SEMIDUALITIES FROM PRODUCTS OF TREES

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Abstract. Let K be a global function field of characteristic p, and let Γ be a finite-index subgroup of an arithmetic group defined with respect to K and such that any torsion element of Γ is a p-torsion element. We define semiduality groups, and we show that Γ is a $\mathbb{Z}[1/p]$-semiduality group if Γ acts as a lattice on a product of trees. We also give other examples of semiduality groups, including lamplighter groups, Diestel-Leader groups, and countable sums of finite groups.

1. Introduction

1.1. **Arithmetic groups.** Let K be a global field (number or function field), and let S be a nonempty set of finitely many inequivalent valuations of K including each archimedean valuation. The ring $\mathcal{O}_S \subseteq K$ will denote the corresponding ring of S-integers. For any $v \in S$, we let K_v be the completion of K with respect to v so that K_v is a locally compact field.

We let $\mathbf{G}$ be a noncommutative, absolutely almost simple algebraic K-group, so that $\mathbf{G}(\mathcal{O}_S)$ is a lattice, included diagonally, in the product of simple Lie groups $\prod_{v \in S} \mathbf{G}(K_v)$. For each $v \in S$, we let X_v be the symmetric space or Euclidean building (depending on whether K_v is an archimedean or nonarchimedean field) associated with $\mathbf{G}(K_v)$, and we let $X_S = \prod_{v \in S} X_v$ so that $\mathbf{G}(\mathcal{O}_S)$ acts on X_S as a lattice.

We let

$$k(\mathbf{G}, S) = \sum_{v \in S} \mathrm{rank}_{K_v} \mathbf{G}$$

If $\mathbf{G}$ is K-anisotropic — that is, if $\mathbf{G}(\mathcal{O}_S)$ acts cocompactly on X_S — then there is a finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ that is a duality group, and if K_v is an archimedean field — that is, if X_v is a symmetric space — for all $v \in S$, then there is a finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ that is a Poincaré duality group.

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Borel-Serre [5] [6] showed that $\mathbf{G}(\mathcal{O}_S)$ is also a virtual duality group when $\mathbf{G}$ is K-isotropic, as long as K is a number field. In particular, Borel-Serre construct a bordification of X_S, which we denote as $\widehat{X_S}$, on which $\mathbf{G}(K)$ acts and $\mathbf{G}(\mathcal{O}_S)$ acts properly and cocompactly, and such that the compactly supported cohomology groups $H_c^*(\widehat{X_S};\mathbb{Z})$ are nontrivial in some single dimension, $\ell(\mathbf{G},S)$. The result is that any finite-index torsion-free subgroup of $\mathbf{G}(\mathcal{O}_S)$ is a duality group of dimension $\ell(\mathbf{G},S)$ with dualizing module $H_c^{\ell(\mathbf{G},S)}(\widehat{X_S};\mathbb{Z})$.

The purpose of this paper is to suggest a possible analogue of Borel-Serre for arithmetic groups $\mathbf{G}(\mathcal{O}_S)$ when K is a global function field.

1.2. **Function field case.** Throughout the remainder of this paper, K denotes a global function field of characteristic p, and we suppose that $\mathbf{G}$ is K-isotropic — that is, that $\mathbf{G}(\mathcal{O}_S)$ does not act cocompactly on X_S.

Any finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ contains torsion, so it cannot be a duality group, as duality groups have finite cohomological dimension. However, there are finite-index subgroups of $\mathbf{G}(\mathcal{O}_S)$ whose only torsion elements are p-elements and whose cohomological dimension over $\mathbb{Z}[1/p]$ is bounded above by $k(\mathbf{G},S)$. We let Γ denote such a subgroup.

The group Γ still has an obstruction to being a $\mathbb{Z}[1/p]$-duality group. Indeed, it is not of type $FP_{k(\mathbf{G},S)}$ over $\mathbb{Z}[1/p]$ (see Kropholler [15], Bux-Wortman [11], Gandini [13], and Bux-Köhl-Witzel [10]). However, Γ is of type $FP_{k(\mathbf{G},S)-1}$, and we conjecture that the discrepancy between type $FP_{k(\mathbf{G},S)}$ and $FP_{k(\mathbf{G},S)-1}$ is the only, and in some ways a minor, obstruction to Γ being a $\mathbb{Z}[1/p]$-duality group. Before making this precise, we'll need a definition.

For a commutative ring R, we say that a group Λ is an R-semiduality group of dimension d if

(*i*) $\mathrm{cd}_R(\Lambda) \leq d$,
(*ii*) Λ is of type FP_{d-1} over R,
(*iii*) $H^k(\Lambda; R\Lambda) = 0$ if $k \neq d$, and
(*iv*) $H^d(\Lambda; R\Lambda)$ is a flat R-module.

In the above definition, $H^d(\Lambda; R\Lambda)$ is called the *dualizing module*, and if the ring R and the group Λ are understood, then we'll often denote the dualizing module simply as D.

In Section 2 of this paper we'll show the following consequence of a group being a semiduality group.

Proposition 1. *If Λ is an R-semiduality group of dimension d, then for any $0 \leq n \leq d$ and any left $R\Lambda$-module M, there are natural* *homomorphisms of R-modules*

$$\varphi_n^M : H_n(\Lambda; D \otimes_R M) \to H^{d-n}(\Lambda; M)$$

The φ_n^M are compatible with the connecting homomorphisms in the long exact homology and cohomology sequences associated to a short exact sequence of coefficient modules, and if $\cdots \to Q_1 \to Q_0 \to M \to 0$ is a projective resolution of M by left $R\Lambda$-modules, then φ_n^M is injective if Q_{n+1} and Q_n are finitely generated, and surjective if Q_n and Q_{n-1} are finitely generated. By convention, Q_{-1} is always finitely generated.

With the definition of semiduality and its immediate consequences listed above, we propose the following

Conjecture 2. *Let $\mathcal{O}_S$ be a ring of S-integers in a global function field K of characterstic p, and let $\mathbf{G}$ be a noncommutative, absolutely almost simple algebraic K-group. If Γ is a finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ such that any torsion element of Γ is a p-element, then Γ is a $\mathbb{Z}[1/p]$-semiduality group of dimension $k(\mathbf{G}, S)$, and the dualizing module admits an action by $\mathbf{G}(K)$.*

Note that Bux-Köhl-Witzel [10] shows that any Γ as in Conjecture 2 is of type $FP_{k(\mathbf{G},S)-1}$ over $\mathbb{Z}[1/p]$, and it is well-known that $\mathrm{cd}_{\mathbb{Z}[1/p]} \Gamma \leq k(\mathbf{G}, S)$ since the dimension of X_S equals $k(\mathbf{G}, S)$ (see Lemma 43 below). Therefore, proving Conjecture 2 would amount to proving that $H^m(\Gamma; \mathbb{Z}[1/p]\Gamma) = 0$ if $m < k(\mathbf{G}, S)$, and that $D = H^{k(\mathbf{G},S)}(\Gamma; \mathbb{Z}[1/p]\Gamma)$ is flat as a $\mathbb{Z}[1/p]$-module, the latter condition being equivalent to D being torsion-free since $\mathbb{Z}[1/p]$ is a principal ideal domain.

Furthermore, we conjecture that D contains, and is an inverse limit of quotients of, $H_c^{k(\mathbf{G},S)}(X_S; \mathbb{Z}[1/p])$. That is, we can view D as an augmentation of $H_c^{k(\mathbf{G},S)}(X_S; \mathbb{Z}[1/p])$. Thus, whereas Borel-Serre exhibits duality groups whose dualizing modules are cohomology groups of augmentations of the spaces on which arithmetic groups act, we conjecture that over function fields, arithmetic groups are semiduality groups whose dualizing modules are augmentations of cohomology groups of spaces on which the arithmetic groups act.

As an illustration, let L be a field whose characteristic is not equal to p. Recall that $\mathrm{cd}_L(\Gamma) \leq k(\mathbf{G}, S)$. By Bux-Köhl-Witzel [10], L is of type $FP_{k(\mathbf{G},S)-1}$ as a $\mathbb{Z}[1/p]\Gamma$-module. Therefore if Conjecture 2 is true, then $H^1(\Gamma; L)$ is a quotient of $H_{k(\mathbf{G},S)-1}(\Gamma; D)$, and if $2 \leq n \leq k(\mathbf{G}, S)$ then

$$H_{k(\mathbf{G},S)-n}(\Gamma; D) \cong H^n(\Gamma; L)$$

Note that the only dimension of $H^*(\Gamma; L)$ which semiduality would not be able to help determine is dimension 0, but we know $H^0(\Gamma; L) = L$.

1.3. **Main result.** What we prove in this paper is a first case of Conjecture 2. Namely

Theorem 3. *Conjecture 2 is true if* $\mathrm{rank}_{K_v}\mathbf{G} = 1$ *for all* $v \in S$. *In particular, if* $\mathbf{P}$ *is a proper* K*-parabolic subgroup of* $\mathbf{G}$*, then there is an exact sequence of* $\mathbb{Z}[1/p]\mathbf{G}(K)$*-modules*

$$0 \longrightarrow H_c^{k(\mathbf{G},S)}(X_S;\mathbb{Z}[1/p]) \longrightarrow D \longrightarrow \bigoplus_{z\in(\mathbf{G}/\mathbf{P})(K)} M_z \longrightarrow 0$$

where M_z *is an uncountable* $\mathbb{Z}[1/p]$*-module for each* $z \in (\mathbf{G}/\mathbf{P})(K)$, $M_z \cong M_w$ *as* $\mathbb{Z}[1/p]$*-modules for any* $z, w \in (\mathbf{G}/\mathbf{P})(K)$*, and* $g(M_z) = M_{gz}$ *for all* $g \in \mathbf{G}(K)$ *and* $z \in (\mathbf{G}/\mathbf{P})(K)$.

For example, $\mathbf{SL_2}(\mathbb{F}_p[t])$ is a semiduality group of dimension 1, $\mathbf{SL_2}(\mathbb{F}_p[t,t^{-1}])$ is a semiduality group of dimension 2, and $\mathbf{SL_2}(\mathcal{O}_S)$ is a semiduality group of dimension $|S|$ whose dualizing module incorporates the action of $\mathbf{SL_2}(K)$ on $\mathbb{P}^1(K)$.

Our proof of Theorem 3 is geometric. That is, we will use strongly that, under the hypotheses of Theorem 3, X_S is a product of trees.

1.4. **Solvable groups.** Let $\mathbf{B_2}$ be the group of upper triangular matrices of determinant 1. Thus, $\mathbf{B_2}(\mathbb{F}_p[t])$ is commensurable to $\mathbb{F}_p[t]$, and $\mathbf{B_2}(\mathbb{F}_p[t,t^{-1}])$ is commensurable to the lamplighter group $\mathbb{F}_p \wr \mathbb{Z}$. This paper will also show

Theorem 4. $\mathbf{B_2}(\mathcal{O}_S)$ *is virtually a* $\mathbb{Z}[1/p]$*-semiduality group of dimension* $|S|$.

Thus, as the discrete group Solv is known to be a Poincaré duality group, and as the solvable Baumslag-Solitar groups are known to be duality groups, the lamplighter groups with prime order cyclic base are semiduality groups. Notice that the three groups from the previous sentence are commensurable respectively to $\mathbf{B_2}(\mathbb{Z}[\sqrt{2}])$, $\mathbf{B_2}(\mathbb{Z}[1/p])$, and $\mathbf{B_2}(\mathbb{F}_p[t,t^{-1}])$.

We also show that certain generalizations of $\mathbf{B_2}(\mathbb{F}_p[t])$ and $\mathbf{B_2}(\mathbb{F}_p[t,t^{-1}])$ are semiduality groups, namely countable sums of finite groups and Diestel-Leader groups, respectively.

1.5. **Outline of proof.** In Section 2 we'll prove Proposition 1. In Section 3 we'll show how the cohomology of a discrete group with group ring coefficients can be, in some cases, interpreted from the topology of a contractible space on which it acts properly, and perhaps noncocompactly. In Section 4 we'll detail how the groups $\mathbf{G}(\mathcal{O}_S)$ from Theorem 3 act cocompactly on the complement of a pairwise disjoint collection of horoballs in a product of trees, and in Section 5 we'll show that such a complement has trivial compactly supported cohomology in dimension $d-1$, where d is the number of factors in the product. Section 6 shows that $\varprojlim^1$ of the compactly supported cohomology of a nested sequence of regular horospheres in a product of trees is torsion-free in dimension $d-1$, and the final section of this paper, Section 7, will combine the ingredients collected in earlier sections to prove that certain groups are semiduality groups, including a proof of Theorem 3.

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2. Homological Algebra

This section is divided in two parts. First, we'll prove Proposition 1 from the introduction. Second, we'll prove an equivalent characterization of semiduality groups in the form of Proposition 11. While Proposition 11 is not directly applied in this paper to other results, it seems to be of independent interest. Both the statement and proof of Proposition 11 were provided to us by an anonymous referee to whom we are grateful.

In this section we let R be a commutative ring.

2.1. **Proof of Proposition 1.** We'll prove Proposition 1 in four steps. First, we'll define φ_n^M. Second, we'll show how the injectivity and surjectivity of φ_n^M can be deduced from the finiteness properties of M. Third, we'll demonstrate the required naturality properties of φ_n^M. Last, and not until the final sentence of Section 2.1, we'll invoke the assumption from Proposition 1 that $H^d(\Gamma; R\Gamma)$ is a flat R-module.

To define φ_n^M, let Γ be a group of type FP_{d-1} over R with $\mathrm{cd}_R\,\Gamma \leq d$. Then there is a projective resolution of the trivial left $R\Gamma$-module R by left $R\Gamma$-modules

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to R \to 0$$

where P_i is finitely generated if $i \leq d-1$.

We let $P_i^* = \mathrm{Hom}_{R\Gamma}(P_i, R\Gamma)$ and $D = H^d(\Gamma; R\Gamma)$. Assuming that $H^k(\Gamma; R\Gamma) = 0$ if $k \neq d$, we have the below exact sequence of right $R\Gamma$-modules

$$0 \to P_0^* \to P_1^* \to \cdots \to P_{d-1}^* \to P_d^* \to D \to 0$$

Let $\cdots \to A_1 \to A_0 \to D \to 0$ be a projective resolution of D by right $R\Gamma$-modules. Let $h : A_\bullet \to P_{d-\bullet}^*$ be a chain map over the identity map on D. Thus, for any left $R\Gamma$-module M, there are chain maps

$$A_\bullet \otimes_{R\Gamma} M \to P_{d-\bullet}^* \otimes_{R\Gamma} M \to \mathrm{Hom}_{R\Gamma}(P_{d-\bullet}, M)$$

that induce natural homomorphisms

$$\mathrm{Tor}_n^{R\Gamma}(D, M) \xrightarrow{\kappa_n^M} H_n(P_{d-\bullet}^* \otimes_{R\Gamma} M) \xrightarrow{\nu_n^M} H^{d-n}(\Gamma; M)$$

We define $\varphi_n^M = \nu_n^M \circ \kappa_n^M$.

To deduce when φ_n^M is injective or surjective, we'll deduce when those properties are satisfied by κ_n^M and ν_n^M separately, beginning with κ_n^M. But before that, we'll need the following

Lemma 5. *Let P be a projective, left $R\Gamma$-module, and P^* its dual. If $\cdots \to Q_1 \to Q_0 \to M \to 0$ is a projective resolution of a left $R\Gamma$-module M, and if Q_{n+1} and Q_n are finitely generated, then $\mathrm{Tor}_n^{R\Gamma}(P^*, M) = 0$.*

Proof. The following diagram commutes

$$\begin{array}{ccccc} P^* \otimes_{R\Gamma} Q_{n+1} & \longrightarrow & P^* \otimes_{R\Gamma} Q_n & \longrightarrow & P^* \otimes_{R\Gamma} Q_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{R\Gamma}(P, Q_{n+1}) & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P, Q_n) & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P, Q_{n-1}) \end{array}$$

Since Q_{n+1} and Q_n are finitely generated projective, they are finitely presented, so the two vertical maps on the left are isomorphisms. It follows that the homology of the top row injects into the homology of the bottom row. That is, $\mathrm{Tor}_n^{R\Gamma}(P^*, M)$ injects into 0, since P is projective, and thus $\mathrm{Hom}(P, -)$ is exact. □

Now we are prepared to address the injectivity and surjectivity of the homomorphisms $\kappa_n^M : \mathrm{Tor}_n^{R\Gamma}(D, M) \to H_n(P_{d-\bullet}^* \otimes_{R\Gamma} M)$.

Lemma 6. *Let $\cdots \to Q_1 \to Q_0 \to M \to 0$ be a projective resolution of a left $R\Gamma$-module M. Then κ_0^M is an isomorphism, κ_1^M is surjective, κ_n^M is injective if Q_{n+1} and Q_n are finitely generated, and κ_n^M is surjective if Q_n and Q_{n-1} are finitely generated.*

Proof. Recall that $A_\bullet \to D$ is a projective resolution of D with a chain map $h : A_\bullet \to P_{d-\bullet}^*$ over the identity on D, so that h is a quasi-isomorphism. We let $P_{d-\geq 1}^*$ be the truncated complex of $P_{d-\bullet}^*$, so that the bottom row of the below diagram of chain maps is exact.

$$\begin{array}{ccccccccc} & & & & A_\bullet & & & & \\ & & & & \downarrow h & & & & \\ 0 & \longrightarrow & P^*_d & \longrightarrow & P^*_{d-\bullet} & \longrightarrow & P^*_{d-\geq 1} & \longrightarrow & 0 \end{array}$$

Because each Q_n is projective, the below diagram of bicomplexes has an exact row and $(h \otimes id)_*$ is a quasi-isomorphism.

$$\begin{array}{ccccccccc} & & & & A_\bullet \otimes_{R\Gamma} Q_\bullet & & & & \\ & & & & \downarrow (h\otimes id)_* & & & & \\ 0 & \longrightarrow & P^*_d \otimes_{R\Gamma} Q_\bullet & \longrightarrow & P^*_{d-\bullet} \otimes_{R\Gamma} Q_\bullet & \longrightarrow & P^*_{d-\geq 1} \otimes_{R\Gamma} Q_\bullet & \longrightarrow & 0 \end{array}$$

Passing to homology, we have the following diagram whose row is exact and whose vertical map, $(h \otimes id)_*$, is an isomorphism.

$$\begin{array}{ccccccc} & & \mathrm{Tor}_n^{R\Gamma}(D, M) & & & & \\ & & \downarrow (h\otimes id)_* & & & & \\ \mathrm{Tor}_n^{R\Gamma}(P^*_d, M) & \longrightarrow & H_n(P^*_{d-\bullet} \otimes_{R\Gamma} Q_\bullet) & \xrightarrow{\iota_*} & H_n(P^*_{d-\geq 1} \otimes_{R\Gamma} Q_\bullet) & \xrightarrow{\partial} & \mathrm{Tor}_{n-1}^{R\Gamma}(P^*_d, M) \end{array}$$

Notice that the domain of ∂ is $H_n(P^*_{d-\geq 1} \otimes_{R\Gamma} M)$ since $P^*_{d-\geq 1}$ is a deleted projective resolution.

For $n \geq 2$, we have $H_n(P^*_{d-\geq 1} \otimes_{R\Gamma} M) = H_n(P^*_{d-\bullet} \otimes_{R\Gamma} M)$ and $\kappa_n^M = \iota_* \circ (h \otimes id)_*$. Thus, when $n \geq 2$, the lemma follows from Lemma 5.

For $n = 1$, we have that $\mathrm{Tor}_{n-1}^{R\Gamma}(P^*_d, M) = P^*_d \otimes_{R\Gamma} M$ and thus the image of ι_*, which is the kernel of ∂, is $H_1(P^*_{d-\bullet} \otimes_{R\Gamma} M)$.

For $n = 0$, notice that κ_0^M is the identity map on $D \otimes_{R\Gamma} M$. □

For $\nu_n^M : H_n(P^*_{d-\bullet} \otimes_{R\Gamma} M) \to H^{d-n}(\Gamma; M)$ we have the following

Lemma 7. *Let M be a left $R\Gamma$-module M. Then ν_1^M is injective, and ν_n^M is bijective if $n \geq 2$. If M is finitely presented, then ν_0^M and ν_1^M are bijective. If M is finitely generated, then ν_0^M is surjective.*

Proof. If $n \geq 2$, then we see that the vertical maps in the below commutative diagram are isomorphisms since P_{d-i} is finitely generated and projective if $i \geq 1$.

$$\begin{array}{ccccc} P^*_{d-(n+1)} \otimes_{R\Gamma} M & \longrightarrow & P^*_{d-n} \otimes_{R\Gamma} M & \longrightarrow & P^*_{d-(n-1)} \otimes_{R\Gamma} M \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{R\Gamma}(P_{d-(n+1)}, M) & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P_{d-n}, M) & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P_{d-(n-1)}, M) \end{array}$$

Therefore, ν_n^M is an isomorphism if $n \geq 2$.

For $n = 1$, observe the below commutative diagram

$$\begin{array}{ccccc} P_{d-2}^* \otimes_{R\Gamma} M & \longrightarrow & P_{d-1}^* \otimes_{R\Gamma} M & \longrightarrow & P_d^* \otimes_{R\Gamma} M \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{R\Gamma}(P_{d-2}, M) & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P_{d-1}, M) & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P_d, M) \end{array}$$

Since P_{d-2} and P_{d-1} are finitely generated and projective, the two vertical maps on the left are isomorphisms so ν_1^M is injective. If, in addition, M is finitely presented, then the vertical map on the right is bijective, so ν_1^M is bijective.

As for ν_0^M, the sequence

$$\mathrm{Hom}_{R\Gamma}(P_{d-1}, M) \to \mathrm{Hom}_{R\Gamma}(P_d, M) \to H^d(\Gamma; M) \to 0$$

is exact by the definition of $H^d(\Gamma; M)$. Using the above for M and $M = R\Gamma$, and using that tensor product is right exact, we see that the rows of the commutative diagram

$$\begin{array}{ccccccc} P_{d-1}^* \otimes_{R\Gamma} M & \longrightarrow & P_d^* \otimes_{R\Gamma} M & \longrightarrow & D \otimes_{R\Gamma} M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}_{R\Gamma}(P_{d-1}, M) & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P_d, M) & \longrightarrow & H^d(\Gamma; M) & \longrightarrow & 0 \end{array}$$

are exact.

The vertical map on the left is an isomorphism since P_{d-1} is finitely generated and projective. Thus, ν_0^M is surjective (resp. bijective) if the second vertical map from the left is by the 5-Lemma. Now note that the second vertical map on the left is surjective if M is finitely generated, and bijective if M is finitely presented. □

We now have

Proposition 8. *Suppose Γ is a group of type FP_{d-1} over R, that $\mathrm{cd}_R(\Gamma) \leq d$, and that $H^k(\Gamma; R\Gamma)$ if $k \neq d$. Let $D = H^d(\Gamma; R\Gamma)$.*

If $\cdots \to Q_1 \to Q_0 \to M \to 0$ is a projective resolution of a left $R\Gamma$-module M, then

$$\varphi_n^M : \mathrm{Tor}_n^{R\Gamma}(D, M) \to H^{d-n}(\Gamma; M)$$

is injective if Q_{n+1} and Q_n are finitely generated, and surjective if Q_n and Q_{n-1} are finitely generated.

Proof. Combine Lemmas 6 and 7. □

Having established conditions for the injectivity and surjectivity of the φ_n^M, we turn to naturality properties of these homomorphisms, beginning with

Lemma 9. *For a left $R\Gamma$-module N and an exact sequence of left $R\Gamma$-modules $0 \to M \to M' \to M'' \to 0$, the composition*

$$\mathrm{Tor}_1^{R\Gamma}(N^*, M'') \to N^* \otimes_{R\Gamma} M \to \mathrm{Hom}_{R\Gamma}(N, M)$$

is 0*, where $N^* = \mathrm{Hom}_{R\Gamma}(N, R\Gamma)$.*

Proof. The proof follows from the below commutative diagram with exact rows

$$\begin{array}{ccccc} \mathrm{Tor}_1^{R\Gamma}(N^*, M'') & \longrightarrow & N^* \otimes_{R\Gamma} M & \longrightarrow & N^* \otimes_{R\Gamma} M' \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_{R\Gamma}(N, M) & \longrightarrow & \mathrm{Hom}_{R\Gamma}(N, M') \end{array}$$

□

We will use the previous lemma to construct a diagram of chain complexes in our proof of the following

Proposition 10. *Suppose Γ is a group of type FP_{d-1} over R, that $\mathrm{cd}_R(\Gamma) \leq d$, and that $H^k(\Gamma; R\Gamma)$ if $k \neq d$. Let $D = H^d(\Gamma; R\Gamma)$.*

For any left $R\Gamma$-module M, the R-module homomorphisms

$$\varphi_n^M : \mathrm{Tor}_n^{R\Gamma}(D, M) \to H^{d-n}(\Gamma; M)$$

are natural, and they are compatible with the connecting homomorphisms in the long exact homology and cohomology sequences associated to a short exact sequence of coefficient modules.

Proof. That the φ_n^M are natural follows from their definition, since homology is a functor.

To show that the φ_n^M are compatible with connecting homomorphisms, we let $0 \to M \to M' \to M'' \to 0$ be a short exact sequence of left $R\Gamma$-modules, and we let J be the image of $\mathrm{Tor}_1^{R\Gamma}(P_d^*, M'') \to P_d^* \otimes_{R\Gamma} M$, so that we have an exact sequence

$$0 \to (P_d^* \otimes_{R\Gamma} M)/J \to P_d^* \otimes_{R\Gamma} M' \to P_d^* \otimes_{R\Gamma} M'' \to 0$$

Furthermore, if $i \geq 1$, then P_{d-i} is finitely generated so P_{d-i}^* is projective, and thus, if we let $(P_{d-i}^* \otimes_{R\Gamma} M)_J = P_{d-i}^* \otimes_{R\Gamma} M$ for $i \geq 1$ and $(P_d^* \otimes_{R\Gamma} M)_J = (P_d^* \otimes_{R\Gamma} M)/J$, then we have a short exact sequence of chain complexes

$$0 \to (P_{d-\bullet}^* \otimes_{R\Gamma} M)_J \to P_{d-\bullet}^* \otimes_{R\Gamma} M' \to P_{d-\bullet}^* \otimes_{R\Gamma} M'' \to 0$$

By the previous lemma, J is contained in the kernel of $P_d^* \otimes_{R\Gamma} M \to \mathrm{Hom}_{R\Gamma}(P_d, M)$ so the composite

$$A_0 \otimes_{R\Gamma} M \to P_d^* \otimes_{R\Gamma} M \to \mathrm{Hom}_{R\Gamma}(P_d, M)$$

factors through $(P^*_{d-\bullet} \otimes_{R\Gamma} M)/J$. Thus, we have a commutative diagram of chain complexes with exact columns

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
A_\bullet \otimes_{R\Gamma} M & \longrightarrow & (P^*_{d-\bullet} \otimes_{R\Gamma} M)_J & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P_{d-\bullet}, M) \\
\downarrow & & \downarrow & & \downarrow \\
A_\bullet \otimes_{R\Gamma} M' & \longrightarrow & P^*_{d-\bullet} \otimes_{R\Gamma} M' & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P_{d-\bullet}, M') \\
\downarrow & & \downarrow & & \downarrow \\
A_\bullet \otimes_{R\Gamma} M'' & \longrightarrow & P^*_{d-\bullet} \otimes_{R\Gamma} M'' & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P_{d-\bullet}, M'') \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

Therefore, for all n, we have the following commutative diagram on homology, whose vertical maps are connecting homomorphisms

$$\begin{array}{ccccc}
\mathrm{Tor}^{R\Gamma}_n(D, M'') & \longrightarrow & H_n(P^*_{d-\bullet} \otimes_{R\Gamma} M'') & \longrightarrow & H^{d-n}(\Gamma; M'') \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Tor}^{R\Gamma}_{n-1}(D, M) & \longrightarrow & H_{n-1}((P^*_{d-\bullet} \otimes_{R\Gamma} M)_J) & \longrightarrow & H^{d-n+1}(\Gamma; M)
\end{array}$$

The top row of the above diagram is $\varphi^{M''}_n$. The bottom row is φ^M_{n-1} as can be seen by noting that the below commutative diagram of chain maps

$$\begin{array}{ccccc}
 & \nearrow & P^*_{d-\bullet} \otimes_{R\Gamma} M & \searrow & \\
 & & \downarrow & & \\
A_\bullet \otimes_{R\Gamma} M & \longrightarrow & (P^*_{d-\bullet} \otimes_{R\Gamma} M)_J & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P_{d-\bullet}, M)
\end{array}$$

yields the commutative diagram on homology

$$\begin{array}{ccccc}
 & \nearrow & H_{n-1}(P^*_{d-\bullet} \otimes_{R\Gamma} M) & \searrow & \\
 & & \downarrow & & \\
\mathrm{Tor}^{R\Gamma}_{n-1}(D, M) & \longrightarrow & H_{n-1}((P^*_{d-\bullet} \otimes_{R\Gamma} M)_J) & \longrightarrow & H^{d-n+1}(\Gamma; M)
\end{array}$$

and noticing that the top row of the above diagram is φ_{n-1}^M while the bottom row coincides with the bottom row of the preceding commutative rectangle. □

Last, if we assume that D is flat as an R-module, so that $\mathrm{Tor}_n^{R\Gamma}(D,M) \cong H_n(\Gamma; D \otimes_R M)$, then Proposition 1 is Propositions 8 and 10.

2.2. **Alternative characterization of semiduality.** In what remains of this section, we'll prove the following result, given to us by a referee.

Proposition 11. *Let* Γ *be a group, and suppose* $\mathrm{cd}_R \Gamma = d$. *The following are equivalent:*

(i) Γ *is of type* FP_{d-1} *over* R, *and* $H^k(\Gamma; R\Gamma) = 0$ *if* $k \neq d$.

(ii) *There is a right* $R\Gamma$*-module* C *such that the natural map* $C \to C^{**}$ *is an injection, and such that if* $0 \leq n \leq d-2$ *then there are natural isomorphisms*

$$\xi_n : \mathrm{Tor}_{d-1-n}^{R\Gamma}(C, M) \to H^n(\Gamma; M)$$

and

$$\psi_n : H_n(\Gamma; N) \to \mathrm{Ext}_{R\Gamma}^{d-1-n}(C, N)$$

for left $R\Gamma$*-modules* M *and right* $R\Gamma$*-modules* N, *and such that there is an exact sequence*

$$0 \to H^{d-1}(\Gamma; M) \to C \otimes_{R\Gamma} M \to \mathrm{Hom}_{R\Gamma}(C^*, M)$$

(iii) *all of the conditions in* (ii) *hold. In addition,* $C^{**}/C \cong H^d(\Gamma; R\Gamma)$, *the exact sequence from* (ii) *extends to an exact sequence*

$$0 \to H^{d-1}(\Gamma; M) \to C \otimes_{R\Gamma} M \to \mathrm{Hom}_{R\Gamma}(C^*, M) \to H^d(\Gamma, M) \to 0$$

and there is an exact sequence

$$0 \to H_d(\Gamma; N) \to B \otimes_{R\Gamma} C^* \to \mathrm{Hom}_{R\Gamma}(C, N) \to H_{d-1}(\Gamma; N) \to 0$$

Proof. Since (iii) is a strictly larger collection of conditions than (ii), (iii) implies (ii), so in this proof we'll show that (i) implies (iii) and that (ii) implies (i). We begin with (i) implies (iii).

As in Section 2.1, there is a projective resolution of the trivial left $R\Gamma$-module R

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to R \to 0$$

where P_i is finitely generated if $i \leq d-1$, and we let C be the image of $P_{d-1}^* \to P_d^*$. The map ξ_n is what we had previously named ν_n^M in the proof of Proposition 10. The proof of the existence of the ψ_n is similar. Indeed, since the P_i are finitely generated projective if $i \leq d-1$, the vertical maps of the commutative diagram below are isomorphisms.

$$\begin{array}{ccccccc} N \otimes_{R\Gamma} P_{d-1} & \longrightarrow & \cdots & \longrightarrow & N \otimes_{R\Gamma} P_0 & \longrightarrow & 0 \\ \downarrow & & & & \downarrow & & \\ \mathrm{Hom}_{R\Gamma}(P_{d-1}^*, N) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_{R\Gamma}(P_0^*, N) & \longrightarrow & 0 \end{array}$$

The isomorphisms ψ_n are the isomorphisms from the homologies of the two rows.

Now note that because $\mathrm{Hom}_{R\Gamma}(-, R\Gamma)$ is left-exact, the bottom row is exact in the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_d & \longrightarrow & P_{d-1} & \longrightarrow & P_{d-2} \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C^* & \longrightarrow & P_{d-1}^{**} & \longrightarrow & P_{d-2}^{**} \end{array}$$

The top row is exact by stipulation, and since P_{d-1} and P_{d-2} are finitely generated projective, the natural vertical maps are isomorphisms. It follows that $C^* \cong P_d$.

Applying the left-exact $\mathrm{Hom}_{R\Gamma}(-, N)$ to $P_{d-2}^* \to P_{d-1}^* \to C \to 0$ yields the exact sequence

$$0 \to \mathrm{Hom}_{R\Gamma}(C, N) \to \mathrm{Hom}_{R\Gamma}(P_{d-1}^*, N) \to \mathrm{Hom}_{R\Gamma}(P_{d-2}^*, N)$$

The isomorphism $N \otimes_{R\Gamma} P_i \cong \mathrm{Hom}_{R\Gamma}(P_i^*, N)$ when $i = d-1, d-2$ implies $\mathrm{Hom}_{R\Gamma}(C, N)$ is the kernel of $N \otimes_{R\Gamma} P_{d-1} \to N \otimes_{R\Gamma} P_{d-2}$. Therefore we have exact sequences

$$N \otimes_{R\Gamma} P_d \to \mathrm{Hom}_{R\Gamma}(C, N) \to H_{d-1}(\Gamma; N) \to 0$$

and

$$0 \to H_d(\Gamma; N) \to N \otimes_{R\Gamma} P_d \to \mathrm{Hom}_{R\Gamma}(C, N)$$

Combining these two sequences and replacing P_d with C^* yields the exact sequence

$$0 \to H_d(\Gamma; N) \to N \otimes_{R\Gamma} C^* \to \mathrm{Hom}_{R\Gamma}(C, N) \to H_{d-1}(\Gamma; N) \to 0$$

Similarly, we apply the right-exact $- \otimes_{R\Gamma} M$ to $P_{d-2}^* \to P_{d-1}^* \to C \to 0$ for the exact sequence $P_{d-2}^* \otimes_{R\Gamma} M \to P_{d-1}^* \otimes_{R\Gamma} M \to C \otimes_{R\Gamma} M \to 0$. When $i = d-1, d-2$, we have $P_i^* \otimes_{R\Gamma} M \cong \mathrm{Hom}_{R\Gamma}(P_i, M)$. Hence, $C \otimes_{R\Gamma} M$ is the cokernel of $\mathrm{Hom}_{R\Gamma}(P_{d-2}, M) \to \mathrm{Hom}_{R\Gamma}(P_{d-1}, M)$ so we have the exact sequences

$$0 \to H^{d-1}(\Gamma; M) \to C \otimes_{R\Gamma} M \to \mathrm{Hom}_{R\Gamma}(P_d, M)$$

and

$$C \otimes_{R\Gamma} M \to \mathrm{Hom}_{R\Gamma}(P_d, M) \to H^d(\Gamma, M) \to 0$$

Therefore, the below is exact

$$0 \to H^{d-1}(\Gamma; M) \to C \otimes_{R\Gamma} M \to \mathrm{Hom}_{R\Gamma}(C^*, M) \to H^d(\Gamma, M) \to 0$$

The final conditions of (iii) to be checked are that $0 \to C \to C^{**} \to H^d(\Gamma, R\Gamma) \to 0$ is exact. For this, let $M = R\Gamma$ in the sequence immediately preceding this paragraph. Our proof that (i) implies (iii) is complete.

To show that (ii) implies (i), note that the isomorphisms $\psi_n : H_n(\Gamma; N) \to \mathrm{Ext}^{d-n+1}_{R\Gamma}(C, N)$ for $0 \leq n \leq d-2$ and any right $R\Gamma$-module N show that $H_n(\Gamma; -)$ commutes with direct products when $0 \leq n \leq d-2$. In particular, $H_n(\Gamma; \prod R\Gamma) \cong \prod H_n(\Gamma; R\Gamma)$ for $0 \leq n \leq d-2$. That is, for $n = 0$ we have $R \otimes_{R\Gamma} (\prod R\Gamma) \cong \prod R$ and for $1 \leq n \leq d-2$ we have $\mathrm{Tor}^{R\Gamma}_n(R; \prod R\Gamma) = 0$, so that Γ is of type FP_{d-1} by Lemma 1.1 and Proposition 1.2 of [4].

For $0 \leq n \leq d-2$, the existence of isomorphisms $\xi_n : \mathrm{Tor}^{R\Gamma}_{d-1-n}(C, R\Gamma) \to H^n(\Gamma; R\Gamma)$ shows that $H^n(\Gamma; R\Gamma) = 0$ since $R\Gamma$ is free. That $H^{d-1}(\Gamma; R\Gamma) = 0$ follows by recalling that C injects into C^{**} and letting $M = R\Gamma$ in the exact sequence

$$0 \to H^{d-1}(\Gamma; M) \to C \otimes_{R\Gamma} M \to \mathrm{Hom}_{R\Gamma}(C^*, M)$$

□

3. Translation from topology

3.1. **Cohomology compactly supported over each compact subcomplex.** If a group Γ has a finite Eilenberg-Maclane complex $X = K(\Gamma, 1)$ with universal cover $\tilde{X}$ then for any ring R there is an isomorphism $H^*(\Gamma; R\Gamma) = H^*_c(\tilde{X}; R)$. In this section we provide an alternative topological characterization of $H^*(\Gamma, R\Gamma)$ in the case that X is not finite. Our proof uses standard techniques, which we include for completeness.

Suppose X is a locally finite cell complex with an action by a group Γ, and let $\pi : X \to \Gamma\backslash X$ denote the quotient map. Let $C^*(X; R)$ denote the cellular cochain complex of X with coefficients in a ring R. Define a subcomplex $C^k_{cc}(X; R) \leq C^k(X; R)$ to contain cochains $\phi \in C^k(X; R)$ such that for every k-cell $\sigma \in \Gamma\backslash X$ we have $\phi(\tilde{\sigma}) = 0$ for all but finitely many $\tilde{\sigma} \in \pi^{-1}(\sigma)$. Then $d(C^k_{cc}(X; R)) \subseteq C^{k+1}_{cc}(X; R)$, and we let $H^*_{cc}(X; R)$ be the cohomology of this complex. We suppress the dependence on the action of Γ from the notation.

Proposition 12. *Suppose X is a locally finite, acyclic cell complex and Γ is a group acting on X with cell stabilizers that are finite and* *preserve orientation. Then*

$$H^*(\Gamma; R\Gamma) \cong H^*_{cc}(X; R)$$

Proof. Recall that the equivariant cohomology of the pair (X, Γ) with coefficients in $R\Gamma$ is defined as

$$H^*_\Gamma(X; R\Gamma) = H^*(\Gamma; C^*(X; R\Gamma))$$

There is an isomorphism (cf. [7, VII.7.3, p173])

$$H^*(\Gamma; R\Gamma) \cong H^*_\Gamma(X; R\Gamma)$$

There is a spectral sequence (cf. [7, p169])

$$E_1^{pq} = H^q(\Gamma; C^p(X; R\Gamma)) \implies H^{p+q}_\Gamma(X; R\Gamma).$$

We will show $H^q(\Gamma; C^p(X; R\Gamma)) = 0$ for all $q > 0$. Let X_p denote the set of p-cells in X and let Σ_p be a set of representatives for $\Gamma \backslash X_p$. Letting Γ_σ denote the stabilizer of $\sigma \in \Sigma_p$, there is a decomposition

$$\begin{aligned} C^p(X; R\Gamma) &= \mathrm{Hom}(C_p(X), R\Gamma) \\ &\cong \prod_{\sigma \in X_p} R\Gamma \\ &\cong \prod_{\sigma \in \Sigma_p} \mathrm{Coind}^\Gamma_{\Gamma_\sigma}(R\Gamma). \end{aligned}$$

Therefore there is a decomposition of cohomology

$$\begin{aligned} H^q(\Gamma; C^p(X; R\Gamma)) &\cong H^q\Big(\Gamma; \prod_{\sigma \in \Sigma_p} \mathrm{Coind}^\Gamma_{\Gamma_\sigma}(R\Gamma)\Big) \\ &\cong \prod_{\sigma \in \Sigma_p} H^q(\Gamma; \mathrm{Coind}^\Gamma_{\Gamma_\sigma}(R\Gamma)) \end{aligned}$$

Applying Shapiro's lemma yields

$$H^q(\Gamma; C^p(X; R\Gamma)) \cong \prod_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma; R\Gamma)$$

Because Γ_σ is finite, there is an isomorphism of $R\Gamma_\sigma$-modules $R\Gamma \cong \mathrm{Coind}^{\Gamma_\sigma}_{\{1\}}(\oplus_{\Sigma_p} R)$. Therefore another use of Shapiro's lemma shows that $H^q(\Gamma_\sigma; R\Gamma) = 0$ for $q > 0$. Recall that $H^0(\Gamma_\sigma; R\Gamma) \cong (R\Gamma)^{\Gamma_\sigma}$.

It follows from the above that $H^*(\Gamma; R\Gamma)$ is the cohomology of the cochain complex

$$\prod_{\sigma \in \Sigma_0} (R\Gamma)^{\Gamma_\sigma} \to \prod_{\sigma \in \Sigma_1} (R\Gamma)^{\Gamma_\sigma} \to \prod_{\sigma \in \Sigma_2} (R\Gamma)^{\Gamma_\sigma} \to \cdots \tag{1}$$

We will show this chain complex is isomorphic to the chain complex $\{C^k_{cc}(X,R)\}$. First we compute the coboundary maps of (1). The isomorphism

$$C^p(X;R\Gamma)^\Gamma \to \prod_{\sigma\in\Sigma_p}(R\Gamma)^{\Gamma_\sigma}$$

sends a map $\phi : C_p(X;R) \to R\Gamma$ to the function $\xi : \Sigma_p \to R\Gamma$ defined by $\xi(\sigma) = \phi([\sigma])$ for any p-cell $\sigma \in \Sigma_p$. From Γ-equivariance of ϕ computation shows that the coboundary operator on the complex (1) is given by

$$d\xi(\sigma) = \sum_i r_i\gamma_i\xi(\sigma_i)$$

if $\partial[\sigma] = \sum n_i\gamma_i[\sigma_i]$ for ring elements $r_i \in R$, group elements $\gamma_i \in \Gamma$ and simplices $\sigma_i \in \Sigma_p$.

Define an isomorphism

$$\Theta : \prod_{\sigma\in\Sigma_p}(R\Gamma)^{\Gamma_\sigma} \to C^p_{cc}(X,R)$$

as follows: given $\phi \in \prod_{\sigma\in\Sigma_p}(R\Gamma)^{\Gamma_\sigma}$, for any p-simplex ρ in X choose $\sigma \in \Sigma_p$ and $\gamma \in \Gamma$ such that $\rho = \gamma\sigma$ and set

$$\Theta\phi(\rho) = [\phi(\sigma)]_{\gamma^{-1}}$$

Here $[x]_{\gamma^{-1}}$ denotes the coefficient of $[\gamma^{-1}]$ in the formal sum $x \in R\Gamma$. Note σ is uniquely specified by ρ and γ is unique up to right multiplication by elements of Γ_σ. Because each $\phi(\sigma)$ is Γ_σ-invariant, Θ does not depend on choice of γ. Moreover, any two p-cells in X that belong to the same Γ orbit will correspond to the same cell σ in the above equation. Since there are only finitely many terms in the formal sum $\phi(\sigma)$, the map $\Theta\phi$ is finitely supported above each cell in X. Therefore Θ determines a well-defined homomorphism of Γ-modules.

It is clear that Θ is injective. To see that Θ is surjective, define an inverse Θ^{-1} by setting $[\Theta^{-1}\xi(\sigma)]_\gamma = \xi(\gamma^{-1}\sigma)$. It remains only to see that Θ is compatible with the coboundary maps. Suppose ρ is a $(p+1)$-cell in X and $\partial[\rho] = \sum_i r_i[\delta_i]$ for ring elements $r_i \in R$ and p-cells δ_i. For each i, write $\delta_i = \gamma_i\sigma_i$ for $\gamma_i \in \Gamma$ and $\sigma_i \in \Sigma_p$. Then

$$\begin{aligned} d[\Theta\phi](\rho) &= \Theta\phi(\partial\rho) \\ &= \sum_i r_i[\Theta\phi](\delta_i) \\ &= \sum_i r_i[\phi(\sigma_i)]_{\gamma_i^{-1}}. \end{aligned}$$

On the other hand, note that if $\rho = \gamma\sigma$ for $\gamma \in \Gamma$ and $\sigma \in \Sigma_{p+1}$, then

$$\partial[\sigma] = \sum_i r_i(\gamma^{-1}\gamma_i)[\sigma_i]$$

Therefore

$$\begin{aligned}\Theta(d\phi)(\rho) &= [(d\phi)(\sigma)]_{\gamma^{-1}} \\ &= [\phi(\partial\sigma)]_{\gamma^{-1}} \\ &= \left[\sum_i r_i(\gamma^{-1}\gamma_i)\phi(\sigma_i)\right]_{\gamma^{-1}} \\ &= \sum_i r_i[(\gamma^{-1}\gamma_i)\phi(\sigma_i)]_{\gamma^{-1}} \\ &= \sum_i r_i[\phi(\sigma_i)]_{\gamma_i^{-1}}\end{aligned}$$

Thus Θ commutes with the coboundary operators and hence is an isomorphism of chain complexes. This completes the proof. □

Lemma 13. *Let X and Γ be as in Proposition 12. If G is a locally compact group acting cellulary on X and $\Gamma \leq G$ then $\mathrm{Comm}_G(\Gamma)$ acts on $H^*_{cc}(X;R)$.*

Proof. Given $\phi \in C^k_{cc}(X)$ and $g \in \mathrm{Comm}_G(\Gamma)$, define $(g\phi)(\sigma) = \phi(g^{-1}\sigma)$. The condition that $g\phi \in C^k_{cc}(X)$ is equivalent to the condition that $\mathrm{supp}(g\phi) \cap \Gamma K$ is compact for any compact set $K \subseteq X$, which is equivalent to $\mathrm{supp}(\phi) \cap g^{-1}\Gamma K$ being compact for any compact K. Fix a compact $K \subseteq X$. Because $g \in \mathrm{Comm}_G(\Gamma)$, there is some compact $K' \subseteq X$ such that $g\Gamma K \subseteq \Gamma K'$. Therefore $\mathrm{supp}(\phi) \cap g^{-1}\Gamma K \subseteq \mathrm{supp}(\phi) \cap \Gamma K'$. The latter is compact because $\phi \in C^k_{cc}(X)$, so $g\phi \in C^k_{cc}(X)$. This action commutes with coboundary maps, so induces an action on cohomology. □

3.2. **Computing $H^*_{cc}(X)$.** Let X and Γ be as in Proposition 12. Suppose there are subcomplexes $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X$ such that each X_k is closed and Γ-invariant, each quotient $\Gamma\backslash X_k$ is compact, and $X = \bigcup_i X_i$. The compactly supported cellular cochain complexes $C^*_c(X_n;R)$ form a codirected system under the restriction maps $r^*_{i,j}$ induced by inclusions $r_{i,j} : X_i \to X_j$ for $i < j$.

The chain complex $C^k_{cc}(X;R)$ is the inverse limit of the system of chain complexes $C^k_c(X_n;R)$ and each restriction map $r_{i,j}$ is surjective on the chain level. It follows (see for example the "Variant" following [18, 3.5.8, p84]) that for any k there is a short exact sequence of cohomology

$$0 \to \varprojlim{}^1 H^{k-1}_c(X_n;R) \to H^k_{cc}(X;R) \to \varprojlim H^k_c(X_n;R) \to 0 \tag{2}$$

Recall that $\varprojlim^1 H_c^{k-1}(X_n;R)$ is the cokernel of the map

$$\Delta : \prod_{n=0}^{\infty} H_c^{k-1}(X_n;R) \to \prod_{n=0}^{\infty} H_c^{k-1}(X_n;R)$$

defined by

$$\Delta(x_0, x_1, x_2, \ldots) = (x_o - r_{0,1}(x_1), x_1 - r_{1,2}(x_2), \ldots)$$

As a straightforward application of the short exact sequence (2) we have:

Proposition 14. *Suppose X is a locally finite cell complex with an action of a group Γ. Suppose X is the union of an increasing sequence of Γ-invariant subcomplexes X_n each with cocompact Γ action. If there is some integer d such that $H_c^*(X_n;R)$ is concentrated in dimension d for all n then $H_{cc}^*(X;R)$ is concentrated in dimensions d and $d+1$. In particular, if X is d-dimensional then*

$$H_{cc}^k(X;R) = \begin{cases} \varprojlim_n H_c^d(X_n;R) & k = d \\ 0 & k \neq d. \end{cases}$$

4. Statement of reduction theory

In this section we'll review the necessary results needed from reduction theory for our proof of Theorem 3. The results in this section are not new, and can be derived from Behr [2] and Harder [14], although there are some minor differences between our treatment of reduction theory here and other versions already existing in the literature. A point of difference in the proof of our formulation of these results compared with formulations in other papers, is that we'll use the reduction theory from Bestvina-Eskin-Wortman [3] as an input, which has the advantage, though not directly applied in this paper, of being equally applicable to arithmetic groups defined with respect to a number field. See also Bux-Wortman [12] and Bux-Köhl-Witzel [10].

4.1. **Algebraic form of reduction theory.** In this section and the next we assume that K is a global field with a ring of S-integers $\mathcal{O}_S \leq K$ and that $\mathbf{G}$ is a noncommutative, absolutely almost simple, K-isotropic, K-group with $\mathrm{rank}_{K_v}(\mathbf{G}) = 1$ for all $v \in S$.

Let $\mathbf{P}$ be a proper K-parabolic subgroup of $\mathbf{G}$. Let $\mathbf{A}$ be a maximal K-split torus in $\mathbf{P}$.

From the root system for $(\mathbf{G}, \mathbf{A})$, we denote the simple root for the positive roots with respect to $\mathbf{P}$ by α_0.

We let $\mathbf{Z_G}(\mathbf{A})$ be the centralizer of $\mathbf{A}$ in $\mathbf{G}$ so that $\mathbf{Z_G}(\mathbf{A}) = \mathbf{MA}$ where $\mathbf{M}$ is a reductive K-group with K-anisotropic center. We let $\mathbf{U}$

be the unipotent radical of $\mathbf{P}$, so that $\mathbf{P} = \mathbf{UMA}$. The Levi subgroup $\mathbf{MA}$ normalizes the unipotent radical $\mathbf{U}$, and elements of $\mathbf{A}$ commute with those of $\mathbf{M}$.

We denote the product over S of local points of a K-group by "unbolding", so that, for example,

$$G = \prod_{v \in S} \mathbf{G}(K_v)$$

We let $\mathcal{P}$ be the set of proper K-parabolic subgroups of $\mathbf{G}$. If $\mathbf{Q} \in \mathcal{P}$, then $\mathbf{Q}$ is conjugate in $\mathbf{G}(K)$ to $\mathbf{P}$. We let

$$\Lambda_{\mathbf{Q}} = \{\, \gamma f \in \mathbf{G}(\mathcal{O}_S)F \mid (\gamma f)\mathbf{P}(\gamma f)^{-1} = \mathbf{Q} \,\}$$

where $F \subseteq \mathbf{G}(K)$ is a finite set of coset representatives for $\mathbf{G}(\mathcal{O}_{\mathcal{S}})\backslash\mathbf{G}(K)/\mathbf{P}(K)$. Note that if $\gamma_1 f_1, \gamma_2 f_2 \in \Lambda_{\mathbf{Q}}$, then $f_1 = f_2$.

Given any $a = (a_v)_{v \in S} \in A$, we let

$$|\alpha_0(a)| = \prod_{v \in S} |\alpha_0(a_v)|_v$$

where $|\cdot|_v$ is the v-adic norm on K_v.

Given any $t > 0$, we let

$$A^+(t) = \{\, a \in A \mid |\alpha_0(a)| \geq t \,\}$$

and for $t > 0$, we let

$$R_{\mathbf{Q}}(t) = \Lambda_{\mathbf{Q}} UMA^+(t)$$

The following is a special case of Proposition 9 from Bestvina-Eskin-Wortman [3].

Proposition 15. *There exists a bounded set $B_0 \subseteq G$, and given any $N_0 \geq 0$, there exists $t_0 > 1$ and a second bounded set $B_1 \subseteq G$ such that*

(i) $G = \bigcup_{\mathbf{Q} \in \mathcal{P}} R_{\mathbf{Q}}(1) B_0$*;*

(ii) if $\mathbf{Q}, \mathbf{Q}' \in \mathcal{P}$ *and* $\mathbf{Q} \neq \mathbf{Q}'$*, then the distance between* $R_{\mathbf{Q}}(t_0)B_0$ *and* $R_{\mathbf{Q}'}(t_0)B_0$ *is at least* N_0*;*

(iii) $\mathbf{G}(\mathcal{O}_S) \cap R_{\mathbf{Q}}(t_0)B_0 = \emptyset$*; and*

(iv) $G - \left(\bigcup_{\mathbf{Q} \in \mathcal{P}} R_{\mathbf{Q}}(2t_0)B_0\right)$ *is contained in* $\mathbf{G}(\mathcal{O}_S)B_1$*.*

4.2. **Geometric form of reduction theory.** We will now reformulate Proposition 15 into a more explicit geometric statement in the form of Proposition 19 below.

For $v \in S$, we let X_v be the Euclidean building for $\mathbf{G}(K_v)$, so that X_v is a tree. We let $X_S = \prod_{v \in S} X_v$.

Let $\Sigma_v \subseteq X_v$ be the geodesic that $\mathbf{A}(K_v)$ acts on by translations. We let $\Sigma_S = \prod_{v \in S} \Sigma_v$, so that Σ_S is isometric to the Euclidean space $\mathbb{R}^{|S|}$.

We define a linear functional $\widehat{\alpha_0} : \Sigma_S \to \mathbb{R}$ by associating a basepoint $e \in \Sigma_S$ with the origin as follows:

$$\widehat{\alpha_0}(ae) = \log_p |\alpha_0(a)|$$

for $a \in A$. The action of A on e factors through $\mathbb{Z}^{|S|}$, where $\widehat{\alpha_0}$ is linear, so $\widehat{\alpha_0}$ extends to a functional on all of Σ_S. Furthermore, $\widehat{\alpha_0}$ is nonzero since there is some a with $|\alpha_0(a)| \neq 1$.

For any $r \in \mathbb{R}$, we let $\Sigma_{S,r} \subseteq \Sigma_S$ be

$$\Sigma_{S,r} = \{\, x \in \Sigma_S \mid \widehat{\alpha_0}(x) = r \,\}$$

Thus, $\Sigma_{S,r}$ is a hyperplane in Σ_S that is a finite Hausdorff distance from $\mathbf{A}(\mathcal{O}_S)e \subseteq \Sigma_{S,0}$.

Note that $\Sigma_{S,r}$ is not singular if $|S| > 1$. That is to say, the projection of $\Sigma_{S,r}$ to each Σ_v is surjective if $|S| > 1$. Indeed, to verify this claim observe that if $v \in S$, then $\mathbf{A}(\mathcal{O}_S)$ has dense projection to $\mathbf{A}(K_v)$, and thus acts cocompactly on Σ_v.

Now consider the geodesics Σ_v to be parameterized as unit speed $\Sigma_v : \mathbb{R} \to X_v$ with $\Sigma_v(\infty) = \mathbf{P}$. From our description of $\widehat{\alpha_0} : \Sigma_S \to \mathbb{R}$, we see that there are positive real numbers λ_v such that if $\rho_S : \mathbb{R} \to X_S$ is given by $\rho_S(t) = (\Sigma_v(\lambda_v t))_{v \in S}$, and if $\beta_{\rho_S} : X_S \to \mathbb{R}$ is the Busemann function for ρ_S – that is if $x \in X_S$, and d is the distance function on X_S, then

$$\beta_{\rho_S}(x) = \lim_{t \to \infty} (t - d(x, \rho_S(t)))$$

– then β_{ρ_S} restricted to Σ_S is exactly $\widehat{\alpha_0}$.

Let

$$\begin{aligned} \Sigma^+_{S,r} &= \{\, x \in \Sigma_S \mid \widehat{\alpha_0}(x) \geq r \,\} \\ &= \{\, x \in \Sigma_S \mid \beta_{\rho_S}(x) \geq r \,\} \end{aligned}$$

so that $\Sigma^+_{S,r}$ is a half space in Σ_S whose boundary equals $\Sigma_{S,r}$.

We let

$$B_{\mathbf{P},S,r} = \{\, x \in X_S \mid \beta_{\rho_S}(x) \geq r \,\}$$

and

$$Y_{\mathbf{P},S,r} = \{\, x \in X_S \mid \beta_{\rho_S}(x) = r \,\}$$

Lemma 16. *$B_{\mathbf{P},S,r} = UM\Sigma^+_{S,r}$ and $Y_{\mathbf{P},S,r} = UM\Sigma_{S,r}$.*

Proof. $\mathbf{M}$ is contained in both $\mathbf{P}$ and the parabolic group opposite to $\mathbf{P}$ with respect to $\mathbf{A}$. Also note that $\mathbf{M}(K_v)$ is compact for all $v \in S$. It follows that $\mathbf{M}(K_v)$ fixes Σ_v pointwise, and thus that M fixes Σ_S pointwise. Therefore, $UM\Sigma^+_{S,r} = U\Sigma^+_{S,r}$.

Elements of $\mathbf{U}(K_v)$ fix unbounded positive rays in Σ_v, thus elements of U fix pointwise a subray of ρ_S, thus β_{ρ_S} is invariant under multiplication by U. Therefore $UB_{\mathbf{P},S,r} = B_{\mathbf{P},S,r}$, so $UM\Sigma^+_{S,r}\subseteq B_{\mathbf{P},S,r}$ follows from $\Sigma^+_{S,r}\subseteq B_{\mathbf{P},S,r}$.

To see that $B_{\mathbf{P},S,r}\subseteq U\Sigma^+_{S,r}$, let $x \in B_{\mathbf{P},S,r}$. Since $X_v = \mathbf{U}(K_v)\Sigma_v$, we see that $x = u(x_v)_{v\in S}$ for some $u \in U$ and $x_v \in \Sigma_v$. Thus, $x \in U\Sigma^+_{S,r}$, again, since β_{ρ_S} is invariant under multiplication by U.

That $Y_{\mathbf{P},S,r} = UM\Sigma_{S,r}$ follows similarly. □

Given $t \in \mathbb{R}$, let $r_t \in \mathbb{R}$ be the supremum of all $r \in \mathbb{R}$ such that $\Sigma^+_{S,r}$ contains $A^+(t)e$. Notice that there is some $C > 0$, independent of t, such that the Hausdorff distance between $A^+(t)e$ and Σ^+_{S,r_t} is bounded by C. Notice also that $t \mapsto r_t$ is an increasing function.

Lemma 17. *The Hausdorff distance between $UMA^+(t)e$ and $B_{\mathbf{P},S,r_t}$ is bounded independent of t.*

Proof. Because the Hausdorff distance between $A^+(t)e$ and Σ^+_{S,r_t} is bounded, the Hausdorff distance between $UMA^+(t)e$ and $UM\Sigma^+_{S,r_t} = B_{\mathbf{P},S,r}$ is bounded. □

Lemma 18. *Let $\mathbf{Q} \in \mathcal{P}$. If $\gamma \in \mathbf{G}(\mathcal{O}_S)$ and $f \in F$ are such that $\gamma f \in \Lambda_{\mathbf{Q}}$, then for any r, we have $\mathbf{Q}(\mathcal{O}_S)\gamma f B_{\mathbf{P},S,r} = \gamma f B_{\mathbf{P},S,r}$.*

Proof. Note that as $B_{\mathbf{P},S,r}$ is given by the Busemann function for ρ_S, $\gamma f B_{\mathbf{P},S,r}$ is given by the Busemann function for $\gamma f \rho_S$.

Since $\gamma f \in \Lambda_{\mathbf{Q}}$, the positive end of each $\gamma f \Sigma_v$ limits to $\mathbf{Q}$. Thus, if $g \in \mathbf{Q}(K_v)$, then $\gamma f \Sigma_v$ and $g\gamma f \Sigma_v$ intersect in a positive ray. Hence, if $g \in Q$, then $g\gamma f \rho_S$ is a finite Hausdorff distance from $\gamma f \rho_S$ and $g\gamma f B_{\mathbf{P},S,r} = \gamma f B_{\mathbf{P},S,r_g}$ for some $r_g \in \mathbb{R}$. By replacing g with its inverse, we may assume that $r_g \geq r$.

We may assume that the set B_0 from Proposition 15 is a sufficiently large neighborhood of $1 \in G$, independent of g, so that, in particular there is a set $B'\subseteq B_0$ containing the point stabilizer of 1 and such that $B'B'\subseteq B_0$, and by the previous lemma, such that $UMA^+(t)B'e$ contains every vertex of $B_{\mathbf{P},S,r_t}$.

Let t_0 be as in Proposition 15. If $r_g \neq r$, then for sufficiently large n we have $g^n e \in \gamma f B_{\mathbf{P},S,r_{t_0}}$. Hence, $g^n e \in \gamma f UMA^+(t_0)B'e$. Therefore, $g^n \in \gamma f UMA^+(t_0)B'B'\subseteq R_{\mathbf{Q}}(t_0)B_0$. We conclude, by Proposition 15 part (iii), that $g \notin \mathbf{G}(\mathcal{O}_S)$. □

If $\mathbf{Q} \in \mathcal{P}$, we define

$$B_{\mathbf{Q},S,r} = \gamma f B_{\mathbf{P},S,r}$$

for any $\gamma f \in \Lambda_Q$. This is well-defined by the previous lemma, and we also see that $\mathbf{Q}(\mathcal{O}_S)B_{\mathbf{Q},S,r} = B_{\mathbf{Q},S,r}$ and that the Hausdorff distance between $R_{\mathbf{Q}}(t)B_0e$ and $B_{\mathbf{Q},S,r_t}$ is bounded independent of t or $\mathbf{Q}$. Using this and that the orbit map $G \to Ge \subseteq X_S$ is proper, we deduce from Proposition 15 the following

Proposition 19. *There exists some $r_0 \in \mathbb{R}$, and given any $N \geq 0$, there is some $r_1 > r_0$ such that*

(i) $\bigcup_{\mathbf{Q}\in\mathcal{P}} B_{\mathbf{Q},S,r_0} = X_S$*;*

(ii) if $\mathbf{Q}, \mathbf{Q}' \in \mathcal{P}$ and $\mathbf{Q} \neq \mathbf{Q}'$, then the distance between $B_{\mathbf{Q},S,r_1}$ and $B_{\mathbf{Q}',S,r_1}$ is at least N;

(iii) $\mathbf{G}(\mathcal{O}_S)e \cap B_{\mathbf{Q},S,r_1} = \emptyset$*; and*

(iv) $X_S - \left(\bigcup_{\mathbf{Q}\in\mathcal{P}} B_{\mathbf{Q},S,r_1}\right)$ *is a finite Hausdorff distance from* $\mathbf{G}(\mathcal{O}_S)e$*.*

For any $r \in \mathbb{R}$, we let $X_{S,r}$ be the closure in X_S of $X_S - (\cup_{\mathbf{Q}\in\mathcal{P}} B_{\mathbf{Q},S,r})$.

Lemma 20. *For $r \gg 0$, $\mathbf{G}(\mathcal{O}_S)$ acts properly and cocompactly on $X_{S,r}$.*

Proof. Let $\gamma \in \mathbf{G}(\mathcal{O}_S)$. Then $\gamma B_{\mathbf{Q},S,r} = B_{\gamma\mathbf{Q}\gamma^{-1},S,r}$ so $\mathbf{G}(\mathcal{O}_S)$ acts on $\cup_{\mathbf{Q}\in\mathcal{P}} B_{\mathbf{Q},S,r}$ and thus on $X_{S,r}$.

Since $\mathbf{G}(\mathcal{O}_S)$ acts properly on X_S, it acts properly on $X_{S,r}$.

That $\mathbf{G}(\mathcal{O}_S)$ acts cocompactly on $X_{S,r}$ follows from (iv) of Proposition 19. □

5. Cohomology of the complement of disjoint horoballs

In this section, we'll examine the cohomology of subspaces of X_S that include spaces of the form $X_{S,r}$, but are slightly more general in that we will allow ourselves to set the height of each horoball individually, rather than use a single parameter to define the height of all horoballs simultaneously. Precisely, for any tuple $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}} \in (\mathbb{R} \cup \{\infty\})^{\mathcal{P}}$, we let $X_{S,(r_{\mathbf{Q}})}$ be the closure of $X_S - (\cup_{\mathbf{Q}\in\mathcal{P}} B_{\mathbf{Q},S,r_{\mathbf{Q}}})$ in X_S, where $B_{\mathbf{Q},S,\infty}$ is taken to be the empty set.

We shall call a tuple $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}} \in (\mathbb{R} \cup \{\infty\})^{\mathcal{P}}$ *sufficiently large* if the resulting sets $B_{\mathbf{Q},S,r_{\mathbf{Q}}}$ are pairwise disjoint, and if their pairwise distance is bounded below by a constant that is sufficiently large. It's known that if $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ is sufficiently large then $X_{S,(r_{\mathbf{Q}})}$ is $(|S| - 2)$-connected but not $(|S| - 1)$-connected (see Stuhler [17], Bux-Wortman [12], and Bux-Köhl-Witzel [10]), but these topological properties are not directly relevant to this paper. What we require in this paper, and what we will prove in this section, is that $H_c^k(X_{S,(r_{\mathbf{Q}})}) = 0$ if $k \leq |S|-1$ and $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ is sufficiently large. (See Proposition 34 below.) We will begin an inductive proof of this claim by observing that the claim is true when $|S| = 1$.

Lemma 21. *If $|S| = 1$, and if $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ is sufficiently large, then the group $H_c^0(X_{S,(r_{\mathbf{Q}})})$ is trivial, where coefficients are in a ring R.*

Proof. In this case, X_S is a tree, and we want to show that the components of $X_{S,(r_{\mathbf{Q}})}$ are unbounded. Indeed, choose an edge $e_0 \in X_{S,(r_{\mathbf{Q}})}$. Because $(r_{\mathbf{Q}})_{\mathbf{Q}}$ is sufficiently large, there is an adjacent edge $e_1 \in X_{S,(r_{\mathbf{Q}})}$, and we can continue in this fashion to create an path of infinite length in $X_{S,(r_{\mathbf{Q}})}$ that begins with e_0. $\square$

Our proof of Proposition 34 will include an investigation of spaces that are quite similar to the space $X_{S,(r_{\mathbf{Q}})}$. Precisely, for any $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$, let $W_{S,(r_{\mathbf{Q}})}$ be the subcomplex of X_S consisting of all cells of X_S that are contained in $X_{S,(r_{\mathbf{Q}})}$. To see that there isn't much difference between $X_{S,(r_{\mathbf{Q}})}$ and $W_{S,(r_{\mathbf{Q}})}$ we have

Lemma 22. *If the tuple $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ is sufficiently large, then there is a proper homotopy equivalence between $W_{S,(r_{\mathbf{Q}})}$ and $X_{S,(r_{\mathbf{Q}})}$.*

Proof. The proof is an observation through Morse theory. Suppose that $\mathfrak{C}\subseteq X_S$ is a chamber that intersects $X_{S,(r_{\mathbf{Q}})}$ nontrivially, but is not contained in $X_{S,(r_{\mathbf{Q}})}$, and thus is not contained in $W_{S,(r_{\mathbf{Q}})}$.

Because $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ is sufficiently large, $\mathfrak{C}$ intersects $B_{\mathbf{Q},S,r_{\mathbf{Q}}}$ for a unique $\mathbf{Q}$. Recall that $B_{\mathbf{Q},S,r_{\mathbf{Q}}}$ is defined as the inverse image of a positive ray with respect to the Busemann function $\beta_{\gamma f \rho_S} : X_S \to \mathbb{R}$ associated to the geodesic $\gamma f \rho_S \subseteq X_S$ where $\gamma f \in \Lambda_{\mathbf{Q}}$.

Let $(x_v)_{v\in S}$ be the maximum point of $\mathfrak{C}$ with respect to $\beta_{\gamma f \rho_S}$. Let $\mathcal{L}_v$ be the descending link of x_v in the tree X_v with respect to $\beta_{\gamma f \Sigma_v} : X_v \to \mathbb{R}$. We let $\mathcal{C}_v$ be the cone on $\mathcal{L}_v$ taken at x_v in the tree X_v.

For $T\subseteq S$, we let $K_T = \prod_{v\in T} \mathcal{C}_v \times \prod_{v\notin T} \mathcal{L}_v$.

Now we are assuming that $(x_v)_{v\in S} \notin X_{S,(r_{\mathbf{Q}})}$, and note that $K_S - (x_v)_{v\in S}$ deformation retracts onto $\cup_{v\in S} K_{S-v}$ in such a way that the homotopy is nonincreasing with respect to $\beta_{\gamma f \rho_S}$. Note further that the maximum points in any K_{S-v_0} with respect to $\beta_{\gamma f \rho_S}$ are points of the form $(y_v)_{v\in S}$ where $y_v = x_v$ if $v \neq v_0$, and if these points are not in $X_{S,(r_{\mathbf{Q}})}$, then we can further retract K_{S-v_0} minus these maximums onto $\cup_{v\in S-v_0} K_{S-\{v_0,v\}}$. We continue in this fashion until all of K_S has been retracted onto some union of K_T with $K_T \subseteq X_{S,(r_{\mathbf{Q}})}$.

$\square$

In particular, the previous two lemmas show that

Lemma 23. *If* $|S| = 1$, *and if* $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ *is sufficiently large, then* $H_c^0(W_{S,(r_{\mathbf{Q}})}) = 0$.

This lemma will serve as the base step for our inductive proof that $H_c^k(W_{S,(r_{\mathbf{Q}})}) = 0$ if $k \leq |S| - 1$ and $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ is sufficiently large, which implies that $H_c^k(X_{S,(r_{\mathbf{Q}})}) = 0$ if $k \leq |S| - 1$ and $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ is sufficiently large.

5.1. **Proper products.** Now we will focus on the case when $|S| \geq 2$. We choose some $w \in S$ and let $\pi_w : X_S \to X_w$ be the projection.

Note that by definition of $W_{S,(r_{\mathbf{Q}})}$, if e is an edge in X_w, and if e° is the interior of e, then $\pi_w|_{W_{S,(r_{\mathbf{Q}})}} : W_{S,(r_{\mathbf{Q}})} \to X_w$ has $\pi_w^{-1}(e^\circ) = e^\circ \times Z_e$ for some complex $Z_e \subseteq X_{S-w}$. Our inductive proof in the remainder of this section is aided by observing that the fibers π_w restricted to one of these "W spaces" is another "W space".

Lemma 24. *For any edge* $e \subseteq X_w$, $Z_e = W_{S-w,(s^e_{\mathbf{Q}})}$ *for some tuple* $(s^e_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$. *Furthermore, by choosing* $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ *sufficiently large we may assume that* $(s^e_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ *is sufficiently large for each edge* $e \subseteq X_w$.

Proof. Let $x_w \in X_w$ be the endpoint of e that maximizes $\beta_{\gamma f \Sigma_w}$ for $\gamma f \in \Lambda_{\mathbf{Q}}$. Then a cell $\mathfrak{F} \subseteq X_{S-w}$ is contained in Z_e exactly if

$$\beta_{\gamma f \rho_S}(e \times \mathfrak{F}) \leq r_{\mathbf{Q}}$$

which is equivalent to

$$\beta_{\gamma f \rho_S}(x_w \times \mathfrak{F}) \leq r_{\mathbf{Q}}$$

and thus to $\beta_{\gamma f \rho_{(S-w)}}(\mathfrak{F}) \leq s^e_{\mathbf{Q}}$ for some $s^e_{\mathbf{Q}}$ depending $\beta_{\gamma f \Sigma_w}(x_w)$, and thus on e. □

Lemma 25. *Let* $\gamma f \in \Lambda_{\mathbf{Q}}$. *If* $e_1, e_2 \in X_w$ *are edges, and if the maximum of* $\beta_{\gamma f \Sigma_w}(e_1)$ *is greater than or equal to the maximum of* $\beta_{\gamma f \Sigma_w}(e_2)$, *then* $s^{e_1}_{\mathbf{Q}} \leq s^{e_2}_{\mathbf{Q}}$. *If* $\beta_{\gamma f \Sigma_w}(e_1) = \beta_{\gamma f \Sigma_w}(e_2)$, *then* $s^{e_1}_{\mathbf{Q}} = s^{e_2}_{\mathbf{Q}}$.

Proof. Let $\chi_w \subseteq X_w$ be a geodesic that limits to $\mathbf{Q}$, and suppose that $e_1 \subseteq \chi_w$.

First assume that that $e_2 \subseteq \chi_w$. Then since $\beta_{\gamma f \Sigma_w}(e_2) \leq \beta_{\gamma f \Sigma_w}(e_1)$ we see that $s^{e_2}_{\mathbf{Q}} \geq s^{e_1}_{\mathbf{Q}}$ as desired.

If e_2 is not contained in χ_w, then there is some $u \in \mathbf{U}(K_w)$ such that $u\chi_w$ does contain e_2. The result follows from the above as $\beta_{\gamma f \rho_{S-w}}$ and $\beta_{\gamma f \Sigma_w}$ are invariant by translations of $\mathbf{U}(K_w)$. □

Given a vertex $y \in X_w$, we let E_y be the set of edges in X_w that contain y. Then the previous lemma produces

Lemma 26. *For any vertex $y \in X_w$, and any parabolic $\mathbf{Q} \in \mathcal{P}$, either $\{s^e_{\mathbf{Q}}\}_{e \in E_y}$ contains a single value, or else $\{s^e_{\mathbf{Q}}\}_{e\in E_y}$ contains exactly two values, and the minimum value is realized by a unique edge in E_y.*

Proof. For $\gamma f \in \Lambda_{\mathbf{Q}}$, observe that there is a unique edge containing y that maximizes the Busemann function $\beta_{\gamma f \Sigma_w}$, and that the remaining edges minimize $\beta_{\gamma f \Sigma_w}$. $\square$

In what follows, we'll denote the unique edge in E_y from the proof of the previous lemma as $e(y, \mathbf{Q})$. Thus if $e, \epsilon \in E_y$, then $s^e_{\mathbf{Q}} \leq s^\epsilon_{\mathbf{Q}}$ if $e = e(y, \mathbf{Q})$, and $s^e_{\mathbf{Q}} = s^\epsilon_{\mathbf{Q}}$ if $e, \epsilon \neq e(y, \mathbf{Q})$.

We will need one more related observation about the fibers of π_w in the form of the following

Lemma 27. *If there is a vertex $y \in X_w$, and a cell $\mathfrak{F} \subseteq X_{S-w}$ such that $y \times \mathfrak{F} \subseteq W_{S,(r_{\mathbf{Q}})}$, then $e \times \mathfrak{F} \subseteq W_{S,(r_{\mathbf{Q}})}$ for each $e \in E_y - e(y, \mathbf{Q})$.*

Proof. Let $\gamma f \in \Lambda_{\mathbf{Q}}$. Since $y \times \mathfrak{F} \subseteq W_{S,(r_{\mathbf{Q}})}$, the values of $\beta_{\gamma f \rho_S}(y \times \mathfrak{F})$ are bounded above by $r_{\mathbf{Q}}$. Since y maximizes the values of e under $\beta_{\gamma f \Sigma_w}$, the values of $\beta_{\gamma f \rho_S}(e \times \mathfrak{F})$ are bounded above by $r_{\mathbf{Q}}$ as well. That is, $e \times \mathfrak{F} \subseteq W_{S,(r_{\mathbf{Q}})}$. $\square$

5.2. **Cover by fibers.** Having collected some information about the fibers of $\pi_w|_{W_{S,(r_{\mathbf{Q}})}}$, we will now use a collection of fibers to create a cover for $W_{S,(r_{\mathbf{Q}})}$.

For any edge $e \subseteq X_w$, let $F_e = e \times W_{S-w,(s^e_{\mathbf{Q}})}$ where $W_{S-w,(s^e_{\mathbf{Q}})}$ is as in Lemma 24.

Lemma 28. *The collection $\{F_e\}$ taken over all edges $e \subseteq X_w$ is a cover for $W_{S,(r_{\mathbf{Q}})}$.*

Proof. Suppose $\sigma \times \mathfrak{F}$ is a cell in $W_{S,(r_{\mathbf{Q}})}$, where σ is a cell in an edge $e \subseteq X_w$ and $\mathfrak{F}$ is a cell in X_{S-w}.

If $\sigma = e$, then $\sigma \times \mathfrak{F} \subseteq F_e$ by Lemma 24. If σ is a vertex of e, say y, then by Lemma 27, there is some e' such that $y \times \mathfrak{F} \subseteq e' \times \mathfrak{F} \subseteq F_{e'}$. $\square$

For any vertex $y \in X_w$, let $F_y = \cup_{e \in E_y} F_e$. Note that there is a proper homotopy equivalence between F_y and

$$\cup_{y \in e} W_{S-w,(s^e_{\mathbf{Q}})} = W_{S-w,(\max_{e \in E_y}\{s^e_{\mathbf{Q}}\})}$$

given by retracting the star of y in X_w to the point y.

Further, if $e \in E_y$, then the inclusion $F_e \to F_y$, after proper homotopy equivalence, is the inclusion $W_{S-w,(s^e_{\mathbf{Q}})} \to W_{S-w,(\max_{\epsilon \in E_y}\{s^\epsilon_{\mathbf{Q}}\})}$. In particular, if $e \in E_y$, the we can, and we shall, identify the map induced by inclusion

$$\rho_{y,e} : H^{|S|-1}_c(F_y) \to H^{|S|-1}_c(F_e)$$

with the map

$$\rho_{y,e} : H_c^{|S|-1}(W_{S-w,(\max_{\epsilon\in E_y}\{s^\epsilon_{\mathbf{Q}}\})}) \to H_c^{|S|-1}(W_{S-w,(s^e_{\mathbf{Q}})})$$

5.3. **Maps between the cohomology of the fibers.** For an edge $e \subseteq X_w$, and a parabolic group $\mathbf{R} \in \mathcal{P}$, we let $\mathcal{S}_{e,\mathbf{R}} \subseteq W_{S-w,(s^e_{\mathbf{Q}})}$ be the complex comprised of all cells $\mathcal{F} \subseteq W_{S-w,(s^e_{\mathbf{Q}})}$ such that there is a cell $\mathcal{G} \subseteq X_{S-w}$ containing $\mathcal{F}$ with $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{G}) \nleq s^e_{\mathbf{R}}$ where $\gamma f \in \Lambda_{\mathbf{R}}$. Thus we may informally think of the boundary of $W_{S-w,(s^e_{\mathbf{Q}})}$ as $\amalg_{\mathbf{Q}\in\mathcal{P}} \mathcal{S}_{e,\mathbf{Q}}$.

Let $y \in X_w$ be a vertex, $e \in E_y$, and $\mathbf{R} \in \mathcal{P}$. We define $\mathcal{J}_{y,e,\mathbf{R}}$ to be the union of cells $\mathcal{F} \subseteq W_{S-w,(\max_{\epsilon\in E_y}\{s^\epsilon_{\mathbf{Q}}\})}$ such that the maximum value of $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{F})$ is greater than $s^e_{\mathbf{R}}$ for $\gamma f \in \Lambda_{\mathbf{R}}$. Notice that if $\mathcal{J}_{y,e,\mathbf{R}} \neq \emptyset$, then $s^e_{\mathbf{R}} < s^\epsilon_{\mathbf{R}}$ for some $\epsilon \in E_y$, which, by Lemma 26, implies that $e = e(y, \mathbf{R})$.

Lemma 29. *If $y \in X_w$, $e \in E_y$, and $\mathbf{R} \in \mathcal{P}$, then $H_c^{|S|-1}(\mathcal{J}_{y,e,\mathbf{R}}) = 0$.*

Proof. We may assume that $\mathcal{J}_{y,e,\mathbf{R}} \neq \emptyset$, so that $s^e_{\mathbf{R}} < s^\epsilon_{\mathbf{R}}$ for $\epsilon \in E_y - e$. Then $\mathcal{J}_{y,e,\mathbf{R}}$ is the complex of cells $\mathcal{F}$ such that the maximum value of $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{F})$ is greater than $s^e_{\mathbf{R}}$ but bounded above by $s^\epsilon_{\mathbf{R}}$.

Let $\mathcal{F}$ be a cell as in the above paragraph of dimension $|S| - 1$ and assume that $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{F})$ attains the minimal value for all such $\mathcal{F}$. Then we can retract $\mathcal{F}$ into $\partial\mathcal{F}$ along the direction of the geodesic $\gamma f \rho_{(S-w)}$. Repeat this process until $\mathcal{J}_{y,e,\mathbf{R}}$ is retracted onto a complex of dimension $|S| - 2$. □

We let $K_{y,e} \subseteq \mathcal{P}$ be the set of all $\mathbf{R} \in \mathcal{P}$ such that $\mathcal{J}_{y,e,\mathbf{R}} \neq \emptyset$. If $e, \epsilon \in E_y$, and if $\mathcal{J}_{y,e,\mathbf{R}}$ and $\mathcal{J}_{y,\epsilon,\mathbf{R}}$ are each nonempty, then $e = e(y, \mathbf{R}) = \epsilon$. Therefore, if e and ϵ are distinct, we have $K_{y,e} \cap K_{y,\epsilon} = \emptyset$ so that if we let $K_y = \cup_{e\in E_y} K_{y,e} \subseteq \mathcal{P}$, then

$$K_y = \amalg_{e\in E_y} K_{y,e}$$

Lemma 30. *Given a vertex $y \in X_w$ and $e \in E_y$,*

$$W_{S-w,(\max_{\epsilon\in E_y}\{s^\epsilon_{\mathbf{Q}}\})} = W_{S-w,(s^e_{\mathbf{Q}})} \cup \Big(\amalg_{\mathbf{R}\in K_{y,e}} \mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}} \Big)$$

Furthermore

$$W_{S-w,(s^e_{\mathbf{Q}})} \cap \Big(\amalg_{\mathbf{R}\in K_{y,e}} \mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}} \Big) = \amalg_{\mathbf{R}\in K_{y,e}} \mathcal{S}_{e(y,\mathbf{R}),\mathbf{R}}$$

Proof. By definition, for all $\mathbf{R} \in \mathcal{P}$, we have that $\mathcal{J}_{y,e,\mathbf{R}}$ and $W_{S-w,(s^e_{\mathbf{Q}})}$ are contained in $W_{S-w,(\max_{\epsilon\in E_y}\{s^\epsilon_{\mathbf{Q}}\})}$.

If $\mathcal{F} \subseteq W_{S-w,(\max_{\epsilon\in E_y}\{s^\epsilon_{\mathbf{Q}}\})}$ is a cell, and if $\mathcal{F}$ is not contained in $W_{S-w,(s^e_{\mathbf{Q}})}$, then the maximum value of $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{F})$ is greater than $s^e_{\mathbf{R}}$ for some

$\mathbf{R} \in \mathcal{P}$ and $\gamma f \in \Lambda_{\mathbf{R}}$, so that $\mathcal{F} \subseteq \mathcal{J}_{y,e,\mathbf{R}}$ which is to say that

$$W_{S-w,(\max_{\epsilon \in E_y}\{s^{\epsilon}_{\mathbf{Q}}\})} \subseteq W_{S-w,(s^e_{\mathbf{Q}})} \cup \bigcup_{\mathbf{R} \in \mathcal{P}} \mathcal{J}_{y,e,\mathbf{R}}$$

so we have equality. Furthermore, by the definition of $K_{y,e}$, and since $(s^e_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}}$ is sufficiently large, we have

$$W_{S-w,(\max_{\epsilon \in E_y}\{s^{\epsilon}_{\mathbf{Q}}\})} = W_{S-w,(s^e_{\mathbf{Q}})} \cup \amalg_{\mathbf{R} \in K_{y,e}} \mathcal{J}_{y,e,\mathbf{R}}$$

Now suppose that there is a cell $\mathcal{F}$ contained in both $W_{S-w,(s^e_{\mathbf{Q}})}$ and $\mathcal{J}_{y,e,\mathbf{R}}$ for some $\mathbf{R} \in K_{y,e}$. The latter inclusion implies that there is some $\mathcal{G} \subseteq X_{S-w}$ such that the maximum value of $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{G})$ is greater than $s^e_{\mathbf{R}}$ for $\gamma f \in \Lambda_{\mathbf{R}}$. That is, $\mathcal{F} \subseteq \mathcal{S}_{e,\mathbf{R}}$.

To show the other inclusion, let $\mathcal{F} \subseteq W_{S-w,(s^e_{\mathbf{Q}})}$ be such that there is a cell $\mathcal{G} \subseteq X_{S-w}$ containing $\mathcal{F}$ with $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{G}) \not\leq s^e_{\mathbf{R}}$ for some $\mathbf{R} \in K_{y,e}$ where $\gamma f \in \Lambda_{\mathbf{R}}$. Then $\mathcal{F} \subseteq \mathcal{J}_{y,e,\mathbf{R}}$. □

We also have the following lemma whose proof is similar.

Lemma 31. *Given a vertex $y \in X_w$,*

$$W_{S-w,(\max_{\epsilon \in E_y}\{s^{\epsilon}_{\mathbf{Q}}\})} = W_{S-w,(s^{e(y,\mathbf{Q})}_{\mathbf{Q}})} \cup \left(\amalg_{\mathbf{R} \in K_y} \mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}} \right)$$

Furthermore

$$W_{S-w,(s^{e(y,\mathbf{Q})}_{\mathbf{Q}})} \cap \left(\amalg_{\mathbf{R} \in K_y} \mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}} \right) = \amalg_{\mathbf{R} \in K_y} \mathcal{S}_{e(y,\mathbf{R}),\mathbf{R}}$$

The Mayer-Vietoris sequence for the pair in Lemma 30 yields the coboundary homomorphism

$$\delta_{y,e} : \oplus_{\mathbf{R} \in K_{y,e}} H^{|S|-2}_c(\mathcal{S}_{e(y,\mathbf{R}),\mathbf{R}}) \to H^{|S|-1}_c(W_{S-w,(\max_{\epsilon \in E_y}\{s^{\epsilon}_{\mathbf{Q}}\})})$$

Similarly, Lemma 31 yields

$$\delta_y : \oplus_{\mathbf{R} \in K_y} H^{|S|-2}_c(\mathcal{S}_{e(y,\mathbf{R}),\mathbf{R}}) \to H^{|S|-1}_c(W_{S-w,(\max_{\epsilon \in E_y}\{s^{\epsilon}_{\mathbf{Q}}\})})$$

so that $\delta_y = \oplus_{e \in E_y} \delta_{y,e}$.

Lemma 32. *Suppose that $H^{|S|-2}_c(W_{S-w,(r_{\mathbf{Q}})}) = 0$ for any sufficiently large sequence $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}}$. If $\sum_{\mathbf{R} \in K_y} v_{\mathbf{R}} \in \oplus_{\mathbf{R} \in K_y} H^{|S|-2}_c(\mathcal{S}_{e(y,\mathbf{R}),\mathbf{R}})$ and $\delta_y(\sum v_{\mathbf{R}}) = 0$, then $\delta_y(v_{\mathbf{R}}) = 0$ for all $\mathbf{R} \in K_y$.*

Proof. Since $H^{|S|-2}_c(W_{S-w,(\max_{\epsilon \in E_y} s^{\epsilon}_{\mathbf{Q}})})$ and $H^{|S|-2}_c(W_{S-w,(s^{e(y,\mathbf{Q})}_{\mathbf{Q}})})$ are trivial by assumption, we have the following portion of the Mayer-Vietoris sequence for the pair from Lemma 31.

$$0 \to \oplus H^{|S|-2}_c(\mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}}) \to \oplus H^{|S|-2}_c(\mathcal{S}_{e(y,\mathbf{R}),\mathbf{R}}) \to H^{|S|-1}_c(W_{S-w,(\max_{\epsilon \in E_y}\{s^{\epsilon}_{\mathbf{Q}}\})})$$

where δ_y is the rightmost map on the line above. Thus, if $\delta_y(\sum v_{\mathbf{R}}) = 0$, then $\sum v_{\mathbf{R}} \in \oplus H_c^{|S|-2}(\mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}})$, and in particular, for each $\mathbf{R}$ we have $v_{\mathbf{R}} \in H_c^{|S|-2}(\mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}})$ so that $\delta_y(v_{\mathbf{R}}) = 0$ for each $\mathbf{R}$. □

Lemma 33. *Suppose that* $H_c^{|S|-2}(W_{S-w,(r_{\mathbf{Q}})}) = 0$ *for any sufficiently large sequence* $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$. *Let* $y \in X_w$ *be a vertex, and suppose* $x \in H_c^{|S|-1}(F_y)$ *is nonzero. Then there is at most one* $e \in E_y$ *such that* $\rho_{y,e}(x) = 0$.

Proof. Suppose that $\rho_{y,e}(x) = 0$, and let $\epsilon \in E_y$. We will show that $\rho_{y,\epsilon}(x) = 0$ implies $e = \epsilon$, thus proving the lemma. Applying the Mayer-Vietoris sequence to the sets from Lemma 30, and using Lemma 29, we have

$$\oplus_{\mathbf{R}\in K_{y,e}} H_c^{|S|-2}(\mathcal{S}_{e,\mathbf{R}}) \to H_c^{|S|-1}(W_{S-w,(\max_{\epsilon\in E_y}\{s^\epsilon_{\mathbf{Q}}\})}) \to H_c^{|S|-1}(W_{S-w,(s^e_{\mathbf{Q}})})$$

where the map on the left is $\delta_{y,e}$ and the map on the right is $\rho_{y,e}$. Therefore, $\rho_{y,e}(x) = 0$ implies $x = \delta_{y,e}(\sum v_{\mathbf{R}}) = \delta_y(\sum v_{\mathbf{R}})$ for some $\sum v_{\mathbf{R}}$.

Now if $\rho_{y,\epsilon}(x) = 0$, then similarly, $x = \delta_y(\sum w_{\mathbf{R}})$ for some $\sum w_{\mathbf{R}}$. Therefore, $\delta_y(\sum(v_{\mathbf{R}} - w_{\mathbf{R}})) = x - x = 0$ which implies that $\delta_y(v_{\mathbf{R}} - w_{\mathbf{R}}) = 0$ for each $\mathbf{R}$ by the previous lemma.

Now fix some $\mathbf{R}$ with $\delta_{y,e}(v_{\mathbf{R}}) \neq 0$. Then

$$\delta_{y,\epsilon}(w_{\mathbf{R}}) = \delta_y(w_{\mathbf{R}}) = \delta_y(v_{\mathbf{R}}) = \delta_{y,e}(v_{\mathbf{R}}) \neq 0$$

from which we deduce that $\mathbf{R}$ is contained in $K_{y,e}$ and $K_{y,\epsilon}$. Thus, $e = \epsilon$. □

We are now ready to prove the main result of this section.

Proposition 34. *If* $(r_{\mathbf{Q}})_{\mathbf{Q}\in\mathcal{P}}$ *is sufficiently large, then* $H_c^k(W_{S,(r_{\mathbf{Q}})}) = 0$ *if* $k \neq |S|$ *and thus* $H_c^k(X_{S,(r_{\mathbf{Q}})}) = 0$ *if* $k \neq |S|$.

Proof. If $|S| = 1$, then we have proved this lemma in Lemma 23, so we assume the lemma is true for $S - w$ and prove it is true for S.

By Lemma 28, and since for any vertex $y \in X_w$ we have $F_y = \cup_{e\in E_y} F_e$, we see that the collection $\{F_y\}$ taken over all vertices $y \in X_w$ is a cover for $W_{S,(r_{\mathbf{Q}})}$. Note also that if y and z are the endpoints of an edge $e \subseteq X_w$, then $F_e = F_y \cap F_z$. Thus, the nerve of $\{F_y\}$ can be identified with X_w, and there's an associated spectral with $E_2^{pq} = H_c^p(X_w, \{H_c^q(F_*)\})$ that converges to $H_c^{p+q}(W_{S,(r_{\mathbf{Q}})})$. (See, e.g. [7] VII.4 for the analogous homology sequence. The derivation of the sequence we use here is a straightforward adaptation of that one.)

Since $F_e = e \times W_{S-w,(s^e_{\mathbf{Q}})}$, our induction hypothesis implies that $H_c^q(F_e) = 0$ for $q \neq |S| - 1$ and, together with X_w being 1-dimensional, that implies $H_c^p(X_w, \{H_c^q(F_*)\}) = 0$ if $q \neq |S|-1$ or if $p \geq 2$. Thus, we will have $H_c^k(W_{S,(r_{\mathbf{Q}})}) = 0$ for $k \neq |S|$ if $H_c^0(X_w, \{H_c^{|S|-1}(F_*)\}) = 0$, so we will verify that the kernel of the map

$$d : \oplus_{y \in X_w^{(0)}} H_c^{|S|-1}(F_y) \to \oplus_{e \in X_w^{(1)}} H_c^{|S|-1}(F_e)$$

is trivial.

To do this, suppose $\sum g_y \in \oplus H_c^{|S|-1}(F_y)$ is nonzero. Choose some vertex $y \in X_w$ with $g_y \neq 0$ and such that y is contained in an edge e and in the component of $X_w - e^\circ$ all of whose vertices $y' \neq y$ have $g_{y'} = 0$.

By Lemma 33, there is an edge $\epsilon \in E_y - e$ such that $\rho_{y,\epsilon}(g_y) \neq 0$, Therefore, the $H_c^{|S|-1}(F_\epsilon)$ component of $d(\sum g_y)$ is nonzero, and thus $d(\sum g_y) \neq 0$.

We have seen that $H_c^p(X_w, \{H_c^q(F_*)\}) = 0$ if $(p,q) \neq (1, |S|-1)$. □

If the sequence $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}}$ from Proposition 34 is constant, then we have the immediate

Corollary 35. *If $r \gg 0$ and $k \neq |S|$, then $H_c^k(X_{S,r}) = 0$.*

The proof of Proposition 34 given above applies to horosphere complements in products of trees that are more general than those arising from arithmetic groups. In particular, suppose $d \in \mathbb{N}$ and that T_i, for $1 \leq i \leq d$, is a locally finite tree with no vertices of valence 1. Choose geodesics $\Sigma_i : \mathbb{R} \to T_i$ pararmeterized such that the integer values of Σ_i are exactly the vertices in T_i in the image of Σ_i. For any collection of positive numbers λ_i, let $\beta : \prod_{i=1}^d T_i \to \mathbb{R}$ be the Busemann function for $\rho : \mathbb{R} \to \prod_{i=1}^d T_i$ given by $\rho(t) = (\Sigma_i(\lambda_i t))_{i=1}^n$. Let $Z = \beta^{-1}((-\infty, r])$ for any given $r \in \mathbb{R}$.

Corollary 36. *If $k \neq d$, then $H_c^k(Z) = 0$.*

Proof. Apply Proposition 34 to a sequence $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}}$ that has exactly one finite value, and the result is the statement of this corollary. The only exception is that Proposition 34 applies to trees whose valences are dictated by an arithmetic group, and it applies to Busemann functions for rays whose slopes (the λ_i) are determined by an arithmetic group. But neither of these explicit data are used in the proof of Proposition 34. □

6. Topology of horospheres

Let $X = \prod_{i=1}^d T_i$ where each T_i is a locally finite tree with no vertices of valence 1. Suppose each edge length in T_i equals 1. For each tree T_i, choose a geodesic $\Sigma_i \subseteq T_i$ and label its vertices $x_{i,n}$ for $n \in \mathbb{Z}$. This induces a height function h_i on the vertices of T_i where $h_i(x_{i,0}) = 0$ and $h_i(v) = n - d(v, x_{i,n})$ if the closest vertex of Σ_i is $x_{i,n}$. Extend each h_i linearly over edges to produce a height function h_i defined on all of T_i. For $1 \leq i \leq d$, we choose $\lambda_i > 0$ and we define a Busemann function $\beta : X \to \mathbb{R}$ by $\beta(x_1 \ldots, x_d) = \sum_i \lambda_i h_i(x_i)$.

Say that a vertex $v \in T_i$ is *below* a vertex $w \in T_i$ if there is a path γ from v to w such that $h_i \circ \gamma$ is strictly increasing. In this case we say w is *above* v. Note that for any $x_i \in T_i$ and $t > 0$ there is a unique point $y_i \in T_i$ above x_i such that $h_i(y_i) = h_i(x_i) + t$. Using this notation, the assignment $x_i \mapsto y_i$ defines a flow $\phi_{i,t} : T_i \to T_i$. These then define a flow on X by

$$\phi_t(x_1, \ldots, x_d) = \left(\phi_{1,t/(\lambda_1\sqrt{d})}(x_1), \ldots, \phi_{d,t/(\lambda_d\sqrt{d})}(x_d)\right)$$

Note that $\beta(\phi_t(x)) = \beta(x) + t$.

For $r \in \mathbb{R}$, we define

$$\begin{aligned} Y_r &= \beta^{-1}(r) \\ X_r &= \beta^{-1}(-\infty, r], \text{ and} \\ B_r &= \beta^{-1}[r, \infty) \end{aligned}$$

The space X naturally has the structure of a cube complex. Subdivide this structure to give X the structure of a cell complex such that Y_r and X_r are subcomplexes. In particular, for each $(d-1)$-cell e of Y_r there is a unique d-cell $\hat{e}$ of X lying above e such that $\mathring{e} \cap \hat{e} \neq \emptyset$.

In this section, all cohomology groups will be understood to have coefficients in some ring R.

6.1. **Horoball cohomology.**

Lemma 37. *For all r and k, $H^k_c(B_r) = 0$.*

Proof. For any number $m > n$ let

$$C(m) = \{\, x = (x_1, \ldots, x_d) \in X \mid \beta(x) \geq r \text{ and } x_i \text{ lies below } x_{i,m} \,\}$$

and let

$$\partial^{\uparrow} C(m) = \{\, x \in C(m) \mid h_i(x) = m \text{ for some } i \,\}$$

Note that $C(m)$ deformation retracts onto $\partial^{\uparrow} C(m)$. Thus we have that $H^k(C(m), \partial^{\uparrow} C(m)) = 0$.

Note that the sets $C(m)$ form an exhaustion of B_r by compact sets. Let $c(m)$ be the closed subset of $C(m)$ consisting of points whose distance from $\partial^{\uparrow} C(m)$ is at least ε, for some small $\varepsilon > 0$. The compact sets $c(m)$ also form an exhaustion of B_r, so it suffices to show

$H^k(B_r, B_r - c(m)) = 0$ for all m. By excision we have

$$H^k(B_r, B_r - c(m)) \cong H^k(C(m), C(m) - c(m))$$

Because $C(m) - c(m)$ deformation retracts onto $\partial^{\uparrow} C(m)$, we have

$$H^k(C(m), C(m) - c(m)) \cong H^k(C(m), \partial^{\uparrow} C(m)) = 0$$

$\square$

6.2. **Horosphere cohomology.** For $n \in \mathbb{N}$, the collection $\{H_c^{d-1}(Y_n)\}_{n\in\mathbb{N}}$ forms a directed system under the maps $(\phi_1)^* : H_c^{d-1}(Y_{n+1}) \to H_c^{d-1}(Y_n)$, where ϕ_1 is the time 1 flow on X. The goal of this section is to show $\varprojlim_n H_c^{d-1}(Y_n)$ is trivial and $\varprojlim_n^1 H_c^{d-1}(Y_n)$ is torsionfree.

There is a Mayer-Vietoris exact sequence

$$\cdots \to H_c^{d-1}(X_n) \oplus H_c^{d-1}(B_n) \to H_c^{d-1}(Y_n) \to H_c^d(X) \to \cdots$$

We know $H_c^{d-1}(X_n) = 0$ by Corollary 36 and $H_n^{d-1}(B_n) = 0$ by Lemma 37. Therefore the connecting map $H_c^{d-1}(Y_n) \to H_c^d(X)$ is injective. In this way we consider each module $H_c^{d-1}(Y_n)$ as a submodule of $H_c^d(X)$. Note that $(\phi_1)^* : H_c^d(X) \to H_c^d(X)$ is the identity map. It therefore follows from naturality of the Mayer–Vietoris sequence that $(\phi_1)^* : H_c^{d-1}(Y_{n+1}) \to H_c^{d-1}(Y_n)$ is the inclusion map of subgroups of $H_c^d(X)$.

We will prove $\varprojlim_n H_c^{d-1}(Y_n) = 0$, for which we set up notation. For any vertex $v \in T_i$ let $g_i(v)$ be the unique vertex above v such that $h_i(g_i(v)) = h_i(v) + 1$. Under the identification of $X^{(0)}$ with $\prod_i T_i^{(0)}$, let $g : X^{(0)} \to X^{(0)}$ be the function defined to be g_i in the i^{th} coordinate, so that $g(w)(i) = g_i(w(i))$ for any $w \in \prod_i T_i^{(0)}$.

Given i and n, let $C_{i,n} \subseteq T_i$ be the subtree of T_i spanned by the set of vertices

$$\{v \in T_i \mid v \text{ is below } x_{i,n} \text{ and } h_i(v) \geq -n\}$$

Let $K_n = \prod_{i=1}^d C_{i,n}$. The collection $\{K_n\}_{n=0}^{\infty}$ forms an exhaustion of X by compact sets, and so $H_c^k(X) = \varinjlim_n H^k(X, X \setminus K_n)$.

We compute each relative cohomology group $H^k(X, X \setminus K_n)$ as the cohomology of the quotient space $X/(X \setminus K_n)$. Let $EC_{i,n}$ be the set of vertices $v \in C_{i,n}$ such that $h_i(v) = -n$. Each set $C_{i,n}$ is homotopy equivalent relative $EC_{i,n} \cup \{x_{i,n}\}$ to a star with $\#(EC_{i,n})$ leaves, and so if ∂K_n is comprised of points in K_n whose i^{th} coordinate for some i is contained in $EC_{i,n} \cup \{x_{i,n}\}$, then K_n is homotopy equivalent relative ∂K_n to a cube complex with a top dimensional cube for each vertex in

$\prod_{i=1}^{d} EC_{i,n}$. It follows that there is a homotopy equivalence

$$X/(X \setminus K_n) \simeq \bigvee_{v \in \prod_{i=1}^{d} EC_{i,n}} S^d$$

To simplify notation, let Λ_n be the set of vertices in $\prod_{i=1}^{d} EC_{i,n}$. With this notation, it follows that there is an isomorphism

$$H^k(X, X \setminus K_n) \cong \begin{cases} R^{\Lambda_n} & \text{if } k = d \\ 0 & \text{else.} \end{cases}$$

Under this identification, the map $f_n : R^{\Lambda_n} \to R^{\Lambda_{n+1}}$ induced by the map of pairs $(X, X \setminus K_{n+1}) \to (X, X \setminus K_n)$ is described as follows: Given a function $\alpha : \Lambda_n \to R$, define $f_n(\alpha)(w) = \alpha(g(w))$ if $g(w) \in \Lambda_n$ and $f_n(\alpha)(w) = 0$ otherwise.

Given a vertex $v \in \Lambda_n$, let $\overline{x_{i,n}, v(i)}$ denote the geodesic segment between $x_{i,n}$ and $v(i)$, and let F_v be the cube $\prod_{i=1}^{d} \overline{x_{i,n}, v(i)}$. Note there is an equality of spaces $K_n = \bigcup_{v \in \Lambda_n} F_v$. Given a compactly supported cellular cochain $\phi \in Z_c^d(X)$, the assignment $[\phi] \mapsto (v \mapsto \phi(F_v))_n$ gives the isomorphism $H_c^d(X) \cong \varinjlim_n R^{\Lambda_n}$.

Note that the above is a proof of the well-known

Proposition 38. *$H_c^k(X) = 0$ if $k \neq d$ and $H_c^d(X)$ is a free R-module.*

See Borel-Serre [6] for a more general theorem about the compactly supported cohomology of Euclidean buildings.

We now observe that $H_c^*(Y_r)$ is concentrated in dimension $d-1$.

Proposition 39. *$H_c^k(Y_r) = 0$ if $k \neq d-1$ and $H_c^{d-1}(Y_r)$ is a free R-module.*

Proof. There is a Mayer-Vietoris exact sequence

$$\cdots \to H_c^k(X_r) \oplus H_c^k(B_r) \to H_c^k(Y_r) \to H_c^{k+1}(X) \to \cdots$$

By Corollary 36 we know $H_c^k(X_r) = 0$ for $k \leq d-1$. By Lemma 37 we know $H_c^k(B_r) = 0$ for all k. And by Proposition 38 we know $H_c^{k+1}(X) = 0$ for $k \leq d-2$. The result follows. □

Using the notation that we established prior to Proposition 38, we will prove

Lemma 40. $\varprojlim_n H_c^{d-1}(Y_n) = 0$.

Proof. Take any cohomology class $[\phi] \in H_c^d(X)$. Choose $n \in \mathbb{N}$ so that the support of ϕ is contained in K_n.

Consider any vertex $v \in \Lambda_n$, and choose m such that $m > \beta(K_{n+1})$. Suppose $[\phi] = [\delta\psi]$ for some $\psi \in H_c^{d-1}(Y_m)$, where δ is the chain map inducing the connecting homomorphism in the Mayer-Vietoris sequence. Choose $N > n+1$ such that the support of ψ, and hence also the support of $\delta\psi$, is contained in K_N. Because $[\phi]$ and $[\delta\psi]$ are equal in $H^d_c(X)$ their images in R^{Λ_N} are equal.

Choose any $w \in \Lambda_N$ so that $g^{N-n}(w) = v$. Then $F_v \subseteq F_w$. Since the suport of ϕ is contained in K_n and $F_v = F_w \cap K_n$ we have $\phi(F_v) = \phi(F_w)$.

For each $1 \le i \le d$ choose a vertex $e_i \in EC_{i,N}$ such that $g_i^{N-n-1}(e_i) \in EC_{i,n+1}$ but $g_i^{N-n} \notin EC_{i,n}$. Let $P(d)$ be the set of subsets of $\{1, \dots, d\}$. For each $\sigma \in P(d)$ define $w_\sigma \in \Lambda_n$ by

$$w_\sigma(i) = \begin{cases} w(i) & \text{if } i \notin \sigma \\ e_i & \text{if } i \in \sigma. \end{cases}$$

Note that $w_\emptyset = w$. If $\sigma \neq \emptyset$ then $F_{w_\sigma} \cap K_n = \emptyset$ and so $\phi(F_{w_\sigma}) = 0$. Because $[\phi]$ and $[\delta\psi]$ have the same image in R^{Λ_N} we see $\phi(F_{w_\sigma}) = \delta\psi(F_{w_\sigma})$ for all $\sigma \in P(d)$.

We claim the $(d-1)$-chain $\sum_{\sigma \in P(d)} (-1)^{|\sigma|}(F_{w_\sigma} \cap Y_m)$ is the zero chain. Indeed, for $1 \le i \le d$, let u_i be an order 2 isometry of T_i with $u_i(w(i)) = e_i$ and $u_i(x_i) = x_i$ if $h_i(x_i) \ge n+1$. For $\sigma \in P(d)$ we let u_σ be the product of u_i with $i \in \sigma$. In particular, $u_\emptyset = 1$. Notice that $u_\sigma F_w = F_{w_\sigma}$, and that $u_\sigma Y_m = Y_m$.

We let $P(d)^* = P(d) - \{1, \cdots, d\}$, and for each $\tau \in P(d)^*$, we define

$$R_\tau = \{(x_i) \in F_w \cap Y_m \mid h_i(x_i) \le n+1 \text{ precisely when } i \in \tau\}$$

Thus, $\cup_{\tau \in P(d)^*} R_\tau = F_w \cap Y_m$, if $\tau_1 \neq \tau_2$ then R_{τ_1} and R_{τ_2} do not contain a common $(d-1)$-cell, and if $\sigma \in P(d)$ then $u_\sigma R_\tau = u_{\sigma \cap \tau} R_\tau$.

Recall that by the binomial theorem, if $k \in \mathbb{N}$, then $\sum_{\mu \in P(k)} (-1)^{|\mu|} = 0$. Thus we have

$$\begin{aligned}
\sum_{\sigma \in P(d)} (-1)^{|\sigma|} (F_{w_\sigma} \cap Y_m) &= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} (u_\sigma F_w \cap Y_m) \\
&= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} u_\sigma (F_w \cap Y_m) \\
&= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} u_\sigma \Big(\sum_{\tau \in P(d)^*} R_\tau \Big) \\
&= \sum_{\tau \in P(d)^*} \sum_{\sigma \in P(d)} (-1)^{|\sigma|} u_\sigma R_\tau \\
&= \sum_{\tau \in P(d)^*} \sum_{\sigma \in P(d)} (-1)^{|\sigma|} u_{\sigma \cap \tau} R_\tau \\
&= \sum_{\tau \in P(d)^*} \sum_{\rho \in P(|\tau|)} \sum_{\mu \in P(d-|\tau|)} (-1)^{|\rho|+|\mu|} u_\rho R_\tau \\
&= \sum_{\tau \in P(d)^*} \sum_{\rho \in P(|\tau|)} (-1)^{|\rho|} \sum_{\mu \in P(d-|\tau|)} (-1)^{|\mu|} u_\rho R_\tau \\
&= \sum_{\tau \in P(d)^*} \sum_{\rho \in P(|\tau|)} (-1)^{|\rho|} \, 0 \, u_\rho R_\tau \\
&= 0
\end{aligned}$$

This establishes our claim that $\sum_{\sigma \in P(d)} (-1)^{|\sigma|} (F_{w_\sigma} \cap Y_m)$ is the zero chain.

Using the definition of the connecting map δ we therefore have

$$\begin{aligned}
0 &= \psi \left(\sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \cap Y_\sigma \right) \\
&= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} \delta \psi (F_{w_\sigma}) \\
&= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} \phi (F_{w_\sigma}) \\
&= \phi(F_w) \\
&= \phi(F_v)
\end{aligned}$$

This shows that if $[\phi] \in H_c^d(X)$, if $[\phi]$ is contained in $H_c^{d-1}(Y_m) \leq H_c^d(X)$ for some sufficiently large value of m, then $[\phi] = 0$. That is, $\varprojlim_n H_c^{d-1}(Y_n) = \cap_m H_c^{d-1}(Y_m) = 0$. □

Our next goal is to prove that $\varprojlim^1 H_c^{d-1}(Y_m)$ is torsion-free.

Lemma 41. *Suppose $[\psi] \in H_c^{d-1}(Y_m)$ and suppose there are $[\phi] \in H_c^d(X)$ and $r \in R$ such that $r[\phi] = [\delta\psi]$. Then there is some $[\tilde{\phi}] \in H_c^{d-1}(Y_m)$ such that $r[\tilde{\phi}] = [\psi]$.*

Proof. Let $\hat{Y}_m$ be the subcomplex of X containing only the (subdivided) d-cells $e \subseteq X$ such that $\beta(e) \geq m$ and $e \cap Y_m \neq \emptyset$. Let $i : \hat{Y}_m \to X$ be the inclusion map. The Mayer-Vietoris connecting map induces a homomorphism $\hat{\delta} : H_c^{d-1}(Y_m) \to H_c^d(\hat{Y}_m)$ such that the following diagram commutes:

$$\begin{array}{ccc} & H_c^{d-1}(Y_m) & \\ {}^{\delta}\swarrow & & \searrow^{\hat{\delta}} \\ H_c^d(X) & \xrightarrow{\quad i^* \quad} & H_c^d(\hat{Y}_m) \end{array}$$

Given any $(d-1)$-cell $c \subseteq Y_m$, let $\hat{c} \subseteq \hat{Y}_m$ be the unique d-cell in $\hat{Y}_m$ with $c \subseteq \hat{c}$.

If $[\phi] \in H_c^d(\hat{Y}_m)$, then define $[\epsilon\phi] \in H_c^{d-1}(Y_m)$ by $\epsilon\phi(c) = \phi(\hat{c})$. Then $\epsilon : H_c^d(\hat{Y}_m) \to H_c^{d-1}(Y_m)$ is the inverse of $\hat{\delta}$, so there is an isomorphism $H_c^{d-1}(Y_m) \cong H_c^d(\hat{Y}_m)$. The lemma follows by setting $[\tilde{\phi}] = \epsilon i^*([\phi])$. $\square$

Lemma 42. $\varprojlim^1 H_c^{d-1}(Y_m)$ *is torsionfree as an R-module.*

Proof. Recall that $\varprojlim^1 H_c^{d-1}(Y_m)$ is the cokernel of the map

$$\Delta : \prod_m H_c^{d-1}(Y_m) \to \prod_m H_c^{d-1}(Y_m)$$

where $\Delta\big(([\psi_m])_m\big) = ([\psi_m] - [\psi_{m+1}])_m$.

Suppose $([\xi_m])_m \in \prod_m H_c^{d-1}(Y_m)$ and that there is some regular element $r \in R$ such that $(r[\xi_m])_m$ is in the image of Δ. Let $([\psi_m])_m \in \prod_m H_c^{d-1}(Y_m)$ be a sequence such that $r[\xi_m] = [\psi_m] - [\psi_{m+1}]$ for all m. Note that this implies that for any $M > m$ there is some $\zeta_{m,M} \in H_c^{d-1}(Y_m)$ so that $r[\zeta_{m,M}] = [\psi_m] - [\psi_M]$.

Fix any m and choose $n \in \mathbb{N}$ such that ψ_m is supported on K_n. Take any $v \in \Lambda_n$. Choose any $M > m$ such that $M > \beta(K_n)$. Choose $N \in \mathbb{N}$ such that ψ_M is supported on K_N.

Choose any $w \in \Lambda_N$ such that $g^{N-n}(w) = v$. Construct vertices $w_\sigma \in \Lambda_N$ for each $\sigma \in P(d)$ as in the proof of Lemma 40, so that $F_{w_\emptyset} = F_v$ and $\psi_m(F_{w_\sigma}) = 0$ if $\sigma \neq \emptyset$ and $\sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \cap Y_M$ is the zero chain. Then

$$\psi_M\left(\sum_{\sigma\in P(d)}(-1)^{|\sigma|}F_{w_\sigma}\cap Y_M\right)=0$$

so

$$\delta\psi_M\left(\sum_{\sigma\in P(d)}(-1)^{|\sigma|}F_{w_\sigma}\right)=0$$

It follows that r divides the quantity

$$\begin{aligned}(\delta\psi_m-\delta\psi_M)&\left(\sum_{\sigma\in P(d)}(-1)^{|\sigma|}F_{w_\sigma}\right)\\ &=\delta\psi_m\left(\sum_{\sigma\in P(d)}(-1)^{|\sigma|}F_{w_\sigma}\right)-\delta\psi_M\left(\sum_{\sigma\in P(d)}(-1)^{|\sigma|}F_{w_\sigma}\right)\\ &=\delta\psi_m(F_{w_\emptyset})\\ &=\psi_m(F_v\cap Y_M)\end{aligned}$$

This holds for any vertex $v\in\Lambda_n$, so r divides the image of $[\psi_m]$ in R^{Λ_n}. Therefore r divides the image of $[\psi_m]$ in $H^d_c(X)$. By Lemma 41, for each m there is some $[\phi_m]\in H^{d-1}_c(Y_m)$ such that $r[\phi_m]=[\psi_m]$. It follows that $([\xi_m])_m=\Delta([\phi_m])$, which completes the proof. □

7. Examples of semiduality groups

Below we provide examples of semiduality groups. In order to verify the condition on the cohomological dimension, we recall the following standard result.

Lemma 43. *Suppose Λ is a group acting on an acyclic cell complex X with finite cell stabilizers. Suppose R is a commutative ring such that $|\Lambda_\sigma|$ is invertible in R for any cell stabilizer Λ_σ. Then $\mathrm{cd}_R\Lambda\le\dim(X)$.*

Proof. Suppose M is an $R\Lambda$-module. For each j let Σ_j be a set of representatives of Λ-orbits of j-cells of X. There is a spectral sequence (compare to the homology version appearing in [7, VII.7.7, p173])

$$E_1^{jq}=\prod_{\sigma\in\Sigma_j}H^q(\Lambda_\sigma;M_\sigma)\implies H^{j+q}(\Lambda;M)$$

For any $q>0$ the module $H^q(\Lambda_\sigma,M_\sigma)$ is annihilated by $|\Lambda_\sigma|$. But M_σ is an R-module and $|\Lambda_\sigma|$ is invertible in R for any cell σ, so the groups $H^q(\Lambda_\sigma;M_\sigma)$ are trivial if $q>0$. By definition, the groups E_1^{jq} are trivial for $j>\dim(X)$. It follows that $\mathrm{cd}_R\Lambda\le\dim(X)$. □

7.1. **Rank 1 arithmetic groups.** In this section we prove Theorem 3. To that end, suppose $\mathcal{O}_S$ is the ring of S-integers in a global function field K of characteristic p. Suppose $\mathbf{G}$ is a noncommutative, absolutely almost simple algebraic K-group and Γ is a finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ such that any torsion element of Γ is a p-element.

Proposition 44. *Γ is a $\mathbb{Z}[1/p]$-semiduality group of dimension $k(\mathbf{G}, S)$.*

Proof. Γ acts on the product of trees X_S with finite p-group stabilizers. It follows from Lemma 43 that $\mathrm{cd}_{\mathbb{Z}[1/p]} \Gamma \leq \dim(X_S) = k(\mathbf{G}, S)$. It is known that Γ is type $FP_{k(\mathbf{G},S)-1}$ over $\mathbb{Z}[1/p]$ (see [17], [12], [10]). It remains to show that $H^*(\Gamma, \mathbb{Z}[1/p]\Gamma)$ is concentrated in dimension $k(\mathbf{G}, S)$, where it is flat as a $\mathbb{Z}[1/p]$-module.

For sufficiently large $n \in \mathbb{N}$, Γ acts properly cocompactly on each $X_{S,n}$ by Lemma 20. As n tends to infinity, the spaces $X_{S,n}$ exhaust X_S. By Proposition 12 there is an isomorphism $H^k_{cc}(X_S) \cong H^k(\Gamma, \mathbb{Z}[1/p]\Gamma)$, where here and for the rest of the proof we take cohomology of spaces with $\mathbb{Z}[1/p]$ coefficients. We proved in Proposition 34 that $H^*_c(X_{S,n})$ is concentrated in dimension d. It follows from Proposition 14 that $H^*_{cc}(X_S)$ is concentrated in dimension d, where $H^d_{cc}(X_S) = \varprojlim_n H^d_c(X_{S,n})$.

It remains only to show that $\varprojlim_n H^d_c(X_{S,n})$ is $\mathbb{Z}[1/p]$-torsion-free, since $\mathbb{Z}[1/p]$ is a principal ideal domain. The closure of the complement of $X_{S,n}$ is a disjoint union of horoballs $\bigcup_{\mathbf{Q}\in\mathcal{P}} B_{\mathbf{Q},S,n}$. Up to proper homotopy, each set $B_{\mathbf{Q},S,n}$ is a set of the form B_n as defined in §6, and $X_{S,n} \cap B_n = Y_n$, where $Y_n = Y_{\mathbf{Q},S,n}$. There is a Mayer-Vietoris exact sequence

$$H^{d-1}_c(X_{S,n}) \oplus H^{d-1}_c\Big(\bigcup_{\mathcal{P}} B_n\Big) \to H^{d-1}_c\Big(\bigcup_{\mathcal{P}} Y_n\Big) \to$$
$$H^d_c(X_S) \to H^d_c(X_{S,n}) \oplus H^d_c\Big(\bigcup_{\mathcal{P}} B_n\Big) \to H^d_c\Big(\bigcup_{\mathcal{P}} Y_n\Big)$$

Because the unions are disjoint, for each k there are isomorphisms

$$H^k_c\Big(\bigcup_{\mathcal{P}} B_n\Big) \cong \bigoplus_{\mathcal{P}} H^k_c(B_n) \quad \text{and} \quad H^k_c\Big(\bigcup_{\mathcal{P}} Y_n\Big) \cong \bigoplus_{\mathcal{P}} H^k_c(Y_n)$$

We know $H^k_c(B_n) = 0$ for all k by Lemma 37. We also know $H^{d-1}_c(X_{S,n}) = 0$ by Proposition 34. Clearly $H^d_c(Y_n) = 0$ since Y_n is $(d-1)$-dimensional. Therefore we have a short exact sequence

$$0 \to \bigoplus_{\mathcal{P}} H^{d-1}_c(Y_n) \to H^d_c(X_S) \to H^d_c(X_{S,n}) \to 0$$

These maps are compatible with the maps induced by inclusion $i_n : X_{S,n} \to X_{S,n+1}$, the time 1 flow $\phi_1 : Y_n \to Y_{n+1}$, and the identity map $X_S \to X_S$. The above short exact sequence therefore gives rise to a short exact sequence of codirected systems of compactly support cohomology, from which there is an exact sequence

$$0 \to \varprojlim \bigoplus_{\mathcal{P}} H_c^{d-1}(Y_n) \to H_c^d(X_s) \to \varprojlim H_c^d(X_{S,n}) \to \varprojlim{}^1 \bigoplus_{\mathcal{P}} H_c^{d-1}(Y_n) \to 0$$

The maps of the system $\{\oplus H_c^{d-1}(Y_n)\}$ preserve the direct sum structure. We know $\varprojlim H_c^{d-1}(Y_n)$ is trivial by Proposition 40 and $\varprojlim^1 H_c^{d-1}(Y_n)$ is torsionfree by Proposition 42. Since $H_c^d(X_S)$ is torsionfree by Proposition 38, it follows that $\varprojlim H_c^d(X_{S,n})$ is torsionfree by the short exact sequence

$$0 \to H_c^d(X_s) \to \varprojlim H_c^d(X_{S,n}) \to \bigoplus_{\mathcal{P}} \varprojlim{}^1 H_c^{d-1}(Y_n) \to 0 \tag{3}$$

□

This proves Theorem 3, as every module in sequence (3) is a $\mathbb{Z}[1/p]\mathbf{G}(K)$-module by Lemma 13.

7.2. **Solvable groups.** In this section we prove groups of the form $\mathbf{B_2}(\mathcal{O}_S)$ are semiduality groups. We then prove that some generalizations of certain groups of this form are also semiduality groups, namely lamplighter groups, Diestel-Leader groups, and countable direct sums of finite groups. All are straightforward applications of the following lemma.

Lemma 45. *Let X be a product of d trees with Busemann function $\beta : X \to \mathbb{R}$ as described in §6. Suppose a group Λ acts on X cellularly, with finite cell stabilizers, and cocompactly on subsets of the form $\beta^{-1}(I)$ for closed intervals I. Suppose R is a principal ideal domain such that $|\Lambda_\sigma|$ is invertible for every cell stabilizer Λ_σ. Then Λ is an R-semiduality group of dimension d.*

Proof. Define

$$\begin{aligned} Y_n &= \beta^{-1}(\{n\}) \\ X_n &= \beta^{-1}[0,n], \text{ and} \\ B_n &= \beta^{-1}[n,\infty) \end{aligned}$$

The space X is contractible, so by Lemma 43 we know $\mathrm{cd}_R \Lambda \leq \dim(X) = d$. Since Λ acts cocompactly with finite stabilizers on a horosphere Y_n and $\tilde{H}_k(Y_n) = 0$ for $k < n-1$ by [9, 3.1], Brown's criterion implies that Λ is type FP_{d-1} (see for example [8, 1.1]).

The complexes X_n form an exhaustion of B_0 by closed, Γ-invariant sets such that $\Gamma \backslash X_n$ is compact. Therefore by the results of §3 there is an isomorphism $H^*(\Lambda; R\Lambda) \cong H^k_{cc}(B_0)$ and for each k there is an exact sequence

$$0 \to \varprojlim{}^1 H_c^{k-1}(X_n) \to H^k_{cc}(B_0) \to \varprojlim H^k_c(X_n) \to 0 \tag{4}$$

Note that the flow ϕ_t provides a proper deformation retraction of X_n to Y_n so $H_c^*(X_n) \cong H_c^*(Y_n)$. We know $H_c^*(Y_n)$ is concentrated in dimension $d-1$ by Proposition 39. Now Lemma 40 says $\varprojlim H_c^{d-1}(Y_n) = 0$ so $H^*_{cc}(B_0)$ is concentrated in dimension d. In that dimension there is an isomorphism $H^d_{cc}(B_0) \cong \varprojlim{}^1 H_c^{d-1}(Y_n)$, which is torsionfree by Lemma 42 and hence flat as an R-module. □

Suppose $\mathcal{O}_S$ is the ring of S-integers in a global function field K of characteristic p. Let $\mathbf{B_2}$ be the group of upper triangular matrices of determinant 1.

Theorem 46. *Suppose Γ is a finite index subgroup of $\mathbf{B_2}(\mathcal{O}_S)$ such that the order of every finite order element is a power of p. Then Γ is a $\mathbb{Z}[1/p]$-semiduality group of dimension $|S|$.*

Proof. In the notation of §4, we may choose $\mathbf{P} = \mathbf{B_2}$. Then applying Lemma 18 with $\gamma = 1$ and $f = 1$, we see that $\mathbf{P}$ acts on the horoball $B_{\mathbf{P},S,n}$ for all sufficiently large $n \in \mathbb{N}$. In fact $\mathbf{P}(\mathcal{O}_S)$ is the entire stabilizer of $B_{\mathbf{P},S,n}$ in $\mathbf{SL_2}(\mathcal{O}_S)$ since if $\gamma \in \mathbf{SL_2}(\mathcal{O}_S)$ and $\gamma B_n = B_n$ then $B_{\mathbf{P},S,n} = B_{\gamma \mathbf{P} \gamma^{-1},S,n}$, which by Proposition 19(ii) means $\mathbf{P} = \gamma \mathbf{P} \gamma^{-1}$ and so $\gamma \in \mathbf{P}$. It follows that the action of $\mathbf{P}$ on $Y_{\mathbf{P},S,n}$ is proper and cocompact since the action of $\mathbf{SL_2}(\mathcal{O}_S)$ is proper and cocompact on X_S. In particular, cell stabilizers are finite.

Γ acts on the product of trees X_S. Let β be the Busemann function associated to the end $\mathbf{P}$. By the previous paragraph Γ acts cocompactly on $\beta^{-1}(I)$ for any compact interval $I \subset \mathbb{R}$ because it has finite index in $\mathbf{P}(\mathcal{O}_S)$. Since Γ has only p-torsion and its action is proper, Lemma 45 applies. □

Suppose F is a finite group. The *lamplighter group with base group* F is $\Gamma_F = F \wr \mathbb{Z} = (\oplus_{i\in\mathbb{Z}} F) \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts by shifting the indices of a sequence (f_i).

Theorem 47. *The lamplighter group with base group F is a $\mathbb{Z}[1/\,|F|]$-semiduality group of dimension 2.*

Proof. Let T_1 and T_2 be copies of a $(|F|+1)$-regular tree. The lamplighter group Γ_F acts on $T_1 \times T_2$ in a natural way; for description of the action see [19, §4]. This action preserves a Busemann function β and is cocompact on any set of the form $\beta^{-1}(I)$ for closed intervals $I \subseteq \mathbb{R}$. Stabilizers of cells are finite sums of copies of F. Therefore Lemma 45 applies. □

There are "higher rank" generalizations of lamplighter groups known as *Diestel-Leader groups* $\Gamma_d(q)$ which act on a product of d regular trees of valence $q+1$. These are constructed in [1] for any values of d and q such that $d \leq p+1$ for any prime p dividing q; a lamplighter group with base group F is an example of $\Gamma_2(|F|)$. The proof of Theorem 47 easily generalizes to prove:

Theorem 48. *A Diestel-Leader group $\Gamma_d(q)$ is a $\mathbb{Z}[1/q]$-semiduality group of dimension d.*

As a final remark, consider a countable collection of finite groups $\{F_i\}_{i\in\mathbb{N}}$ and let $\Lambda = \oplus_{i\in\mathbb{N}} F_i$. (This is not necessarily solvable.) Then Λ is an R-semiduality group of dimension 1 for any principal ideal domain R in which $|F_i|$ is invertible for every i. (So, for example, any countable sum of finite groups is a $\mathbb{Q}$-semiduality group.) To see this, let $\Lambda_n = \oplus_{i=0}^n F_i$. Form a graph of groups with underlying graph a simplicial ray whose nth vertex and proceeding edge are labeled by Λ_n, with inclusion maps from edge groups to incident vertex groups. Then Λ is the fundamental group of this graph of groups. It acts on the Bass-Serre tree preserving a height function inherited from the base ray, and is cocompact on preimages of closed intervals. Cell stabilizers are isomorphic to some Λ_n, so Lemma 45 produces the desired result.

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