Homework #3

1. The DCT states that for 3 vectors, x, y, z, satisfying

$$\begin{bmatrix} z_0 & z_{N-1} & \cdots & z_1 \\ z_1 & z_0 & \ddots & \vdots \\ z_2 & & \ddots & \vdots \\ \vdots & & & \\ z_{N-1} & \cdots & & z_0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} y \end{bmatrix} \quad \text{or} \quad Z x = y$$

their discrete fourier coefficients will satisfy

$$\begin{bmatrix} \hat{z}_0 & & & 0 \\ & \hat{z}_1 & & \\ & & \ddots & \\ 0 & & & \hat{z}_{N-1} \end{bmatrix} \begin{bmatrix} \hat{x} \end{bmatrix} = \begin{bmatrix} \hat{y} \end{bmatrix} \frac{1}{N} \quad \text{or} \quad \hat{Z}\hat{x} = \frac{1}{N}\hat{y}$$

Letting U be the DFT matrix with $U^{-1} = \frac{1}{N}U^*$ and $U\hat{x} = x$, $U\hat{y} = y$, $U\hat{z} = z$ it is simple to rearrange the theorem by the following steps:

$$Z x = y \qquad \text{the original statement}$$

$$Z(U\hat{x}) = (U\hat{y}) \qquad \text{substitute } x \to U\hat{x} \quad y \to U\hat{y}$$

$$= N U \tfrac{1}{N}\hat{y} \qquad \text{mult. and divide by } N$$

$$= N U \hat{Z}\hat{x} \qquad \text{substitute } \tfrac{1}{N}\hat{y} \to \hat{Z}\hat{x} \text{ (the theorized equivalence)}$$

$$\frac{1}{N}U^* Z U\hat{x} = N\underbrace{\left(\tfrac{1}{N}U^*U\right)}_{I}\hat{Z}\hat{x} \qquad \text{mult. both sides by } U^{-1} = \tfrac{1}{N}U^*$$

$$\Rightarrow \frac{1}{N}U^* Z U = N\hat{Z}$$

Now if this can be verified, the theorem will be ~~Verified~~.

To verify $\frac{1}{N} U^* Z U = N\hat{Z}$

multiply both sides by U, then

$$ZU = NU\hat{Z}$$

writing out the elements of these gives:

$$(NU\hat{Z})_{ij} = N\hat{Z}_j w^{ij} \qquad \text{since } U_{ij} = w^{ij},\ w = e^{2\pi i/N}$$

It's probably best to use an index other than i

$$= N\left(\frac{1}{N}\sum_{k=0}^{N-1} z_k w^{-jk}\right) w^{ij}$$

$$= \sum_{k=0}^{N-1} z_k w^{j(i-k)}$$

and

$$(ZU)_{ij} = \sum_{k=0}^{i} z_k w^{j(i-k)} + \sum_{k=i+1}^{N-1} z_k w^{j\overbrace{(N-1-k+i+1)}^{i-k+N}}$$

$$= \sum_{k=0}^{i} z_k w^{j(i-k)} + \sum_{k=i+1}^{N-1} z_k w^{j(i-k)} \underbrace{w^{jN}}_{1} \qquad \text{since } w^{jN} = w^0 = 1$$

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$$= \sum_{k=0}^{N-1} z_k w^{j(i-k)}$$

$$\Rightarrow (ZU)_{ij} = (NU\hat{z})_{ij}$$

$$\Rightarrow \frac{1}{N} U^* Z U = N\hat{Z}$$

2. a) Use the given algorithm to compute

$$(a_0 + a_1 x + a_2 x^2)(b_0 + b_1 x + b_2 x^2)$$

$a = [a_0, a_1, a_2]$ $\qquad$ $b = [b_0, b_1, b_2]$

Then the solution, C, is found by:

$$\begin{bmatrix} 0 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & a_2 & a_1 & a_0 \\ a_0 & 0 & 0 & 0 & a_2 & a_1 \\ a_1 & a_0 & 0 & 0 & 0 & a_2 \\ a_2 & a_1 & a_0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ 0 \end{bmatrix}$$

$c_0 = a_0 b_0$

$c_1 = b_0 a_1 + b_1 a_0$

$c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$

$c_3 = a_2 b_1 + a_1 b_2$

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$c_4 = a_2 b_2$

directly calculating the product gives:

$$(a_0 + a_1 x + a_2 x^2)(b_0 + b_1 x + b_2 x^2) = a_0 b_0 + a_0 b_1 x + a_0 b_2 x^2 + a_1 b_0 x + a_1 b_1 x^2 + a_1 b_2 x^3 + a_2 b_0 x^2 + a_2 b_1 x^3 + a_2 b_2 x^4$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + (a_1 b_2 + a_2 b_1) x^3 + a_2 b_2 x^4$$

$$= \boxed{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4}$$

Therefore the algorithm correctly computes the coefficients of this polynomial.

2, b)

```
% This function computes the convolution of the two given
% vectors using the algorithm in the assignment.
% takes the two vectors
% returns their convolution vector
function y = dct(x,z)
[n,m] = size(x);
[m,l] = size(z);
a = zeros(n+m,1);               %
b = zeros(n+m,1);               % appending zeros to the vectors
a(1:n,1) = x(1:n,1);            %
b(1:m,1) = z(1:n,1);
y = ifft(fft(a).*fft(b));       % using the DCT, computes the convolution
y = y(1:n+m-1,1);               % eliminates the extra element
```

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2, c)

```
>> % Use the DCT function to compute the product of the two polynomials
>> % the coefficients are stored in the vectors x and z
>> x = ones(17,1);
>> z = 2*x;
>>
>> % These are the coefficients of the product of the polynomials
>> y = dct(x,z)

y =

    2.0000
    4.0000
    6.0000
    8.0000
   10.0000
   12.0000
   14.0000
   16.0000
   18.0000
   20.0000
   22.0000
   24.0000
   26.0000
   28.0000
   30.0000
   32.0000
   34.0000
   32.0000
   30.0000
   28.0000
   26.0000
   24.0000
   22.0000
   20.0000
   18.0000
   16.0000
   14.0000
   12.0000
   10.0000
    8.0000
    6.0000
    4.0000
    2.0000

>>
```

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these are the same as the analytic solution.
Therefore this algorithm works.

Homework #3
Math 5620
Jay Newby, Eric Platt, Andrew Nelson

Problem #3

a)

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```
function y=toepmult(A,x)
%Input: toeplitz matrix A and vector x
%Output: y=Ax

format long;
n=length(x);
a=[A(end,end:-1:1) A(1,end:-1:2)];
x=[x zeros(1,n-1)];
af=fft(a);
xf=fft(x);
axf=af.*xf;
y=ifft(axf);
y=y(1:n)';
```

b)

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```
N=[10 100 500 1000];
for j=1:4
    A=toeplitz([1 -1./(2:N(j))],1./(1:N(j)));
    x=rand(1,N(j));
    err(j)=norm(toepmult(A,x)-(A*x'),inf);
end
```

10	100	500	1000
4.4409e-016	1.5543e-015	3.9968e-015	1.0658e-014

Problem #4

a)

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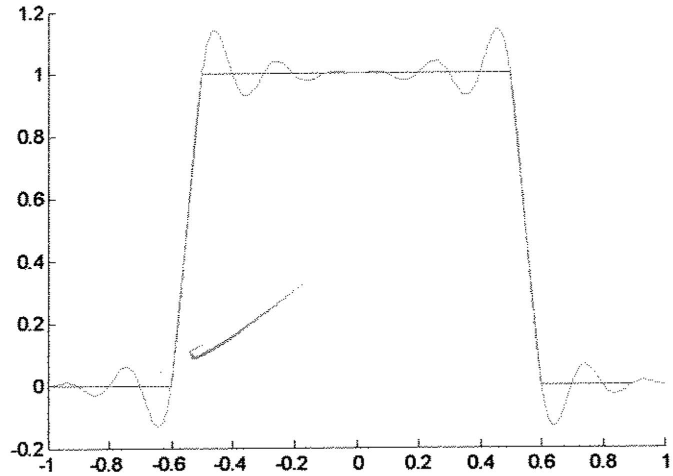
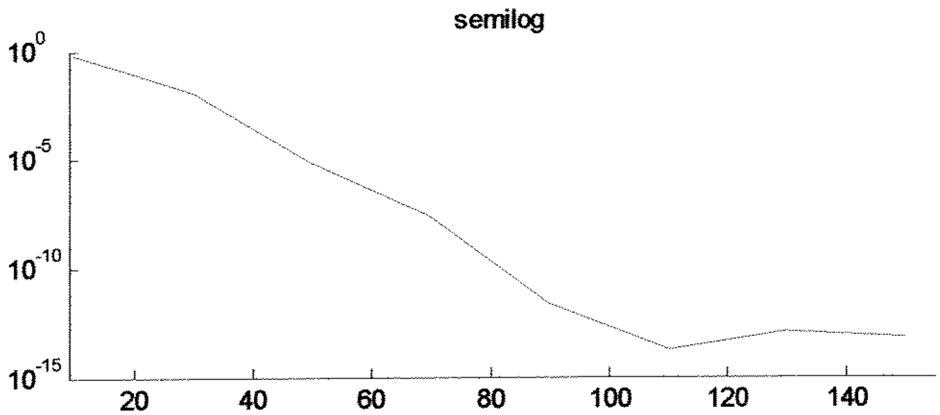
```
x=-1:.1:1;
x=x(1:end-1);
y=abs(x) <= 0.5;
xi=-1:.01:1;
p=trig_interp(x,y,xi);

N= [10 30 50 70 90 110 130 150];
for j=1:8
    x2=-1:2/N(j):1;
    x3=-1:.01:1;
    x2=x2(1:end-1);
    y2= exp(sin(4*pi*x2));
    y3=exp(sin(4*pi*x3));
    p2=trig_interp(x2,y2,x3);
    er(j)=norm(p2-y3,inf);
end
```



b)



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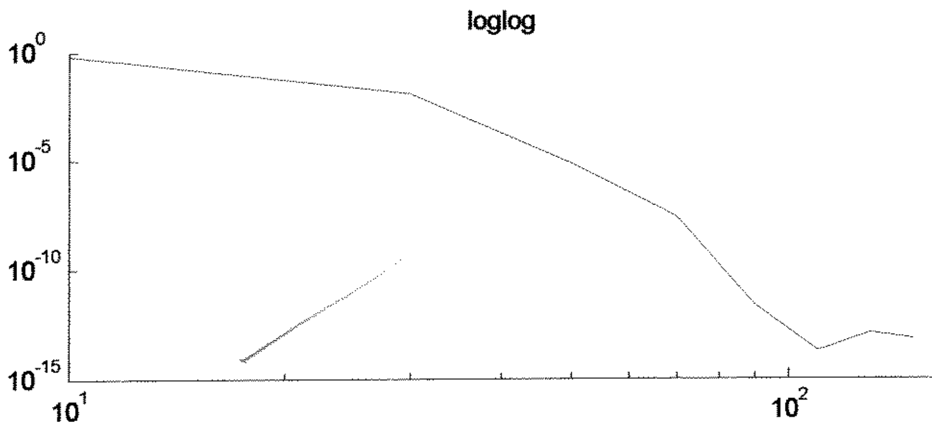

The semilog plot appears to have the linear behavior so the error decreases exponentially and the accuracy is spectral.

c)

The error is zero:

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$P_1(t)=a_0+a_1\cos(2\pi t/3)+b_1\sin(2\pi t/3)$ set $a_0=0, a_1=0, b_1=1, t=3x/2$ and $P_1(x)\equiv\sin(\pi x)$.

5) $f''(x_0-h) + 10f''(x_0) + f''(x_0+h) \approx \frac{1}{h^2}\left[12f(x_0-h) - 24f(x_0) + 12f(x_0+h)\right.$

To determine the order of accy. for this approximation, using $h=1$ and $x_0=0$, I will try some polynomials.

$f(x)=1$, $f''(x)=0$

approx. $= \frac{1}{h^2}(0) = 0$ ✓ correct for order 0

$f(x)=x$, $f''(x)=0$

approx $= \frac{1}{h^2}(-12h+12h) = 0$ ✓ correct for order 1

$f(x)=x^2$, $f''(x)=2$

$$\text{approx} = \frac{1}{h^2}\left(12(-h)^2 + 12(h)^2\right)$$

$$= \frac{1}{h^2}\cdot h^2(12+12)$$

$$= 24$$

$f''(-h)+10f''(0)+f''(h) = 2+20+2 = 24$ ✓ correct for order 2

$f(x)=x^3$, $f''(x)=6x$

$$\text{approx} = \frac{1}{h^2}\left(12\cdot(-h)^3 + 12(h)^3\right)$$

$$= 0$$

$f''(-h)+10f''(0)+f''(h) = -6h+6h = 0$ ✓ correct for order 3

$f(x)=x^4$, $f''(x)=12x^2$

$$\text{approx} = \frac{1}{h^2}\left(12\cdot(-h)^4 + 12(h)^4\right)$$

$$= 24h^2$$

$f''(-h) + 10f''(0) + f''(h) = 12h^2 + 12h^2$

$= 24h^2$ ✓ correct for order 4

$f(x) = x^5$, $f''(x) = 20x^3$

approx $= \frac{1}{h^2}\left(12(-h)^5 + 12(h)^5\right)$

$= 0$

$f''(-h) + 10f''(0) + f''(h) = 20(-h)^3 + 20(h)^3$

$= 0$ ✓ correct for order 5

$f(x) = x^6$, $f''(x) = 30x^4$

approx $= \frac{1}{h^2}\left(12(-h)^6 + 12(+h)^6\right)$

$= 24h^4$

$f''(-h) + 10f''(0) + f''(h) = 30h^4 + 30h^4$ X not correct for order 6

$= 60h^4$

15/15 Since this is correct for up to order 5 polynomials, the order of accuracy is $5-1 = 4$

⑥ Consider $f(t,y) = |y|$ on $D = [-1,1] \times \mathbb{R}$

(a) <u>Claim</u> f is continuous with respect to y on D.

<u>Pf</u> Let $\{y_1, y_2\} \subset \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+$.
We know from the triangle inequality that
$||y_1|-|y_2|| \leq |y_1 - y_2|$ so if $|y_1 - y_2| < \varepsilon$ it follows that
$||y_1|-|y_2|| < \varepsilon \Rightarrow |f(t,y_1) - f(t,y_2)| < \varepsilon$.
Therefore for every $\varepsilon \in \mathbb{R}_+$ $\exists$ a $\delta \in \mathbb{R}_+$ such that
$|y_1 - y_2| < \delta \Rightarrow |f(t,y_1) - f(t,y_2)| < \varepsilon$, namely $\delta = \varepsilon$.
Hence f is continuous with respect to y on D. ∎

(b) <u>Claim</u> f is Lipschitz continuous with respect to y on D with Lipschitz constant $L = 1$.

<u>Pf</u> As we saw
$|f(t,y_1) - f(t,y_2)| = ||y_1| - |y_2|| \leq |y_1 - y_2|$ by the triangle inequality.

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Therefore $|f(t,y_1) - f(t,y_2)| \leq 1 \cdot |y_1 - y_2|$ for $\{(t,y_1),(t,y_2)\} \subset D$
and the Lipschitz condition is seen to hold
for $L = 1$ ∎

(c) From class we know the IVP $y' = f(t,y)$, $t \in [a,b]$, $y(a) = \alpha$
is well-posed if $f(t,y)$ is continuous and Lipschitz continuous in y. We have shown in part (a) that
$|y|$ is continuous and in part (b) that it is Lipschitz continuous in y. Therefore the IVP
$y' = |y|$, $t \in [-1,1]$, $y(-1) = -1$ is well-posed.