liu, 2018)

or else, it has at most $n + 1$ components and each of them has the same rational homotopy type of the following spaces:

$$S^3 \times S^5 \times \ldots \times S^{2n+1} \times S^1 \times S^5 \times \ldots \times S^{2n-3},$$

$$\ldots,$$

$$S^1 \times S^3 \times \ldots \times S^{2i-3} \times \mathbb{C}P^{i-1} \times S^{2i+1} \times \ldots \times S^{2n+1} \times S^1 \times S^5 \times \ldots \times S^{2n-3},$$

$$S^1 \times S^3 \times \ldots \times S^{2n-1} \times \mathbb{C}P^n \times S^1 \times S^5 \times \ldots \times S^{2n-3}.$$
Recall that a space $M$ is (rationally) elliptic if both $H^*(M; \mathbb{Q})$ and $\pi_*(M) \otimes \mathbb{Q}$ are finite dimensional vector spaces over $\mathbb{Q}$.

**Theorem (Hao, Liu and Sun, 2016)**

For an $S^1$-space $M$ which is a nilpotent finite complex, the following conditions are equivalent:

- $M$ is elliptic.
- Each component of $M_{Q}^{hS^1}$ is elliptic.
- One of the components of $M_{Q}^{hS^1}$ is elliptic.

**Remark**

The theorem holds also for $G = S^3$. The proof is similar.
Our main results

**Theorem (Hao, Liu, Sun; 2016)**

For an $S^1$-complex $M$ which is simply connected with

$$\dim \pi_*(M) \otimes \mathbb{Q} < \infty.$$  

Then

$$k : M^S_1 \mathbb{Q} \hookrightarrow M^hS_1 \mathbb{Q}$$

is a rational homotopy equivalence if and only if $M$ is rational homotopy equivalent to a product of $CP^\infty$.  

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If $f$ is a continuous map between simply connected topological spaces then $H_\ast(f; \mathbb{Q})$ is an isomorphism if and only if $\pi_\ast(f) \otimes \mathbb{Q}$ is an isomorphism. In this case $f$ is a rational homotopy equivalence.

Two spaces $X$ and $Y$ have the same rational homotopy type if they are connected by a chain of rational homotopy equivalences in alternating directions, in which case we write

$$X \sim_{\mathbb{Q}} Y.$$ 

A simply connected space is rational if its homotopy groups (or, equivalently, its integral homology groups) are rational vector spaces.
For each simply connected space $X$ there is a relative CW complex $(X_Q, X)$, unique up to homotopy type rel $X$ such that $X_Q$ is a rational space and

$$X \to X_Q$$

is a rational homotopy equivalence. We call such an $X_Q$ a \textit{rationalization} of $X$. The rationalizations $X_Q$ of a simply connected space all have the same weak homotopy type and that the weak homotopy type of $X_Q$ depends only on the weak homotopy type of $X$. The weak homotopy type of $X_Q$ is the \textit{rational homotopy type} of $X$. If $X$ is a CW complex then so is $X_Q$.

If $g : X \to Y$ is any continuous map into a simply connected rational space $Y$ then $g$ extends (uniquely up to homotopy rel $X$) to a map $$g_Q : X_Q \to Y.$$
A Sullivan algebra is a commutative cochain algebra of the form $(\Lambda V, d)$, where

- $V = \{ V^p \}_{p \geq 1}$ and $\Lambda V$ denotes the free graded commutative algebra on $V$;
- $V = \bigcup_{i=0}^{\infty} V(i)$, where $V(0) \subset V(1) \subset \cdots$ is an increasing sequence of graded subspaces such that $d|_{V(0)} = 0$, $d : V(i) \to \Lambda V(i-1)$, $i \geq 1$. 

$\Lambda V$ denotes the free graded commutative algebra on $V$. $\bigcup_{i=0}^{\infty} V(i)$ denotes the union of all subspaces $V(i)$, where $V(0) \subset V(1) \subset \cdots$ forms an increasing sequence of graded subspaces. $d|_{V(0)} = 0$ means that the derivative $d$ acts trivially on the subspace $V(0)$. The derivative $d$ maps elements of $V(i)$ to the graded subspaces of $\Lambda V$ shifted down by one, i.e., $d : V(i) \to \Lambda V(i-1)$ for $i \geq 1$. 
A cochain algebra $\Lambda V$ that is not a Sullivan algebra

Example

Consider the cochain algebra $(A, d) = (\Lambda(v_1, v_2, v_3), d)$, $|v_i| = 1$, with $dv_1 = v_2 v_3$, $dv_2 = v_3 v_1$ and $dv_3 = v_1 v_2$. Here $(A, d)$ is not a Sullivan algebra. If it were, it would have to have cocycle of degree 1. The cocycles $1$ and $v_1 v_2 v_3$ represents a basis for $H(A)$, and so it has a minimal model

$$m : (\Lambda(w), 0) \to (\Lambda V, d),$$

where $|w| = 3$, $m(w) = v_1 v_2 v_3$. 

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A Sullivan model for a commutative cochain algebra \((A, d)\) is a quasi-isomorphism

\[ m : (\Lambda V, d) \xrightarrow{\sim} (A, d) \]

from a Sullivan algebra \((\Lambda V, d)\).

Sullivan (Publ. I. H. E. S., 1977) defined a functor \(A_{PL}(-)\) from topological spaces to commutative differential graded algebras over \(\mathbb{Q}\). The functor

\[ X \mapsto A_{PL}(X; \mathbb{Q}) \]

serves as the fundamental bridge which we use to transfer problems from topology to algebra.
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If $X$ is a path connected topological space then a Sullivan model for $A_{PL}(X)$,

$$m : (\Lambda V, d) \xrightarrow{\sim} A_{PL}(X),$$

is called a Sullivan model for $X$.

A Sullivan algebra (or model), $(\Lambda V, d)$ is called minimal if

$$\text{Im} d \subset \Lambda^+ V \cdot \Lambda^+ V.$$

A nilpotent space $X$ of finite type admits a Sullivan minimal model $(\Lambda V, d)$. 
If $X$ is a path connected topological space then a Sullivan model for $A_{PL}(X)$, 

$$m : (\Lambda V, d) \overset{\sim}{\longrightarrow} A_{PL}(X),$$

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Definition

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A nilpotent space $X$ of finite type admits a Sullivan minimal model $(\Lambda V, d)$. 
The minimal model of $X$, $(\Lambda V, d)$, contains algebraic versions of every invariant of the rational homotopy type of $X$. For instance, we have

$$H^*(X; \mathbb{Q}) \cong H^*(\Lambda V, d),$$

$$\pi_q(X) \otimes \mathbb{Q} \cong \text{Hom}_{\mathbb{Z}}(V^q, \mathbb{Q}).$$
Example (The Sullivan model \((\Lambda V, d)\) of spheres \(S^k\))

When \(k\) is odd, \((\Lambda V, d) = (\Lambda(e), 0), |e| = k\).

When \(k\) is even, \((\Lambda V, d) = (\Lambda(e, e'), de' = e^2), |e'| = 2k - 1\).

For compact matrix Lie groups \(SO(n), SU(n)\) and \(Q(n)\) defined by

\[
SO(n) = \{A|A^t = A^{-1} \text{ and } \det A = 1\} \subset M(n; \mathbb{R}),
\]

\[
SU(n) = \{A|\bar{A}^t = A^{-1} \text{ and } \det A = 1\} \subset M(n; \mathbb{C}),
\]

\[
Q(n) = \{A|\bar{A}^t = A^{-1}\} \subset M(n; \mathbb{H}),
\]

where \(\mathbb{R}, \mathbb{C},\) and \(\mathbb{H}\) are the reals, complex numbers and quaternions, we have the following
Sullivan model

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For compact matrix Lie groups \(SO(n)\), \(SU(n)\) and \(Q(n)\) defined by

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\]
\[
Q(n) = \{ A | \bar{A}^t = A^{-1} \} \subset M(n; \mathbb{H}),
\]

where \(\mathbb{R}\), \(\mathbb{C}\), and \(\mathbb{H}\) are the reals, complex numbers and quaternions, we have the following
Example (The Sullivan models of Matrix Lie groups)

\[ \text{SO}(2n + 1) : \Lambda(x_1, \cdots, x_n), \quad |x_i| = 4i - 1. \]
\[ \text{SO}(2n) : \Lambda(x_1, \cdots, x_{n-1}, x'_n), \quad |x_i| = 4i - 1, \quad |x'_n| = 2n - 1. \]
\[ \text{SU}(n) : \Lambda(x_2, \cdots, x_n), \quad |x_i| = 2i - 1. \]
\[ Q(n) : \Lambda(x_1, \cdots, x_n), \quad |x_i| = 4i - 1. \]
Sullivan model

Example (The Sullivan models of $BG$)

Let $G$ denote a compact connected Lie group, or more generally a path connected topological group of the homotopy type of a finite CW-complex. Then the minimal Sullivan model of $G$ is an exterior algebra $$(\Lambda P, 0)$$ where $P$ is an oddly graded finite dimensional space. Hence, the minimal Sullivan model of $BG$ is $$(\Lambda Q, 0)$$ with $Q^r = P^{r-1}$. [see page 217, RHT]
Suppose $m_X : (\Lambda V, d) \to A_{PL}(X)$ and $m_Y : (\Lambda W, d) \to A_{PL}(Y)$ are Sullivan models for path connected topological spaces $X$ and $Y$, and the rational homology of the one of these spaces has finite type. Then

$$m_X \cdot m_Y : (\Lambda V, d) \otimes (\Lambda W, d) \to A_{PL}(X \times Y)$$

is a Sullivan model for $X \times Y$. Observe that if $(\Lambda V, d)$ and $(\Lambda W, d)$ are minimal models then so is their tensor product.
Recall that a space is a nilpotent space if $X$ is a connected CW complex with $\pi_1(X)$ nilpotent and acting nilpotently on the higher homotopy groups of $X$.

A fibration $X \rightarrow E \rightarrow B$ of nilpotent spaces with $B$ simply connected corresponds to a Koszul-Sullivan extension (KS-extension). This is a sequence of DG algebras

$$(\Lambda W, \delta) \rightarrow (\Lambda W \otimes \Lambda V, D) \rightarrow (\Lambda V, d),$$

in which $(\Lambda W, \delta)$ and $(\Lambda V, d)$ are minimal models for $B$ and $X$, respectively. Furthermore, the DG algebra $(\Lambda W \otimes \Lambda V, D)$ is a model for $E$ but need not be minimal; the differential here satisfies $D(w) = \delta(w)$ for $w \in W$ while $D(v) - d(v) \in \Lambda^+ W \otimes \Lambda V$ for $v \in V$. 
As most of our work rely on the identification of the homotopy fixed point set of a given action with the space of sections of the associated Borel fibration, we make a quick overview of the result we use concerning the rational homotopy type of the space of sections of a fibration. To do so fix a nilpotent fibration,

\[ F \longrightarrow E \longrightarrow B, \]

that is, a fibration of path connected, nilpotent spaces in which the action of \( \pi_1(B) \) in the homotopy groups of the fibre is also nilpotent. By \( \text{sec} \, p \) we denoted the space of continuous sections of \( p \). If \( \sigma \) is such a section, \( \text{sec}_\sigma \, p \) denotes the path component of \( \text{sec} \, p \) containing \( \sigma \). In the pointed category these are \( \text{sec}^* \, p \) and \( \text{sec}_\sigma^* \, p \).
Recall that the space $\text{map}(X, Y)$ (respectively $\text{map}^*(X, Y)$) of continuous maps (respectively pointed continuous maps) is simply the space of sections (respectively pointed sections) of the trivial fibration

$$Y \longrightarrow X \times Y \longrightarrow X.$$  

If $B$ is a finite CW-complex and $F$, $E$ are CW-complexes of finite type, then Hilton, Mislin and Roitberg (Trans. Amer. Math. Soc., 1977) showed $\text{sec } p$ has the homotopy type of a CW-complex of finite type and each of its path component is nilpotent. In this case, it can be deduced that $\text{sec } p_Q$ has the weak homotopy type of $(\text{sec } p)_Q$,

$$\text{sec } p_Q \simeq_w (\text{sec } p)_Q.$$
However, *this is not, in general, a homotopy equivalence* as $\text{sec } p_{\mathbb{Q}}$ may fail to be of the homotopy type of a CW-complex. Smrekar (arXiv:0708.2838v1, 2007) showed that: If $\dim B = n$ and $\pi_{\geq n}(F)$ is torsion for some $n \geq 1$, then $\text{sec } p_{\mathbb{Q}}$ is of the homotopy type of a CW-complex and therefore,

$$\text{sec } p_{\mathbb{Q}} \simeq (\text{sec } p)_{\mathbb{Q}}.$$ 

Furthermore, whenever $B$ is finite dimensional and $\pi_{\dim B} F = 0$, $\text{sec } p$ is a rational space of the weak homotopy type of $\text{sec } p_{\mathbb{Q}}$. 
The space of sections

We now turn to the case in which $B$ is not a finite CW-complex. In this case, and for any $n > 1$, we denote by

$$F \rightarrow E_n \overset{p_n}{\rightarrow} B^{(n)}$$

the pullback fibration of the inclusion $B^{(n)} \hookrightarrow B$ of the $n$-skeleton over $p$,

\[
\begin{array}{ccc}
F & \rightarrow & F \\
\downarrow & & \downarrow \\
E_n & \rightarrow & E \\
\downarrow p_n & & \downarrow p \\
B^{(n)} & \hookrightarrow & B.
\end{array}
\]
The space of sections

Observe that, if $\sigma \in \sec p$, the map

$$B^{(n)} \hookrightarrow B \xrightarrow{\sigma} E$$

induces a section $\sigma_n \in \sec p_n$. This process defines fibrations,

$$\cdots \rightarrow \sec p_n \rightarrow \sec p_{n-1} \rightarrow \cdots \rightarrow \sec p_2 \rightarrow \sec p_1.$$ 

The same applies to each path component and to the pointed case. It is immediate that,

$$\sec p = \lim_{\leftarrow} \sec p_n,$$

$$\sec_\sigma p = \lim_{\leftarrow} \sec_{\sigma_n} p_n,$$

$$\sec^* p = \lim_{\leftarrow} \sec^*_{\sigma_n} p_n.$$
The space of sections

If $F$ is a rational space with $\pi_{\geq N} F = 0$, then $\sec p$ is a rational nilpotent space of the homotopy type of a CW-complex. Moreover,

$$\sec p \simeq \sec p_N \simeq (\sec p_N)_Q \simeq \sec p_{NQ}.$$  

From now on, and unless explicitly stated otherwise, $G$ will denote a compact connected Lie group, or more generally a path connected topological group of the homotopy type of a finite CW-complex. In the same way, by a topological $G$-space $M$ we mean a nilpotent $G$-space of the homotopy type of a CW-complex of finite type.
For such a $G$-space on $M$, we have the corresponding Borel fibration

$$\xi : M \to M_{hG} \to BG.$$ 

Recall that $M_{hG} = M \times EG/G$ and $\xi[x, m] = p(x)$ with $p : EG \to BG$ the Milnor’s universal $G$-bundle. It is a classical fact that the homotopy fixed point set

$$M^{hG} = \text{map}^G(EG, M)$$

is homotopy equivalent to the space of sections of this fibration $\text{sec}(\xi)$. 
We first consider the base of

\[ F \to E \overset{p}{\to} B, \]

to be a finite complex. Fix a model of this fibration

\[ (A, d) \to (A \otimes \Lambda V, D) \to (\Lambda V, d), \tag{1} \]

in which \( A \) is finite dimensional and denote by

\[ (A^\#, \; \delta) = (\text{Hom}(A, \mathbb{Q}), \; d^\#) \]

the differential graded coalgebra dual of \( A \) with the grading

\[ A^\# = B_n^\# = \text{Hom}(A^n, \mathbb{Q}). \]

Consider the free commutative \( \mathbb{Z} \)-graded algebra

\[ \Lambda(A \otimes \Lambda V \otimes A^\#) \]

endowed with the differential induced by \( D \) and \( \delta \).
We first consider the base of

\[ F \to E \xrightarrow{p} B, \]

to be a finite complex. Fix a model of this fibration

\[(A, \ d) \to (A \otimes \Lambda V, \ D) \to (\Lambda V, \ d), \]

in which \( A \) is finite dimensional and denote by

\[(A^\#, \ \delta) = (\text{Hom}(A, \ Q), \ d^\#) \]

the differential graded coalgebra dual of \( A \) with the grading

\[ A^\#_{-n} = B^n_\# = \text{Hom}(A^n, \ Q). \]

Consider the free commutative \( \mathbb{Z} \)-graded algebra

\[ \Lambda(A \otimes \Lambda V \otimes A^\#) \]

endowed with the differential induced by \( D \) and \( \delta \).
Let $J$ be the differential ideal generated by $1 \otimes 1 - 1$ and the elements

$$v_1 v_2 \otimes \beta - \sum_j (-1)^{|v_2||\beta'_j|}(v_1 \otimes \beta'_j)(v_2 \otimes \beta''_j),$$

$$b \otimes \alpha \otimes \beta - \sum_j (-1)^{|\beta_j|(|\alpha|+1)} \beta'_j(b)\alpha \otimes \beta''_j,$$

with $v_1, v_2 \in V$, $\alpha \in \Lambda V$, $b \in A$, $\beta \in A^\#$, $\Delta \beta = \sum_j \beta'_j \otimes \beta''_j$. The map induced by the inclusion $V \hookrightarrow A \otimes \Lambda V$,

$$\rho : \Lambda(V \otimes A^\#) \xrightarrow{\sim} \Lambda(A \otimes \Lambda V \otimes A^\#)/J$$

is an isomorphism of graded algebras.
Let $J$ be the differential ideal generated by $1 \otimes 1 - 1$ and the elements

$$ v_1 v_2 \otimes \beta - \sum_j (-1)^{|v_2||\beta_j'|} (v_1 \otimes \beta_j')(v_2 \otimes \beta_j''), $$

$$ b \otimes \alpha \otimes \beta - \sum_j (-1)^{|\beta_j|(|\alpha|+1)} \beta_j'(b) \alpha \otimes \beta_j'' ,$$

with $v_1, v_2 \in V$, $\alpha \in \Lambda V$, $b \in A$, $\beta \in A^\#$, $\Delta \beta = \sum_j \beta_j' \otimes \beta_j''$. The map induced by the inclusion $V \hookrightarrow A \otimes \Lambda V$,

$$ \rho : \Lambda(V \otimes A^\#) \xrightarrow{\sim} \Lambda(A \otimes \Lambda V \otimes A^\#)/J $$

is an isomorphism of graded algebras.
Sullivan models of the space of sections

It is easy to see that

$$\tilde{d} = \rho^{-1} d\rho$$

defines a differential in $\Lambda(V \otimes A^\#)$, which makes $\rho$ an isomorphism of differential graded algebras. Let $I$ denote the differential ideal generated by $(V \otimes A^\#)^{\leq 0}$ and $\tilde{d}(V \otimes A^\#)^0$. Then the quotient $\Lambda(V \otimes A^\#)/I$ is a model of $\text{sec}_\sigma(p)$.

As for the sections spaces $\text{sec} p$, we have the following


$(\Lambda(V \otimes A^\#), \tilde{d})$ is a model of $\text{sec} p$. 
We now make no finiteness assumptions on the base of \( p \). Recall that, if \( A \) is a CDGA model of the space \( X \), then the inclusion \( X^{(n)} \hookrightarrow X \) of the \( n \)-skeleton is modeled by the projection

\[
A \rightarrow A_n = A/I,
\]

where \( I = A^{\geq n+1} \oplus C^n \) and \( C^n \) is a complement of the cocycles of \( A^n \).

Then if \((A, d) \rightarrow (A \otimes \Lambda V, D) \rightarrow (\Lambda V, d)\) denotes again a model of \( p \), then sequence

\[
A_n \rightarrow (A_n \otimes \Lambda V, D') \rightarrow (\Lambda V, d),
\]

in which

\[
(A_n \otimes \Lambda V, D') = A_n \otimes_A (A \otimes \Lambda V, D),
\]

is a model of the pullback filtration

\[
F \rightarrow E_n \overset{p_n}{\rightarrow} B^{(n)}.
\]
Hence, at the sight of all of the above, we have

**Theorem (U. Buijs, ect, 2015)***

If $\pi \geq N(F) = 0$, then $(\Lambda(V \otimes A^\#_N), \tilde{d})$ is a model of sec $p$. 
$S^1 \times S^3$ actions on the rational n-sphere $S^n_Q$

Theorem

Given an $S^1 \times S^3$-action on the rational n-sphere $S^n_Q$,

- When $n$ is odd, $S^n_Q^{nh}S^1 \times S^3$ is path connected and
  $S^n_Q^{nh}S^1 \times S^3 \sim_Q \text{map}(\mathbb{C}P^{2k+1} \times \mathbb{H}P^k, S^n) \sim_Q S^1 \times S^3 \times \cdots \times S^n \times S^a \times S^5 \times \cdots \times S^{n-4}$. \\
a = \begin{cases} 
1, & n = 4k + 1, \\
3, & n = 4k + 3.
\end{cases}

- When $n$ is even, $M^n_Q^{h}S^1 \times S^3$ may be path connected and
  $S^n_Q^{nh}S^1 \times SO(n + 2)/U(n+2) \times |\Lambda((x_t)_{1 \leq t \leq k}, (y_l)_{1 \leq l \leq 2k}, d)|$, \\
where $| |$ is the spatial realization functor.
Outline of proof  For convenience, let $G = S^1 \times S^3$. For the $G$-space $S^n_Q$, we have the corresponding Borel fibration

$$
\xi : S^n_Q \to (S^n_Q)_{hS^1 \times S^3} \to \mathbb{C}P^\infty \times \mathbb{H}P^\infty,
$$

and

$$
S^{nhG}_Q \simeq \text{sec } \xi.
$$

The model of the corresponding Borel fibration above is

$$(A, 0) \to (A \otimes \Lambda V, D) \to (\Lambda V, d),$$

where $(\Lambda V, d) = (\Lambda(a), 0)$, $(A, 0) = (\Lambda(x, y), 0)$, $|a| = n$, $2|x| = |y| = 4$. 
Outline of proof  For convenience, let \( G = S^1 \times S^3 \). For the \( G \)-space \( S^n_Q \), we have the corresponding Borel fibration

\[
\xi : S^n_Q \rightarrow (S^n_Q)_{hS^1 \times S^3} \rightarrow \mathbb{C}P^\infty \times \mathbb{H}P^\infty,
\]

and

\[ S^{nhG}_Q \simeq \sec \xi. \]

The model of the corresponding Borel fibration above is

\[(A, 0) \rightarrow (A \otimes \Lambda V, D) \rightarrow (\Lambda V, d),\]

where \((\Lambda V, d) = (\Lambda(a), 0), (A, 0) = (\Lambda(x, y), 0), \ |a| = n, 2|x| = |y| = 4.\]
$S^1 \times S^3$ actions on the rational n-sphere $S^n_Q$

(1) $n$ is odd. In this case, $\pi_{\geq n+1}(S^n_Q) = 0$, and

$$S^n_{QhG} \cong \sec \xi \cong \sec \xi_{n+1} \cong \sec \xi_{n+1Q}.$$ 

Consider the model of $\xi_{n+1}$,

$$(A^{n+1}, 0) \rightarrow (A^{n+1} \otimes \Lambda a, D) \rightarrow (\Lambda a, 0).$$
(1) $n$ is odd. In this case, $\pi_{\geq n+1}(S^n) = 0$, and

$$S^n_{\mathbb{Q}} \simeq \sec \xi \simeq \sec \xi_{n+1} \simeq \sec \xi_{n+1\mathbb{Q}}.$$ 

Consider the model of $\xi_{n+1}$,

$$\xi_{n+1} \rightarrow (A^{n+1} \otimes \Lambda a, D) \rightarrow (\Lambda a, 0).$$
(1.1) \( n = 4k + 1 \). In this case, \(|a|\) is odd. Note that \( \xi_{n+1} \) is modeled by the trivial algebraic fibration

\[
(A^{n+1}, 0) \to (A^{n+1} \otimes \Lambda a, 0) \to (\Lambda a, 0),
\]

and

\[
\sec \xi_{n+1} \cong \text{map}(\mathbb{CP}^{2k+1} \times \mathbb{HP}^k, S^n)_{\mathbb{Q}}.
\]

A direct computation shows that the minimal model for the mapping space is

\[
(\Lambda(a \otimes 1, a \otimes x^\#, \cdots, a \otimes (x^{2k})^\#, a \otimes y^\#, \cdots, a \otimes (y^k)^\#), 0).
\]

We observe that this is also the minimal model for the product

\[
S^1 \times S^3 \times \cdots \times S^n \times S^1 \times S^5 \times \cdots \times S^{n-4}.
\]
\((1.2) \ n = 4k + 3. \) In a similar way,

\[
\sec \xi_{n+1} \cong \text{map}(\mathbb{C}P^{2k+1} \times \mathbb{H}P^k, S^n)_Q
\]

and it has the same rational homotopy type with

\[
S^1 \times S^3 \times \ldots \times S^n \times S^3 \times S^7 \times \ldots \times S^{n-4}.
\]

(2) \( n \) is even. In this case,

\[
\pi_{\geq 2n}(S^n_Q) = 0,
\]

then

\[
S^n_{QhG} \cong \sec \xi \cong \sec \xi_{2n}.
\]
(1.2) $n = 4k + 3$. In a similar way,

$$\text{sec} \xi_{n+1} \simeq \text{map}(\mathbb{C}P^{2k+1} \times \mathbb{H}P^k, S^n)_{\mathbb{Q}}$$

and it has the same rational homotopy type with

$$S^1 \times S^3 \times \cdots \times S^n \times S^3 \times S^7 \times \cdots \times S^{n-4}.$$

(2) $n$ is even. In this case,

$$\pi_{\geq 2n}(S^n_{\mathbb{Q}}) = 0,$$

then

$$S^n_{\mathbb{Q}hG} \simeq \text{sec} \xi \simeq \text{sec} \xi_{2n}.$$
(2.1) If $n = 4k$, the model of the corresponding Borel fibration is

$$(A^{2n}, 0) \to (A^{2n} \otimes \Lambda(e, e'), D) \to (\Lambda(e, e'), de' = e),$$

where $A^{2n} = \Lambda(x, y)/(x^{4k+1}, y^{2k+1})$, $|x| = 2$, $|y| = 4$, $De = 0$, $De' = e^2 + \lambda y^{\frac{n}{4}} e + \mu x^{\frac{n}{2}} e + \sum_{i,j,k} \tau_{i,j,k} x^i y^j e$.

We only consider the case that $\lambda = \mu = 0$, and the other cases are omitted. It is easy to show that in this case

$$S_{Q}^{nhS^1 \times S^3} \cong \text{map}(\mathbb{C}P^{2k+1} \times \mathbb{H}P^{k}, S^n).$$

The model for $S_{Q}^{nhS^1 \times S^3}$ is

$$(\Lambda(e \otimes 1, e \otimes x^\#), \cdots, e \otimes (x^{2k-1})^\#, e \otimes y^\#, \cdots, e \otimes (y^{k-1})^\#, e' \otimes 1, e' \otimes x^\#), \cdots, e' \otimes (x^{4k-1})^\#, e' \otimes y^\#, \cdots, e' \otimes (y^{2k})^\#), \tilde{d}).$$
(2.1) If \( n = 4k \), the model of the corresponding Borel fibration is

\[
(A^{2n}, 0) \to (A^{2n} \otimes \Lambda(e, e'), D) \to (\Lambda(e, e'), de' = e),
\]

where \( A^{2n} = \Lambda(x, y)/(x^{4k+1}, y^{2k+1}) \), \(|x| = 2\), \(|y| = 4\), \( De = 0\), \( De' = e^2 + \lambda y^n e + \mu x^n e + \sum_{i,j,k} \tau_i x^j y^k e \).

We only consider the case that \( \lambda = \mu = 0 \), and the other cases are omitted. It is easy to show that in this case

\[
S_Q^{nhS^1 \times S^3} \sim \text{map}(\mathbb{C}P^{2k+1} \times \mathbb{H}P^k, S^n).
\]

The model for \( S_Q^{nhS^1 \times S^3} \) is

\[
(\Lambda(e \otimes 1, e \otimes x^\#), \cdots, e \otimes (x^{2k-1})^\#, e \otimes y^\#, \cdots, e \otimes (y^{k-1})^\#, e' \otimes 1, e' \otimes x^\#, \cdots, e' \otimes (x^{4k-1})^\#, e' \otimes y^\#, \cdots, e' \otimes (y^{2k})^\#), \tilde{d}).
\]
A direct computation shows the model for \( \sec \xi_{2n} \) is
\[
\Lambda((\omega_s)_{1 \leq s \leq 2k}, (\mu_r)_{1 \leq r \leq 4k}, (x_t)_{1 \leq t \leq k}, (y_l)_{1 \leq l \leq 2k}, d),
\]
where
\[
|\omega_s| = 2s, \quad |\mu_r| = 2r - 1, \quad |x_t| = 4t, \quad |y_l| = 4l - 1; \quad d\omega_s = 0, \quad d\mu_1 = 0, \quad d\mu_r = \sum_{s+t=r} \omega_s \omega_t \quad (r > 1), \quad dx_t = 0, \quad dy_1 = 0, \quad dy_l = \sum_{s+t=l} x_s x_t \quad (l > 1).
\]
We know this is the model for
\[
S^1 \times SO(n + 2)/U(n + 2) \times | \otimes \Lambda((x_t)_{1 \leq t \leq k}, (y_l)_{1 \leq l \leq 2k}, d)|,
\]
where \(| |\) is the spatial realization functor.

(2.2) The proof of the case that \( n = 4k + 2 \) is similar, and omitted.
From now on, and unless explicitly stated otherwise, $G$ will denote a compact connected Lie group, or more generally a path connected topological group of the homotopy type of a finite CW-complex. In the same way, by a topological $G$-space we mean a nilpotent $G$-space of a CW-complex of finite type.

Even though the action of $G$ on such a space $X$ induces an action in the rationalization $X_\mathbb{Q}$, the homotopy fixed point set of the resulting action $(X_\mathbb{Q})^{hG}$ may fail to be nilpotent ([Lannes, Lecture Note Series, 1987]). Besides, $(X_\mathbb{Q})^{hG}$ may not have the homotopy type of a CW-complex.
We then start by setting a sufficiently general context in which the homotopy fixed point set of $G$-actions in rational nilpotent spaces has the homotopy type of a nilpotent CW-complex. Identifying $X^{hG}$ with the space $\text{sec} \xi$ of sections of the Borel fibration

$$X \rightarrow X_{hG} \xrightarrow{\xi} BG,$$

we see that, if $\pi_n X$ is torsion for a certain $n > 1$, in particular if $X$ is elliptic, then $(X_{\mathbb{Q}})^{hG}$ is a rational nilpotent complex of the homotopy type of a CW-complex.
Recall that

**Theorem (U. Buijs, ect. 2014)**

If $M$ is a Postnikov piece, that is $\pi_{>N}(M\mathcal{Q}) = 0$ for some $N$, then

- $M^{hG}$ has the homotopy type of a nilpotent CW-complex of finite type.
- $(M^{hG})_\mathcal{Q} \cong (M\mathcal{Q})^{hG}$. 
Remark

If the $G$-space $X$ is not a Postnikov piece, the homotopy fixed point set $X^hG$ may not be of the homotoy type of a CW-complex and thus, $(X^hG)_Q$ makes no sense from the classical point of view. However, if $\pi_{> N}X$ is torsion for some $N$, in particular if $X$ is rationally elliptic, $X_Q$ is a Postnikov piece and by (i) of the above theorem, the homotopy fixed point set of the rational action $(X_Q)^hG$ is a nilpotent space of the homotopy type of a CW-complex. Moreover, $(X_Q)^hG$ is also a rational space for which,

$$(X_Q)^hG \cong \text{sec } \eta \sim \text{sec } \eta_N \cong \text{sec } \eta_{NQ},$$

where $\eta$ is the Borel fibration $X_Q \to (X_Q)^hG \xrightarrow{\eta} BG$ of the associated rational $G$-action on $X_Q$. 

$S^1$ actions on elliptic spaces
From now on we will consider $G$-actions on rationalizations $X_{\mathbb{Q}}$ of elliptic spaces (which are not necessarily arising from actions in $X$). In view of the remark above, as there is no possible confusion, we will denote $(X_{\mathbb{Q}})^{hG}$ simply by $X_{\mathbb{Q}}^{hG}$.


Let $X$ be an elliptic space for which $X_{\mathbb{Q}}$ is a $G$-space. Then, each path component of the homotopy fixed point set $X_{\mathbb{Q}}^{hG}$ is also elliptic.
Example

For an $S^1$-action on $M = K(\mathbb{Z}, n) \times K(\mathbb{Z}, n+1)$, such that the model of it’s Borel fibration is

$$\eta_n : (\Lambda x, 0) \mapsto (\Lambda x \otimes \Lambda(z, y), D) \to (\Lambda(z, y), d),$$

where $|x| = 2, |z| = n, |y| = n+1, D(z) = 0, \text{ and } D(y) = xz$. For $n = 2k$, there is only one retraction $\sigma$: $\sigma(z) = \sigma(y) = 0$. Thus $\text{sec}(\eta_{2k})$ is path connected. A model of $\text{sec}(\eta_{2k})$ is

$$(\Lambda((z_i)_{1 \leq i \leq k}, (y_j)_{1 \leq j \leq k+1}), d),$$

where $|z_i| = 2i, |y_j| = 2j - 1$ and $d(y_i) = z_i$. Since the minimal model of $\text{sec}(\eta_{2k})$ is $(\Lambda y_{k+1}, 0)$, $\text{sec}(\eta_{2k}) \simeq^\mathbb{Q} S^{2k+1}$ is an elliptic space. However, $M$ is not an elliptic space.
$S^1$ actions on elliptic spaces

Theorem (Hao, Liu and Sun, 2016)

For an $S^1$-space $M$ which is a nilpotent finite complex, the following conditions are equivalent:

- $M$ is elliptic.
- Each component of $M_{Q}^{hS^1}$ is elliptic.
- One of the components of $M_{Q}^{hS^1}$ is elliptic.

In fact, the theorem above holds also for $G = S^3$. The proof is similar.
Outline of proof

Recall that a space $M$ is said to be elliptic if both $H^*(M, \mathbb{Q})$ and $\pi_*(M) \otimes \mathbb{Q}$ are finite dimensional vector spaces. Let $(\Lambda V, d)$ is the minimal Sullivan model for $M$. Then we have

$$\pi_q(M) \otimes \mathbb{Q} = \text{Hom}(V^q, \mathbb{Q}),$$

$$\pi_*(M^{hS^1}) \otimes \mathbb{Q} = \pi_*(\text{sec}_\sigma \xi) \otimes \mathbb{Q}.$$

For such an $S^1$-space $M$ we have the corresponding Borel fibration

$$\xi : M \rightarrow M^{hS^1} \xrightarrow{p} BS^1.$$

Recall that $M^{hS^1} = ES^1 \times_{S^1} M$ and $p[x, m] = \bar{p}(x)$ with $\bar{p} : ES^1 \rightarrow BS^1$ the universal $S^1$-bundle. Here, $ES^1 = S^\infty$ and $BS^1 = \mathbb{C}P^\infty$. 
Note that the model of the Borel fibration above is

$$(\Lambda x, 0) \to (\Lambda x \otimes \Lambda V, D) \to (\Lambda V, d)$$

for which a given section $\sigma$ is modeled by the projection

$$\pi : (\Lambda x \otimes \Lambda V, D) \to (\Lambda x, 0), \pi(V) = 0.$$ 

Moreover, if $A_n$ denotes the quotient algebra $\Lambda x/(x^{n+1})$, then a model for the

$$\xi_n : M \to M_{hG}(n) \xrightarrow{p_n} \mathbb{C}P^n$$

is given by $(A_n \otimes \Lambda V, d)$, and a model for $\Gamma^n$ is given by $(\Lambda (V \otimes A_n^\#)/I, D)$.
We now show a description of the rational homotopy Lie algebra of \( \sec_\sigma(p) \) which can be derives from this model [Urtzi. Buijs, ect. 2006]. Let \( A \) denotes \( \Lambda x \). Consider the graded vector space

\[
\mathcal{L}(A \otimes \Lambda V, A) = \{ A-\text{linear map } f : A \otimes \Lambda V \to A, \ f(1) = 0, \ f(\Lambda^{\geq 2} V) = 0 \}.
\]

Define a differential as usual \( \delta \) on \( \mathcal{L}(A \otimes \Lambda V, A) \) by

\[
\delta(f) = d \circ f - (-1)^n f \circ D.
\]

We restrict to positive elements by considering the subcomplex

\[
\mathcal{L}(A \otimes \Lambda V, A)_q = \begin{cases} 
\mathcal{L}(A \otimes \Lambda V, A)_q, & \text{for } q > 1, \\
\mathcal{Z} \mathcal{L}(A \otimes \Lambda V, A)_1, & \text{for } q = 1.
\end{cases}
\]
We will show there exists a graded Lie algebra structure on $s^{-1}H_\ast(\mathcal{L}(A \otimes \Lambda V, A), D)$ as follows.

Decompose the differential $D$ on $A \otimes \Lambda V$ in the form

$$D = D_1 + D_2 + \cdots,$$

and write $D_2(v) = \sum_i b_i v'_i v''_i$. We define a linear map $\{,\}$ of degree 1

$$\mathcal{L}(A \otimes \Lambda V, A) \otimes \mathcal{L}(A \otimes \Lambda V, A) \to \mathcal{L}(A \otimes \Lambda V, A)$$

by

$$\{\varphi, \psi\}(v) = \sum_i b_i((-1)^{||\varphi||+||\psi||+1 \sum_i b_i((-1)^{||v'_i||\psi|||\varphi(v'_i)\psi(v''_i)} + (-1)^{||v''_i||(|\psi|+||v'_i|)\varphi(v''_i)\psi(v'_i)})$$
We obtain a graded Lie algebra structure on $s^{-1}H_*(\mathcal{L}(A \otimes \Lambda V, A), D)$ by

$$[s^{-1}\varphi, s^{-1}\psi] = (-1)^{|\psi|} s^{-1}\{\varphi, \psi\},$$

with respect to which we have:

**Theorem (Urtzi. Buijs, ect. 2006)**

As graded lie algebras,

$$s^{-1}H_*(\mathcal{L}(A \otimes \Lambda V, A), D) \cong \pi_*(\Omega \sec_\sigma \xi) \otimes \mathbb{Q}$$

Recall that the Lie bracket $[,]$ in $\pi_*(\Omega \sec_\sigma \xi)$ is given by

$$[\alpha, \beta] = (-1)^{|\alpha|+1} \partial_*([\partial_*^{-1}\alpha, \partial_*^{-1}\beta]_W), \quad \alpha, \beta \in \pi_*(\Omega \sec_\sigma \xi),$$

where $\partial_* : \pi_*(\sec_\sigma \xi) \xrightarrow{\sim} \pi_{*-1}(\Omega \sec_\sigma \xi)$, the connecting homomorphism for the path space fibration, is an isomorphism.
In order to compare $\dim \pi_\ast (M) \otimes \mathbb{Q}$ with $\dim \pi_\ast (M^{hG}) \otimes \mathbb{Q}$, we only need to compare $\dim H^\ast (\mathcal{L}(A \otimes \wedge V, A), D)$ with $\dim \text{Hom}(V, \mathbb{Q})$. We can show that the vector space $V$ can be decomposed as a direct sum $W \oplus K \oplus S$, where

- $W \oplus K = \ker D_1$ and
- $K$ and $S$ have the same dimension, admitting bases $\{v_i\}_{i \in I}$, $\{s_i\}_{i \in I}$, for any $i \in I$, there exists $n_i \geq 1$ such that $D_1(s_i) = x^{n_i} v_i$ (for $S^1$-action and $S^3$-action this is right).
Applying this decomposition, we can show the following result:

**Theorem**

For a $S^k$-space, $k = 1, 3$, we have

$$\dim \pi_{\leq N+k}(\sec \sigma, \xi) \geq \frac{1}{2} \dim \pi_{\leq N}(M_Q), \ N \geq 1.$$ 

**Note.** The theorem couldn’t hold for $S^1 \times S^3$. 

$S^1$ actions on elliptic spaces
Then we can show

Theorem (Hao, Liu and Sun, 2016)

For an $S^1$-space $M$ which is a nilpotent finite complex, the following conditions are equivalent:

- $M$ is elliptic.
- Each component of $M_{\mathbb{Q}}^{hS^1}$ is elliptic.
- One of the components of $M_{\mathbb{Q}}^{hS^1}$ is elliptic.
The inclusion \( k : M^{S^1} \hookrightarrow M^{hS^1} \)

The rest of the paper is devoted to showing:

**Theorem**

*For an \( S^1 \)-complex \( M \) which is simply connected with*

\[
\dim \pi_\ast(M) \otimes \mathbb{Q} < \infty.
\]

*Then the inclusion*

\[
k : M^{S^1} \hookrightarrow M^{hS^1}
\]

*is a rational homotopy equivalence if and only if \( M \) is rational homotopy equivalent to a product of \( CP^\infty \).*
Let $M$ be an $S^1$-space and $M^G \neq \emptyset$. The equivariant map $M^{S^1} \hookrightarrow M$ induces a map of between the corresponding Borel fibrations,

\[ \begin{array}{ccc}
M^{S^1} & \longrightarrow & M \\
\downarrow & & \downarrow \\
\mathbb{C}P^\infty \times M^{S^1} & \longrightarrow & M_{hS^1} \\
\eta & & \xi \\
\downarrow & & \downarrow \\
\mathbb{C}P^\infty & & .
\end{array} \]
The inclusion $k : M^{S^1} \hookrightarrow M^{hS^1}$

Then, the fundamental inclusion from the fixed point set into the homotopy fixed point set is identified with

$$k : M^{S^1} \hookrightarrow \text{map}(\mathbb{C}P^\infty, M^{S^1}) \xrightarrow{\gamma_*} \text{Sec}(\xi) \simeq M^{hS^1},$$

where $M^{S^1} \hookrightarrow \text{map}(\mathbb{C}P^\infty, M^{S^1})$ and $\gamma_* : \text{sec}\eta \to \text{sec}\xi$ are induced by $BG \to \ast$ and $\gamma$ respectively.
The inclusion $k : M^{S^1} \hookrightarrow M^{hS^1}$

If there exists some $N$ such that $\pi_{\geq N}(M_Q) = 0$ and $\pi_{\geq N}(M^{S^1}_Q) = 0$, then the map $k$ is identified with the corresponding

$$M^{S^1} \hookrightarrow \text{map}((\mathbb{C}P^{\infty})^{(N)}, M^{S^1}) \rightarrow \text{Sec}(\xi_N) \cong M^{hS^1},$$

obtained by truncating in the diagram (2):

$$
\begin{array}{ccc}
M^{S^1} & \rightarrow & M \\
\downarrow & & \downarrow \\
F_N & \rightarrow & E_N
\end{array}
\gamma_N

\left(\begin{array}{c}
\eta_N \\
(\mathbb{C}P^{\infty})^{(N)}.
\end{array}\right)
\xi_N
$$
Now let

$$\begin{align*}
(A, 0) & \longrightarrow (A \otimes V, D) \longrightarrow (\Lambda V, d) \\
& \downarrow \psi \downarrow \phi \\
& (A, 0) \otimes (\Lambda Z, d) \longrightarrow (\Lambda Z, d)
\end{align*}$$

be a model of the above diagram, where \((A, 0) = (\Lambda x/(\Lambda x) >^N, 0)\), \((\Lambda V, d)\) and \((\Lambda Z, d)\) are minimal Sullivan models of \(M\) and \(M^{S^1}\), respectively.
Then we have the following

**Theorem (U. Buijs, ect. 2009)**

The composition

\[
(\Lambda(V \otimes A^\#), \tilde{d}) \xrightarrow{\phi} (\Lambda(Z \otimes A^\#), \tilde{d}) \xrightarrow{\gamma} (\Lambda Z, d)
\]

is a model of \( k : M^{S_1}_Q \hookrightarrow M^{hS_1}_Q \). The morphisms above are defined by

\[
\phi(v \otimes \alpha) = \rho^{-1}[\psi(v) \otimes \alpha], \quad v \otimes \alpha \in V \otimes A^#,
\]

\[
\gamma(z \otimes \alpha) = \begin{cases} 
  z & \alpha = 1, \\
  0 & \alpha \neq 1,
\end{cases} 
\quad z \otimes \alpha \in Z \otimes A^#.
\]
The inclusion $k : M^{S^1} \hookrightarrow M^{hS^1}$

Then we will give some information about $\psi$. First, let $(\Lambda x \otimes \Lambda V, D)$ be a model of the fibration $\xi$, we can decompose the differential $D$ in $A \otimes \Lambda V$ into

$$D = \sum_{i \leq 1} D_i, \quad D_i(V) \subset \Lambda x \otimes \Lambda^i V.$$ 

**Proposition (U. Buijs, ect, 2009)**

The vector space $V$ can be decomposed into a direct sum $W \oplus K \oplus S$, where

- $W \oplus K = \ker D_1$;
- $K$ and $S$ have the same dimension admitting bases $\{v_i\}_{i \in I}$, $\{s_i\}_{i \in I}$, and for any $i \in I$, there exists $n_i \geq 1$ such that $D_1(s_i) = x^{n_i} v_i$. 

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The inclusion $k : M^{S^1} \hookrightarrow M^{hS^1}$

From the lemma above, we can easily get the following

**Lemma**

- $\dim W = \dim Z$.
- There are $\{w_j\}_{j \in J}$, $\{z_j\}_{j \in J}$ which are homogenous basis of $W$ and $Z$ respectively, and non negative integers $\{m_j\}_{j \in J}$ such that
  
  $$\psi(w_j) = x^{m_j} z_j + \Gamma_j, \quad \Gamma_j \in R \otimes \Lambda^{\geq 2} Z, \quad j \in J,$$

  and

  $$\psi(s_i) \in R \otimes \Lambda^{\geq 2} Z, \quad \psi(v_i) \in R \otimes \Lambda^{\geq 2} Z, \quad s_i \in S, \quad v_i \in K, \quad i \in I.$$
The inclusion \( k : M^{S^1} \hookrightarrow M^{hS^1} \)

Then we make use of the model of \( k \) is

\[
\alpha : (\Lambda(V \otimes A^\#), \tilde{d}) \to (\Lambda(Z \otimes A^\#), \tilde{d}) \to (\Lambda Z, d)
\]

to obtain the following

**Theorem (Hao, Liu and Sun, 2016)**

*For an \( S^1 \)-complex \( M \) which is simply connected with*

\[
\dim \pi_\ast(M) \otimes \mathbb{Q} < \infty.
\]

*Then the inclusion*

\[
k : M^{S^1} \hookrightarrow M^{hS^1}
\]

*is a rational homotopy equivalence if and only if \( M \) is rational homotopy equivalent to a product of \( \mathbb{CP}^\infty \).*
Thank you!