Homework #1 - Solutions

Problem #1

Here are two examples of 2x2 matrices that have a Gerschgorin circle without an eigenvalue inside it or on its edge:

Example #1

```
> A := array([[1,-6],[5,-2]]);
  evalf(Eigenvals(A));
```

$$A := \begin{bmatrix} 1 & -6 \\ 5 & -2 \end{bmatrix}$$

$$[-0.500000000 + 5.267826876\, I, -0.500000000 - 5.267826876\, I]$$

```
> r=abs(-.500000000+5.267826876*I):
> plot([[1+6*sin(t),6*cos(t),t=0..2*Pi],[-2+5*sin(t),5*cos(t),t=0..2
  *Pi],[.3*sin(t)-.5,.3*cos(t)+5.267826876,t=0..2*Pi],[.3*sin(t)-.5,
  .3*cos(t)-5.267826876,t=0..2*Pi]],color=[red,blue,green,black]);
```

The two larger circles are the Gerschgorin circles, one centered at 1 with radius 6 and the centered at -2 with radius 5. The two smaller circles are centered around the eigenvalues and it is clear that the eigen values are not inside nor on the edge of the Gerschgorin circle centered at -2.

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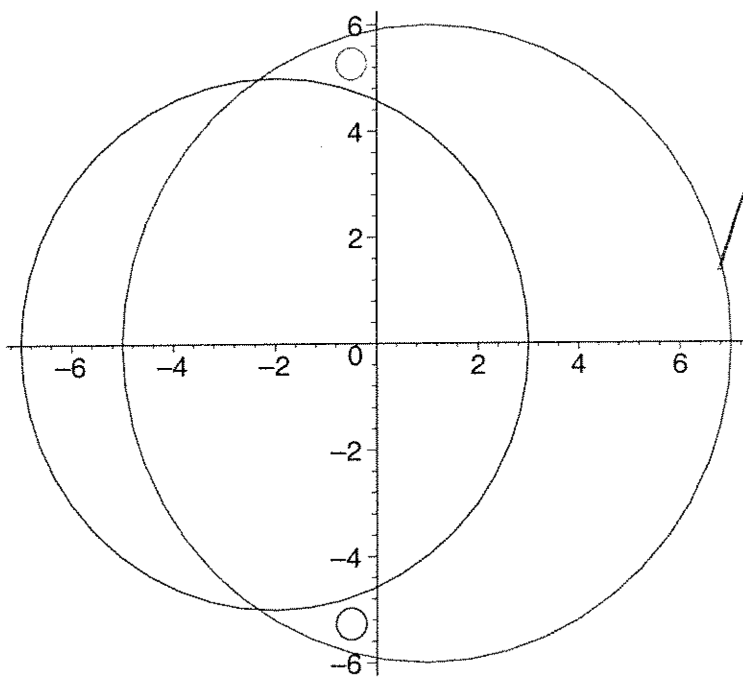


>

Example #2

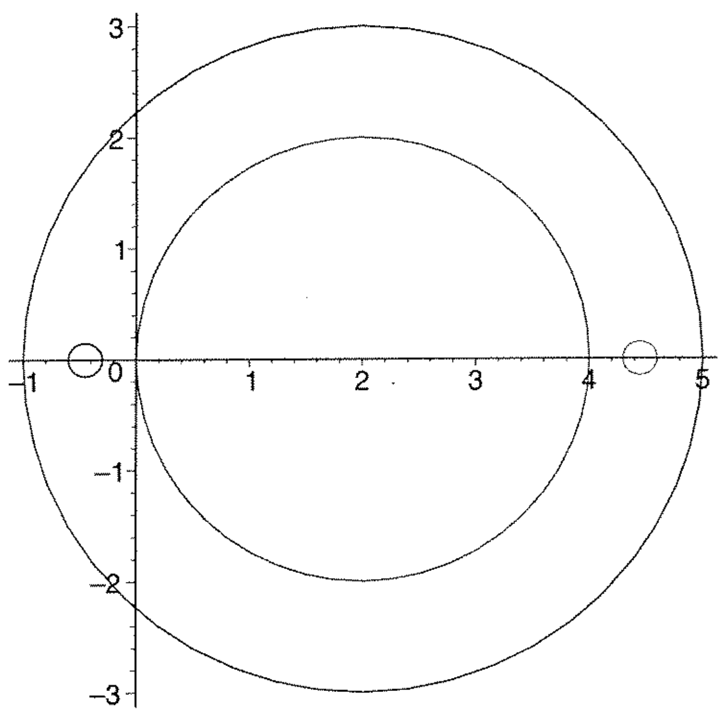
```
> B := array([[2,3],[2,2]]);
  evalf(Eigenvals(B));
```

$$B := \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$$

[4.449489743, -0.449489743]

```
> plot([[2+2*sin(t), 2*cos(t), t=0..2*Pi], [2+3*sin(t), 3*cos(t),
  t=0..2*Pi], [4.449489743+.15*sin(t), .15*cos(t), t=0..2*Pi],
  [-.449489743+.15*sin(t), .15*cos(t),
  t=0..2*Pi]],color=[red,blue,green,black]);
```

The two larger circles are the Gerschgorin circles, both centered at 2 but one with radius 2 and the other with radius 3. The two smaller circles are centered around the eigenvalues and it is clear that the eigen values are not inside nor on the edge of the Gerschgorin circle with radius 2.




```
>
>
>
```

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Math 5620
HW #1

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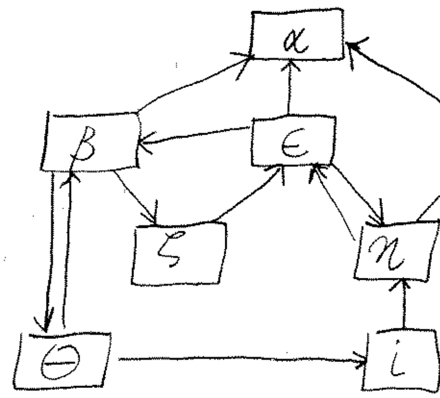
② ⓐ Given the directed graph of a tiny web below, compute the PageRank of each of the pages for $p = 0.85$ and $p = 0.95$. Use the power method discussed in the book and lecture for the computations. Turn in a listing of your code. Compare the number of iterations of the power method needed for convergence for the two different values of p.



The connectivity matrix is given by:

$$\sigma = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\begin{array}{c} \alpha \\ \beta \\ \gamma \\ \epsilon \\ \zeta \\ \eta \\ \theta \\ \iota \end{array}$$

$$\begin{array}{cccccccc} \alpha & \beta & \gamma & \epsilon & \zeta & \eta & \theta & \iota \end{array}$$

- Note that eigenvalues don't need to be calculated. I change the loop execution condition from $\| Ax^{(k)} - \lambda^{(k)} x^{(k)} \| \geq \varepsilon$ to $\| x^{(k)} - x^{(k-1)} \| \geq \varepsilon$. This avoids unnecessary computations.
- See MATLAB code
- The PageRank $\tilde{x}$ for each page is computed to be:

(over) 2

$p = 0.85$

$$\tilde{x} = \begin{bmatrix} 0.21641 \\ 0.12309 \\ 0.10706 \\ 0.17219 \\ 0.076621 \\ 0.15369 \\ 0.076621 \\ 0.074307 \end{bmatrix} \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \epsilon \\ \zeta \\ \eta \\ \theta \\ \iota \end{array}$$

$p = 0.95$

$$\tilde{x} = \begin{bmatrix} 0.22895 \\ 0.12334 \\ 0.10630 \\ 0.17517 \\ 0.07250 \\ 0.15339 \\ 0.07250 \\ 0.06787 \end{bmatrix} \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \epsilon \\ \zeta \\ \eta \\ \theta \\ \iota \end{array}$$

- Using the criteria $\|x^{(k)} - x^{(k-1)}\|_2 \geq \epsilon$ for $\epsilon = 10^{-8}$, I find the # of iterations for the $p=0.85$ case is 27, for the $p=0.95$ case it is 32 iterations.
- This is due to the fact that we know for the power method that

$$\|x^{(k)} - v^{(1)}\|_2 = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$$

$|\lambda_2/\lambda_1|^k$ is smallest when λ_1 is much bigger than λ_2.

- For the $p=0.95$ case λ_2 is larger than λ_2 in the $p=0.85$ case thus convergence is slower.

Good

(b) Suppose that every page on the web has an outgoing link to every other page on the web. What would be the PageRank of each of the pages and why?

- I assume the page also has a link to itself

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- The entries of the transition probability matrix A are all

$$a_{ij} = p\left(\frac{1}{n}\right) + \frac{(1-p)}{n} = \frac{1}{n}$$

(The connectivity matrix is the all ones matrix.)

- The Perron-Frobenius theorem tells us that $\lambda = 1$ is the largest eigenvalue of A (all the rest are 0)

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2 cont.

- We then have

$$Ax = x$$

- Since all the entries in A are given by $1/n$, the entries in x are given by:

$$x_i = \frac{1}{n} \sum_{k=1}^{n} x_k$$

- The solution to this is that $x_i = 1$ for all i
- The PageRank for each page is given by

$$\tilde{x} = \frac{x}{\sum x_i} = \frac{x}{n} \implies \tilde{x}_i = \frac{1}{n} \text{ for all } i$$

$\Rightarrow$ Every page has the same PageRank.

```
% prob2a.m - MATH 5620 Homework 1, problem 2(a), Spring 2006
% Author - Scott Talbot

clear

% connectivity matrix

% the class example for checking results
% G = [0 0 0 1 0 1; 1 0 0 0 0 0; 0 1 0 0 0 0;...
%        0 1 1 0 0 0; 0 0 1 0 0 0; 0 0 1 0 0 0];

G = [0 1 1 1 0 0 0 0; 0 0 0 1 0 0 1 0; 0 0 0 0 0 1 0 0;...
         0 0 0 0 1 1 0 0; 0 1 0 0 0 0 0 0; 0 0 0 1 0 0 0 1;...
         0 1 0 0 0 0 0 0; 0 0 0 0 0 0 1 0];
[rows,cols] = size(G);
n = rows; % number of web pages
p = 0.85; % probability of following a link from a page that has a link
delta = (1-p)/n;
c = sum(G); % column sum of the connectivity matrix

% computing transition probability matrix A
A = zeros(size(G));
for ii = 1:cols
    if c(ii) ~= 0
        A(:,ii) = p*G(:,ii)/c(ii)+delta;
    else
        A(:,ii) = 1/n;
    end
end

iter = 1;
epsilon = 1e-8;
x = ones(rows,1); % initialize to the all ones vector

% first iteration outside the loop
z = A*x;
x_old = x;
x = z/norm(z);
while norm(x-x_old) >= epsilon
    z = A*x;
    x_old = x;
    x = z/norm(z);
    iter = iter+1;
end

x_tilda = x/sum(x)
iter
```

Does not take the advantage of the sparsity of G. This computation can be reduced quite significantly.

③ Companion Matrix

(a) Let $p_n(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ for any a_j, $j = 0, \ldots, n-1$

$$A_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix}$$

We wish to show that $p_n(\lambda) = \det(\lambda I_n - A_n)$ $\forall n \in \mathbb{N}$, where I_n is the $n \times n$ identity matrix.

Proof (i) When $n=1$ we have $p_1(\lambda) = \lambda + a_0$ and $A_1 = [-a_0]$. So $\lambda I_1 - A_1 = [\lambda + a_0]$ and $\det(\lambda I_1 - A_1) = \lambda + a_0 = p_1(\lambda)$. $\forall a_0$

Thus the result holds for $n = 1$.

(ii) Suppose $\exists n \in \mathbb{N}$ such that $p_n(\lambda) = \det(\lambda I_n - A_n)$ $\forall a_j$, $j = 0, \ldots, n-1$

$$A_{n+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & -a_n \end{bmatrix}$$

Thus $\quad |\lambda I_{n+1} - A_{n+1}| =$

$$\begin{vmatrix} \lambda & -1 & 0 & \cdots & & 0 \\ 0 & \lambda & -1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & \\ 0 & & & & \lambda & -1 \\ a_0 & a_1 & & \cdots & & \lambda+a_n \end{vmatrix}$$

Expanding by minors along the first column gives

$$|\lambda I_{n+1} - A_{n+1}| = \lambda \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \ddots & \vdots \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \\ a_1 & a_2 & & \cdots & \lambda+a_n \end{vmatrix}$$

$$+ (-1)^n a_0 \begin{vmatrix} -1 & 0 & \cdots & & & 0 \\ \lambda & -1 & & & & \vdots \\ 0 & \lambda & \ddots & \ddots & & \vdots \\ \vdots & & & \lambda & -1 & 0 \\ 0 & \cdots & & 0 & \lambda & -1 \end{vmatrix}$$

The first determinant in the expansion equals $\lambda^n + a_n \lambda^{n-1} + \cdots + a_1$ by assumption and the second is lower triangular so we only need to multiply its diagonals. The diagonals all equal -1 and there are n of them so the second determinant is $(-1)^n$.

Therefore $|\lambda I_{n+1} - A_{n+1}| = \lambda(\lambda^n + a_n\lambda^{n-1} + \cdots + a_1) + (-1)^n a_0 (-1)^n$

$$= \lambda^{n+1} + a_n\lambda^n + \cdots + a_1\lambda + a_0.$$

Thus $\det(\lambda I_{n+1} - A_{n+1}) = p_{n+1}(\lambda)$.

By induction the result follows:

$\det(\lambda I_n - A_n) = p_n(\lambda) \quad \forall n \in \mathbb{N}$ ■

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(b) The first through $(n-1)^{st}$ row of A all have 0 on the diagonal and only the 1 on the superdiagonal is nonzero. Thus $a_{ii} = 0$ for $i = 1, \dots, n-1$ and

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| = 1 \quad \text{for} \quad i = 1, \dots, n-1.$$

Thus we get $n-1$ Gerschgorin circles centered at 0 with radius 1.

The n^{th} row gives $a_{nn} = -a_{n-1}$ and

$$\sum_{j=1}^{n-1} |a_{nj}| = |a_0| + |a_1| + \cdots + |a_{n-2}|$$

so we get the Gerschgorin circle defined by $\{r \in \mathbb{C} : |r + a_{n-1}| \leq |a_0| + \cdots + |a_{n-2}|\}$.

Combined we get the bound on any root r,

$$|r| \leq 1 \quad \text{or} \quad |r + a_{n+1}| \leq |a_0| + \cdots + |a_{n-2}|.$$

If they give disjoint regions then one root must satisfy $|r + a_{n+1}| \leq |a_0| + \cdots + |a_{n-2}|$ because only one circle defined this region while there must be $n-1$ roots satisfying $|r| \leq 1$ since $n-1$ circles all gave this region.

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(C) The first column has $a_{11} = 0$ and $\sum_{j=2}^{n} |a_{j1}| = |a_0|$

So we get the bound $|r| \le |a_0|$.

For $i = 2, \dots, n-1$ $a_{ii} = 0$ and $\sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ji}| = 1 + |a_i|$

So we get the bound $|r| \le 1 + |a_i|$ for $i = 2, \dots, n-1$.

For the n^{th} column $a_{nn} = -a_{n-1}$ and $\sum_{j=1}^{n-1} |a_{jn}| = 1$

So we get the bound $|r + a_{n-1}| \le 1$.

Combined we have

$|r| \le |a_0|$ or $|r| \le 1 + |a_i|$ for $i = 2, \dots, n-1$

or $|r + a_{n-1}| \le 1$.

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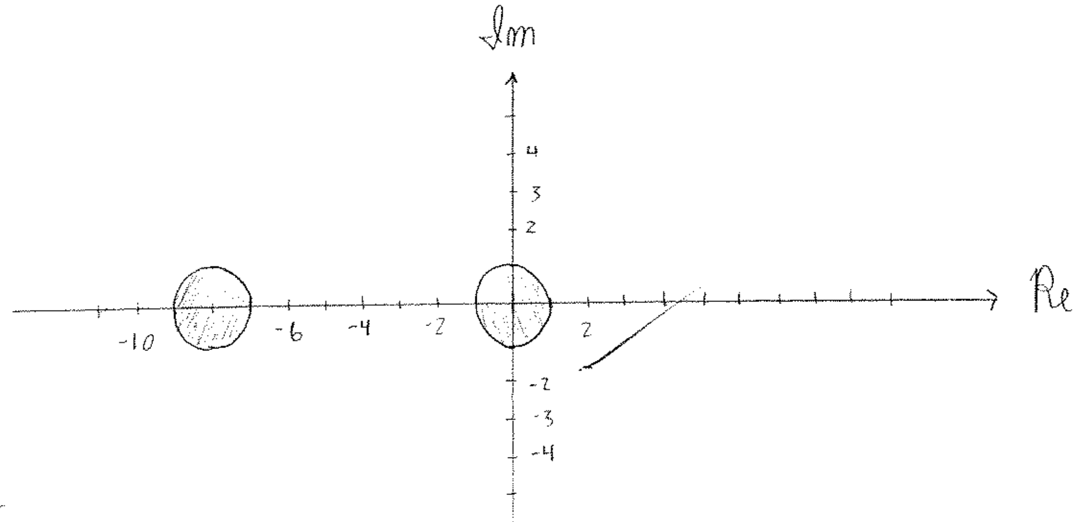
(d) (i) $\lambda^{10} + 8\lambda^{9} + 1 = 0$

From part (b) we get the bounds $|r| \leq 1$ or $|r+8| \leq 1$.

From part (c) we get the bounds $|r| \leq 1$ or $|r+8| \leq 1$.

The graph shows the regions where the roots can be.



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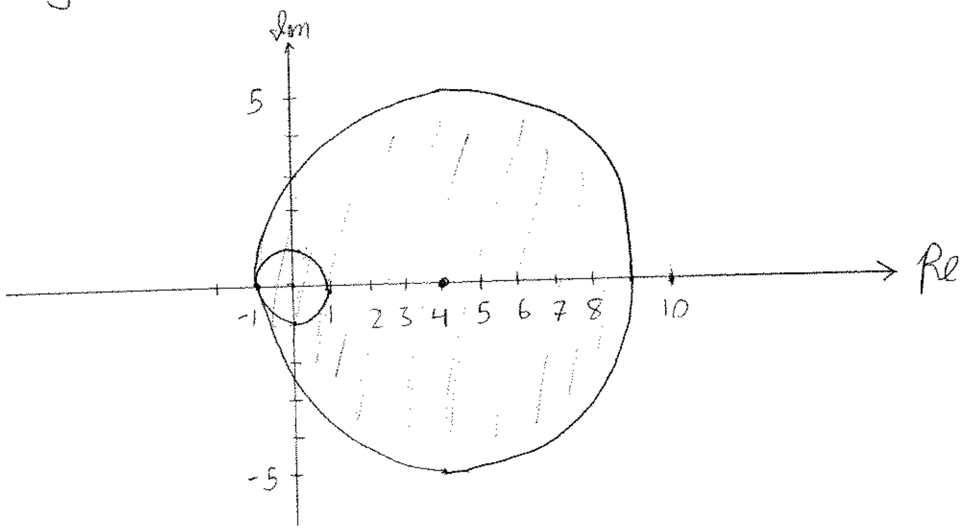
The two regions are disjoint. For (b) and (c) only one circle is centered at -8 so one root is in that region and the other 9 are in the disk of radius 1 about 0.

(ii) $\lambda^6 - 4\lambda^5 + \lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1 = 0$

From part (b) we have $|r| \le 1$ or $|r-4| \le 5$.

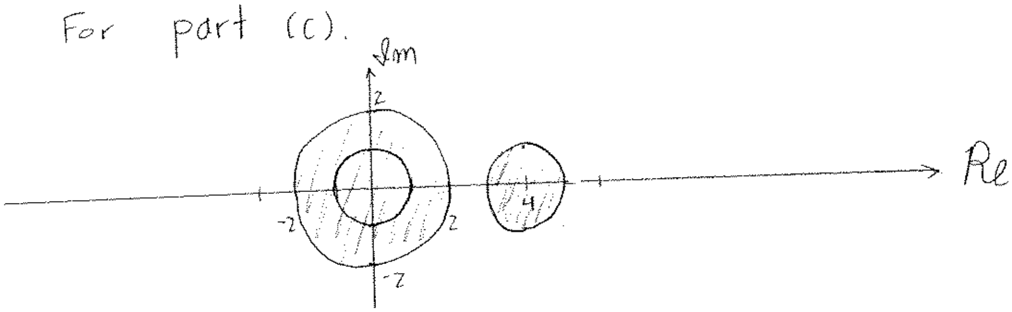
From part (c) we have $|r| \le 1$ or $|r| \le 2$ or $|r-4| \le 1$

The graph for part (b) is

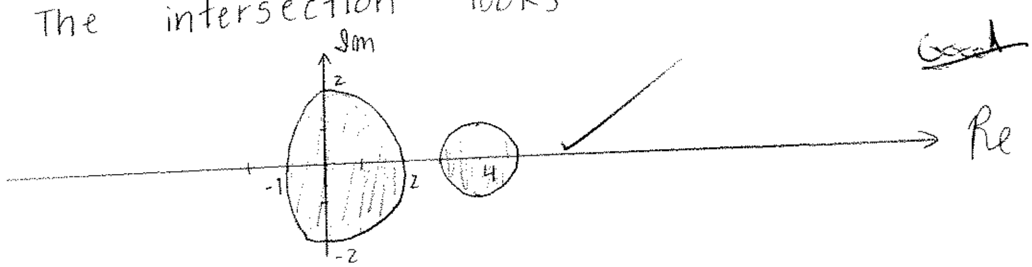


So the roots must all satisfy $|r-4| \le 5$

For part (c).



5/5 The intersection looks like



Good

one root satisfies $|r-4| \le 1$ and the other 5 satisfy $|r| \le 2$ and $|r-4| \le 5$.

Problem 4

a)

runRayleighQuotient.m

```
A = [-2 1 0 0 0 0;
    1 -2 1 0 0 0;
    0 1 -2 1 0 0;
    0 0 1 -2 1 0;
    0 0 0 1 -2 1;
    0 0 0 0 1 -2];
tol = 10^-13;
[vec, lam] = eig(A)
vec(:,4);

display('Initial guess: ');
x0 = [-.5 .2 .4 -.4 -.2 .5]'
[v, lambda] = RayleighQuotient(A, x0, tol)
errorComp = min(norm(v - vec(:,3), inf), norm(v + vec(:,3), inf))
error = norm(A*v - lambda*v, inf)

display('Initial guess: ');
x0 = [1 1 1 1 1 1]'
[v, lambda] = RayleighQuotient(A, x0, tol)
errorComp = min(norm(v - vec(:,6), inf), norm(v + vec(:,6), inf))
error = norm(A*v - lambda*v, inf)

display('Initial guess: ');
x0 = [-1 -1 -1 1 1 1]'
[v, lambda] = RayleighQuotient(A, x0, tol)
errorComp = min(norm(v - vec(:,5), inf), norm(v + vec(:,5), inf))
error = norm(A*v - lambda*v, inf)

display('Initial guess: ');
x0 = [-0.3 0.4 -0.3 -0.3 0.5 -0.3]'
[v, lambda] = RayleighQuotient(A, x0, tol)
errorComp = min(norm(v - vec(:,2), inf), norm(v + vec(:,2), inf))
error = norm(A*v - lambda*v, inf)

display('Initial guess: ');
x0 = [0.3 -0.4 0.4 -0.4 0.5 -0.3]'
[v, lambda] = RayleighQuotient(A, x0, tol)
errorComp = min(norm(v - vec(:,1), inf), norm(v + vec(:,1), inf))
error = norm(A*v - lambda*v, inf)

display('Initial guess: ');
x0 = [0.3 0.4 -0.4 -0.4 0.3 0.4]'
[v, lambda] = RayleighQuotient(A, x0, tol)
errorComp = min(norm(v - vec(:,4), inf), norm(v + vec(:,4), inf))
error = norm(A*v - lambda*v, inf)
```

Nice!

RayleighQuotient.m

```
function [v,lam] = RayleighQuotient(A, x0, eps)
  k = 0;
  v = x0/norm(x0,2);
  lam = v'*A*v;
  cont = 1;
```

```
    I = eye(size(A));
    while (cont)
      vold = v;
      lambdaOld = lam;
      z = (A-lambdaOld*I)\vold;
      v = z/norm(z,2);
      lam = v'*A*v;
      k = k + 1;
      cont = NotDone(A, vold, lambdaOld, eps);
    end
    k = k - 1
end;

function ret = NotDone(A, x, lambda, eps)
    infnorm = norm(A*x - lambda*x, inf);
    if (infnorm >= eps)
      ret = 1;
    else
      ret = 0;
    end;
end;
```

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b)

Answer received by executing Matlab command eig:

Eigenvalue:	Corresponding eigenvector:	Eigenvalue:	Corresponding eigenvector:
-3.80193773580484	0.23192061392433 -0.41790650594127 0.52112088916960 -0.52112088916960 0.41790650594128 -0.23192061392433	-1.55495813208737	0.52112088916960 0.23192061392433 -0.41790650594127 -0.41790650594127 0.23192061392433 0.52112088916960
-3.24697960371747	-0.41790650594128 0.52112088916960 -0.23192061392433 -0.23192061392433 0.52112088916960 -0.41790650594127	-0.75302039628253	-0.41790650594128 -0.52112088916960 -0.23192061392433 0.23192061392433 0.52112088916960 0.41790650594127
-2.44504186791263	-0.52112088916960 0.23192061392433 0.41790650594128 -0.41790650594127 -0.23192061392433 0.52112088916960	-0.19806226419516	-0.23192061392433 -0.41790650594127 -0.52112088916960 -0.52112088916960 -0.41790650594127 -0.23192061392433

Answer received by executing Rayleigh Quotient Iteration (RQI) algorithm:

Initial guess:	Eigenvalue	Corresponding eigenvector	Number of iterations
1	-0.19806226419516	-0.23192061392433	3
1		-0.41790650594128	
1		-0.52112088916960	
1		-0.52112088916960	
1		-0.41790650594127	
1		-0.23192061392433	
Error ($\lvert A^*\lambda - v^*\lambda \rvert$)	6.93889390390723e-017	**Error ($\lvert v_{Matlab}-v_{RQI} \rvert$)**	2.77555756156289e-016
-0.5	-2.44504186791263	0.52112088916960	2
0.2		-0.23192061392433	
0.4		-0.41790650594128	
-0.4		0.41790650594128	
-0.2		0.23192061392433	
0.5		-0.52112088916960	
Error ($\lvert A^*\lambda - v^*\lambda \rvert$)	4.44089209850063e-016	**Error ($\lvert v_{Matlab}-v_{RQI} \rvert$)**	2.22044604925031e-016
-1	-0.75302039628253	0.41790650594128	3
-1		0.52112088916960	
-1		0.23192061392433	
1		-0.23192061392433	
1		-0.52112088916960	
1		-0.41790650594128	
Error ($\lvert A^*\lambda - v^*\lambda \rvert$)	1.11022302462516e-016	**Error ($\lvert v_{Matlab}-v_{RQI} \rvert$)**	4.44089209850063e-016
0.3	-1.55495813208737	-0.52112088916960	3
0.4		-0.23192061392433	
-0.4		0.41790650594127	
-0.4		0.41790650594127	
0.3		-0.23192061392433	
0.4		-0.52112088916960	
Error ($\lvert A^*\lambda - v^*\lambda \rvert$)	2.22044604925031e-016	**Error ($\lvert v_{Matlab}-v_{RQI} \rvert$)**	2.22044604925031e-016
0.3	-3.80193773580484	-0.23192061392433	3
-0.4		0.41790650594127	
0.4		-0.52112088916960	
-0.4		0.52112088916960	
0.5		-0.41790650594127	
-0.3		0.23192061392433	
Error ($\lvert A^*\lambda - v^*\lambda \rvert$)	4.44089209850063e-016	**Error ($\lvert v_{Matlab}-v_{RQI} \rvert$)**	3.88578058618805e-016
-0.3	-3.24697960371747	-0.41790650594127	3
0.4		0.52112088916960	
-0.3		-0.23192061392433	
-0.3		-0.23192061392433	
0.5		0.52112088916960	
-0.3		-0.41790650594127	
Error ($\lvert A^*\lambda - v^*\lambda \rvert$)	4.44089209850063e-016	**Error ($\lvert v_{Matlab}-v_{RQI} \rvert$)**	3.33066907387547e-016

⑤ (a) $x^Tx = y^Ty$; $w = \dfrac{y-x}{\|y-x\|}$

It follows that $w^T = \dfrac{y^T - x^T}{\|y-x\|}$.

So $$ww^Tx = \frac{1}{\|y-x\|^2}(y-x)(y^T-x^T)x = \frac{1}{\|y-x\|^2}(yy^Tx - xy^Tx - yx^Tx + xx^Tx)$$

$$= \frac{1}{\|y-x\|^2}\left[x(x^Tx - y^Tx) - y(x^Tx - y^Tx)\right]$$

$$= \frac{x-y}{\|y-x\|^2}(x^Tx - y^Tx)$$

$$= \frac{x-y}{\|y-x\|^2}\left(\tfrac{1}{2}x^Tx + \tfrac{1}{2}y^Ty - y^Tx\right) \qquad \text{since } y^Ty = x^Tx$$

$$= \frac{x-y}{\|y-x\|^2}\left(\tfrac{1}{2}x^Tx + \tfrac{1}{2}y^Ty - \tfrac{1}{2}y^Tx - \tfrac{1}{2}x^Ty\right) \qquad \text{since } y^Tx = x^Ty$$

$$= \frac{x-y}{2\|y-x\|^2}(x^Tx + y^Ty - y^Tx - x^Ty)$$

$$= \frac{x-y}{2\|y-x\|^2}\left[y^T(y-x) - x^T(y-x)\right]$$

$$= \frac{x-y}{2\|y-x\|^2}(y^T - x^T)(y-x)$$

$$= \frac{x-y}{2\|y-x\|^2}\|y-x\|^2$$

$$= \frac{x-y}{2}.$$

The result is

$$ww^T x = \frac{x-y}{2}$$

$$\Rightarrow \quad 2ww^T x = x-y$$

$$\Rightarrow \quad x - 2ww^T x = y$$

$$\Rightarrow \quad (I - 2ww^T)x = y$$

$$\Rightarrow \quad Hx = y \quad \blacksquare$$

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(b) (i) $H = I - 2ww^T$

So $H^T = I - 2(ww^T)^T = I - 2(w^T)^T w^T$

$= I - 2ww^T$

Therefore $H = H^T$, i.e. H is symmetric.

$$HH^T = (I - 2ww^T)(I - 2ww^T) = H^T H$$

$$= I - 2ww^T - 2ww^T + 4ww^Tww^T$$

$$= I - 4ww^T + 4w(w^Tw)w^T$$

$$= I - 4ww^T + 4ww^T \quad \text{since } w^Tw = 1 \text{ (} w \text{ is normalized)}$$

$$= I$$

1b ~~[illegible]~~/10 Thus $HH^T = H^TH = I$, i.e. H is orthogonal.

Since H is symmetric and orthogonal its only possible eigenvalues are ± 1.

(ii) Let $H \in \mathbb{R}^{n \times n}$

$$Tr(H) = Tr(I - 2ww^T)$$

$$= Tr(I) - 2Tr(ww^T)$$

$$= n - 2Tr(ww^T)$$

So we just need to determine $Tr(ww^T)$.

$$\mathrm{Tr}(ww^T) = \sum_{i=1}^{n} w_i^2$$

$$= \sum_{i=1}^{n} \frac{(y_i - x_i)^2}{\|y-x\|^2}$$

$$= \frac{1}{\|y-x\|^2} \sum_{i=1}^{n} (y_i - x_i)^2$$

$$= \frac{\|y-x\|^2}{\|y-x\|^2}$$

$$= 1$$

Therefore $\boxed{\mathrm{Tr}(H) = n-2}$

Since $H \in \mathbb{R}^{n \times n}$ with $HH^T = H^T H = I$ and $H = H^T$ it follows H has n eigenvalues, counting multiplicity, and the possible values are ± 1. Let P denote the number of eigenvalues which equal -1, then the number which equal $+1$ is $n-P$. The trace is the sum of the eigenvalues so

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$$\mathrm{Tr}(H) = n-2 = -P + (n-P)$$

$$\Rightarrow \quad P = 1.$$

So H has eigenvalue -1 with multiplicity 1 and $+1$ with multiplicity $n-1$.

(iii) The determinant is the product of the eigenvalues so $\det(H) = (-1)(1)^{n-1} = \boxed{-1}$

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