Math 5610 **Review For Final** **Final: Dec. 15, 2005**
Grady Wright **8:00–10:00am**

Administrative Details
Cumulative test weighted toward the last half of the class Open text book (Bradie) only! Open notes

1 Machine Errors (Chapter 1)

- How does a loss-of-significance error occur?
- How does an overflow/underflow error occur?
- What techniques can be used for fixing these types of problems?

2 Linear Systems (Chapter 3)

2.1 Norms

- Definitions for the 1-norm, 2-norm, and ∞-norm for vectors and matrices.
- Spectral radius of a matrix.
- $\lim_{k\to\infty} \|A\|^k = 0$ if and only if what?

2.2 Direct Methods

2.2.1 Gaussian elimination

- How does partial pivoting work and why do we use it?
- How does the operation count scale with the dimension of the linear system?
- How do we compute the $P^T LU$ decomposition and what are its advantages?

2.2.2 Cholesky's method

- What property must the matrices satisfy?
- What decomposition does this method produce?
- How does the operation count scale with the dimension of the linear system?

2.2.3 Tridiagonal matrices

- How does Crout's method work?
- What property must the matrices satisfy?
- How does the operation count of Crout's method scale with the dimension of the linear system?

2.3 Iterative methods

2.3.1 Stationary methods

- Jacobi
- Gauss-Seidel
- Successive Overrelaxation
- What properties guarantee convergence?
- How is convergence related to the spectral radius of the matrix?

2.3.2 Conjugate gradient method

- How does this method differ from the other iterative methods listed above?
- What type of matrices does the method work for?
- What is the fundamental idea behind the method?

2.3.3 Conditioning and stability of linear systems

- Definition of the condition number of a matrix?
- What does the condition number measure?
- How does the relative error in the computed solution depend on the condition number?

3 Rootfinding (Chapter 2)

General Problem: Find the solution to the equation $f(x) = 0$.

3.1 Geometric Convergence

An iterative method is said to converge geometrically to α if

$$|\alpha - x_{n+1}| \approx c|\alpha - x_n|^p$$

for some non-zero constant $c \in R$. What do we call the convergence when $p = 1$,$p = 2$, or $p = 3$?

3.2 Newton's method

- Formula:
- How is it derived?
- What is the convergence rate?
- When does it become much slower, and how do we fix it?

3.3 Secant method

- Formula:
- How is it derived?
- What is the convergence rate?

3.4 Müller's method

- How is it derived?

3.5 Fixed-Point Iteration

Problem: Given some function $g(x)$, find x such that $x = g(x)$. How do we rewrite a root-finding problem into a fixed point iteration problem?

3.5.1 Graphical Convergence Test

It is important to know how to approximately determine the convergence region of a fixed-point iteration using the graphical technique with the line $y = x$.

3.5.2 Important Theorems

The following theorems are important for determing the region of convergence of a fixed-point iteration method:

- Theorem p. 87

- Theorems from the notes

3.5.3 Convergence results

When does a fixed point iteration scheme converge faster than linear?

3.6 Steffenson's method

3.6.1 Formula

What is Steffenson's method applied to the fixed point iteration scheme $x_{n+1} = g(x_n)$?

3.6.2 Convergence

When does Steffenson's method converge and what is the convergence rate?

3.7 Roots of Polynomials

3.7.1 Factor Theorem

Given that the roots of a polynomial $p(x)$ are $x_1, x_2, \ldots, x_n$, we can write $p(x)$ as

$$p(x) = c(x - x_1)(x - x_2)(x - x_n)$$

where c is some non-zero constant.

3.7.2 Horner scheme (polynomial deflation)

How do we use Horner's scheme with Netwon's method to efficiently calculate roots of a polynomial?

3.8 Non-linear systems

Only need to know what Newton's method is for non-linear systems.

4 Interpolation (Chapter 5)

4.1 Polynomial methods

4.1.1 Existence and uniqueness

Very good thing to know.

4.1.2 Methods of generating interpolants

Should know each of the following formulas for computing the interpolating polynomial, and where it is to be applied

- Lagrange's interpolation formula (and error formula)
- Barycentric interpolation
- Newton divided difference formula (and error formula)
- Hermite interpolation (and error formula)
- Neville's method is not on the final.

4.1.3 Runge Phenomenon

Very important!

- What does it look like?
- When does it occur?
- How can we get rid of it?

4.1.4 Horner's scheme

Method for efficiently evaluating a polynomial.

4.2 Piecewise Polynomial Interpolation

What is the general idea behind this technique? What it the effect on the Runge Phenomenon?

4.2.1 Cubic Splines

- How are the following three end-conditions defined:

  $\Longrightarrow$ Clamped:

  $\Longrightarrow$ Natural:

  $\Longrightarrow$ Not-a-Knot:

- What optimality property does a natural cubic spline satisfy and how do we prove it?

4.2.2 Piecewise cubic Hermite

- How is it defined?

- When is it applicable?

5 Numerical Integration or Quadrature (Chapter 6)

General problem: Approximate the definite integral

$$I(f) = \int_a^b f(x)dx$$

5.1 Newton-Cotes Formulas

It is important to know the general idea behind generating these formulas.

5.1.1 Trapezoidal Rule

- Formula with exact error term:

$$\int_{x_0}^{x_1} f(x)dx =$$

- How is the error term derived?

- Composite formula with exact error term:

$$\int_{x_0}^{x_n} f(x)dx =$$

- How is the composite error term derived?

5.1.2 Simpson's Rule

- Formula with exact error term:

$$\int_{x_0}^{x_2} f(x)dx =$$

- How is the error term derived?

- Composite formula with exact error term:

$$\int_{x_0}^{x_n} f(x)dx =$$

- How is the composite error term derived?

5.1.3 Simpson's 3/8 Rule

Not on final.

5.1.4 Boole's Rule

Not on final.

5.1.5 Degree of precision

- Definition:

- How do we determine the degree of precision? Two general methods

  1. If the error formula is known, then look at the order of the derivative that occurs in the error formula and subtract 1.
  2. If the error formula is not known, appy the quadrature formula to the monomials $1, x, x^2, \ldots$ and see which degree of polynomial it stops being exact for.

- Why are Newton-Cotes formulas of very high precision (i.e. above 4) rarely used?

5.1.6 Open Newton-Cotes Formulas

What is the difference between the closed and open formulas?

- Midpoint formula with exact error term:

$$\int_{x_0}^{x_1} f(x)dx =$$

- How is the error term derived?

- Composite midpoint formula with exact error term:

$$\int_{x_0}^{x_n} f(x)dx =$$

5.2 Adaptive Quadrature

Just know the concept behind this method.

5.3 Gaussian Quadrature (GQ)

General Idea: Find nodes $\{x_j\}_{j=1}^n \subseteq [a,b]$ and weights $w_1, w_2, \ldots, w_n$ such that

$$\int_a^b w(x)p(x)dx = \sum_{j=1}^n w_j p(x_j)$$

where $p(x)$ is any polynomial of degree $\leq 2n-1$, i.e the quadrature formula is exact for all polynomials of degree $\leq 2n-1$.

5.3.1 Derivation of GQ nodes and weights

It is important to know the two approaches we studied for deriving the GQ nodes and weights:

1. Brute-Force technique: Write down the equations that arise based on the requirements of the GQ formula. This leads to a non-linear system to solve.
2. Orthogonal polynomials:

5.4 Trapezoidal Rule Enhancements

5.4.1 Euler-MacLaurin Formula

- Formula:

$$\int_{x_0}^{x_n} f(x)dx =$$

- What does this tell us about the Trapezoidal rule applied to periodic functions?

5.4.2 Gregory's Formula

Not on the final.

5.4.3 Richardson Extrapolation

This is a very important concept. General idea is as follows: Given a quadrature formula $I_n(f)$ of the form

$$I_n(f) = \int_{x_0}^{x_n} f(x)dx + ch^p + O(h^{p+1}),$$

where c is some unknown constant, can we somehow eliminate the leading error term ch^p without knowing c. If we can then the new method becomes $O(h^{p+1})$. You should know how to eliminate this leading error term.

5.4.4 Romberg Integration

Repeatedly apply Richardson extrapolation to Trapezoidal rule to remove the leading error terms. It is important to know how this is done, and how we can combine the terms in a nice table known as the Romberg table.

5.5 Improper Integrals

- What is meant by an improper integral?
- Change of variable technique.
- Gaussian quadrature technique.

5.6 Multidimensional quadrature

Not on the final.

6 Approximation Theory (class notes and Chapter 5.8)

6.1 Weierstrass Theorem

6.2 Discrete Least Squares Approximations

Given sample (x_j, y_j), $j = 1, 2, \ldots, n$, determine the polynomial $p_m(x)$ of degree $\leq m$ that minimizes $\sum_{j=1}^{n} [y_j - p_m(x_j)]^2$.

- Why do we call it *least squares*?
- How do we compute the solution

6.3 Inner Products

Given two functions (vectors) f and g, we denote their weighted inner product as

$$(f, g) = \int_a^b w(x) f(x) g(x) dx \ ,$$

where $w(x)$ is a weight function.

- What is the purpose of the weight function?
- We call $w(x)$ a weight function if it satisfies the following 3 properties:
  1. $w(x) \geq 0 \quad \forall x \in (a, b)$.
  2. $w(x) \neq 0$ on an subinterval of (a, b).
  3. $\int_a^b w(x) dx$ exists.

6.4 Continuous Least Squares Approximations

Given $f \in C[a, b]$, find a polynomial $p^*(x)$ of degree $\leq m$ that minimizes

$$\int_a^b w(x)[f(x) - p(x)]^2 dx \tag{1}$$

among all polynomials $p(x)$ of degree $\leq m$.

- Why do we call it least squares?
- What purpose does the weight function serve in a least squares approximation?
- Know the 4 common weight functions we discussed, and the types of problems they are to be applied to.

6.4.1 Solution to the least squares problem

- Bad way: Use monomials, i.e. define $E_2(a_0, a_1, \ldots, a_m) = \int_a^b w(x)[f(x) - \sum_{j=0}^{m} a_j x^j]$ and solve where $\frac{\partial E}{\partial a_k} = 0$ for $k = 0, 1, \ldots, m$. Why is this a bad way to solve the problem?

- Good way: Use orthogonal polynomials. Let $\{\varphi_j\}_{j=0}^{m}$ be a set of orthogonal polynomials with respect to the weighted inner product (f, g) given above. Then the solution to (1) is given by

$$p^*(x) = \sum_{j=0}^{m} \frac{(f, \varphi_j)}{(\varphi_j, \varphi_j)} \varphi_j(x)$$

6.5 Orthogonal Polynomials

Know the two types of orthogonal polynomials we discussed and how they are computed (see p. 489 Table 6.4 of Bradie).

6.5.1 Legendre Polynomials

Denoted by $P_n(x)$. The following properties are important to know:

1. Weight function:
2. Interval:
3. Inner product:
4. Triple recursion:
5. Least squares approximation solution:
6. General properties of the error associated with this least squares approximation.

6.5.2 Chebyshev Polynomials

Denoted by $T_n(x) = \cos(n \arccos(x))$. The following properties are important to know:

1. Weight function:
2. Interval:
3. Inner product:
4. Triple recursion:
5. Least squares approximation solution:
6. General properties of the error associated with this least squares approximation.