Horospherical limit points of $S$-arithmetic groups

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Abstract. Suppose $\Gamma$ is an $S$-arithmetic subgroup of a connected, semisimple algebraic group $G$ over a global field $Q$ (of any characteristic). It is well known that $\Gamma$ acts by isometries on a certain CAT(0) metric space $X_S = \prod_{v \in S} X_v$, where each $X_v$ is either a Euclidean building or a Riemannian symmetric space. For a point $\xi$ on the visual boundary of $X_S$, we show there exists a horoball based at $\xi$ that is disjoint from some $\Gamma$-orbit in $X_S$ if and only if $\xi$ lies on the boundary of a certain type of flat in $X_S$ that we call “$Q$-good.” This generalizes a theorem of G. Avramidi and D. W. Morris that characterizes the horospherical limit points for the action of an arithmetic group on its associated symmetric space $X$.

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1. Introduction

Definition 1.1 ([6, Defn. B]). Suppose the group $\Gamma$ acts by isometries on the CAT(0) metric space $X$, and fix $x \in X$. A point $\xi$ on the visual boundary of $X$ is a horospherical limit point for $\Gamma$ if every horoball based at $\xi$ intersects the orbit $x \cdot \Gamma$. Notice that this definition is independent of the choice of $x$. Also note that if $\Lambda$ is a finite-index subgroup of $\Gamma$, then $\xi$ is a horospherical limit point for $\Lambda$ if and only if it is a horospherical limit point for $\Gamma$.

In the situation where $\Gamma$ is an arithmetic group, with its natural action on its associated symmetric space $X$, the horospherical limit points have a simple geometric characterization:

Theorem 1.2 (Avramidi-Morris [11, Thm. 1.3]). Let

- $G$ be a connected, semisimple algebraic group over $Q$,
- $K$ be a maximal compact subgroup of the Lie group $G(\mathbb{R})$,
- $X = K \backslash G(\mathbb{R})$ be the corresponding symmetric space of noncompact type (with the natural metric induced by the Killing form of $G(\mathbb{R})$), and
- $\Gamma$ be an arithmetic subgroup of $G$.

Then a point $\xi \in \partial X$ is not a horospherical limit point for $\Gamma$ if and only if $\xi$ is on the boundary of some flat $F$ in $X$, such that $F$ is the orbit of a $Q$-split torus in $G(\mathbb{R})$.  

Date: September 25, 2013.
This note proves a natural generalization that allows $\Gamma$ to be $S$-arithmetic (of any characteristic), rather than arithmetic. The precise statement assumes familiarity with the theory of Bruhat-Tits buildings \cite{12}, and requires some additional notation.

**Notation 1.3.**
1. Let
   - $Q$ be a global field (of any characteristic),
   - $G$ be a connected, semisimple algebraic group over $Q$,
   - $S$ be a finite set of places of $Q$ (containing all the archimedean places if the characteristic of $Q$ is 0),
   - $G_v = \mathbf{G}(Q_v)$ for each $v \in S$, where $Q_v$ is the completion of $Q$ at $v$,
   - $K_v$ be a maximal compact subgroup of $G_v$, for each $v \in S$, and
   - $Z_S$ be the ring of $S$-integers in $Q$.
2. Adding the subscript $S$ to any symbol other than $Z$ denotes the Cartesian product over all elements of $S$. Thus, for example, we have $G_S = \prod_{v \in S} G_v = \prod_{v \in S} \mathbf{G}(Q_v)$.
3. For each $v \in S$, let
   \[ X_v = \begin{cases} 
   \text{the symmetric space } K_v \backslash \mathbf{G}(Q_v) & \text{if } v \text{ is archimedean,} \\
   \text{the Bruhat-Tits building of } \mathbf{G}(Q_v) & \text{if } v \text{ is nonarchimedean.}
   \end{cases} \]
   Thus, $G_v = \mathbf{G}(Q_v)$ acts properly and cocompactly by isometries on the CAT(0) metric space $X_v$. So $G_S$ acts properly and cocompactly by isometries on the CAT(0) metric space $X_S = \prod_{v \in S} X_v$.

**Definition 1.4.** We say a family $\{Y_t\}_{t \in \mathbb{R}}$ of subsets of $X_S$ is uniformly coarsely dense in $X_S / G(Z_S)$ if there exists $C > 0$, such that, for every $t \in \mathbb{R}$, each $G(Z_S)$-orbit in $X_S$ has a point that is at distance $< C$ from some point in $Y_t$.

See Definition \cite{3.2} for the definition of a $Q$-good flat in $X_S$.

**Theorem 1.5** (cf. \cite[Cor. 4.5]{1}). For a point $\xi$ on the visual boundary of $X_S = \prod_{v \in S} X_v$, the following are equivalent:
1. $\xi$ is a horospherical limit point for $G(Z_S)$.
2. $\xi$ is not on the boundary of any $Q$-good flat.
3. There does not exist a parabolic $Q$-subgroup $P$ of $G$, such that $P_S$ fixes $\xi$, and $P(Z_S)$ fixes some (or, equivalently, every) horosphere based at $\xi$.
4. The horospheres based at $\xi$ are uniformly coarsely dense in $X_S / G(Z_S)$.
5. The horoballs based at $\xi$ are uniformly coarsely dense in $X_S / G(Z_S)$.
6. $\pi(\mathcal{B}) = X_S / G(Z_S)$ for every horoball $\mathcal{B}$ based at $\xi$, where $\pi : X_S \to X_S / G(Z_S)$ is the natural covering map.

**Remarks 1.6.**
- \cite{1} is a restatement of Definition \cite{1.1}.
- \cite{4} is obvious, because horoballs are bigger than horospheres.
- \cite{5} is well known (see, for example, \cite[Lem. 2.3(\Rightarrow)]{1}).

The remaining implications \cite{1} \Rightarrow \cite{2} \Rightarrow \cite{3} \Rightarrow \cite{4} are proved in the following sections, by fairly straightforward adaptations of arguments in \cite{1}.
Acknowledgments. D. W. M. would like to thank A. Rapinchuk for answering his questions about tori over fields of positive characteristic. K. W. gratefully acknowledges the support of the National Science Foundation.

2. Proof of \((3 \Rightarrow 4)\)

\((3 \Rightarrow 4)\) of Theorem 1.5 is the contrapositive of the following result.

Proposition 2.1 (cf. [1, Thm. 4.3]). If the horospheres based at \(\xi\) are not uniformly coarsely dense in \(X_S/\Gamma\), then there is a parabolic \(Q\)-subgroup \(P\) of \(\Gamma\), such that

1. \(P\) fixes \(\xi\), and
2. \(P(\xi)\) fixes some (or, equivalently, every) horosphere based at \(\xi\).

Proof. We modify the proof of [1, Thm. 4.3] to deal with minor issues, such as the fact that \(G\) is not (quite) transitive on \(X_S\). To avoid technical complications, assume \(G\) is simply connected. We begin by introducing yet more notation:

- (I) Let \(\Gamma = G(\xi)\).
- (x) Let \(x \in X_S\). If \(v \in S\) is a nonarchimedean place, then we choose \(x\) so that its projection to \(X_v\) is a vertex.
- (\(\gamma\)) Let \(\gamma : \mathbb{R} \rightarrow X_S\) be a geodesic with \(\gamma(0) = x\) and \(\gamma(+\infty) = \xi\). Let \(\gamma^+ : [0, \infty) \rightarrow X\) be the forward geodesic ray of \(\gamma\). For each \(v \in S\), let \(\gamma_v\) be the projection of \(\gamma\) to \(X_v\), so \(\gamma_v\) is a geodesic in \(X_v\).
- (\(F_S\)) For each \(v \in S\), choose a maximal flat (or “apartment”) \(F_v\) in \(X_v\) that contains \(\gamma_v\). Then \(F_S\) is a maximal flat in \(X_S\) that contains \(\gamma\).
- (\(A_v\)) For each \(v \in S\), there is a maximal \(Q_v\)-split torus \(A_v\) of \(\Gamma(\xi)\), such that \(A_v\) acts properly and cocompactly on the Euclidean space \(F_v\) by translations. Then \(A_v\) acts properly and cocompactly on \(F_S\) (by translations).
- (\(C_v\)) For each \(v \in S\), choose a compact subset \(C_v\) of \(F_v\), such that \(C_vA_v = F_v\). Then \(C_vA_v = F_v\).
- (\(A_S\)) Let \(A_v = \{ a \in A_v \mid C_v a \cap \gamma \neq \emptyset \}\) and \(A^+_v = \{ a \in A_v \mid C_v a \cap \gamma^+ \neq \emptyset \}\).
- (\(F_\perp, A_\perp\)) Let \(F_\perp\) be the (codimension-one) hyperplane in \(F_S\) that is orthogonal to the geodesic \(\gamma\) and contains \(x\). Let \(A_\perp = \{ a \in A_v \mid C_v a \cap F_\perp \neq \emptyset \}\).
- (\(P_v^\xi, N_v\)) For each \(v \in S\), let \(P_v^\xi = \{ g \in \Gamma(\xi) \mid \{ a g a^{-1} \mid a \in A_\perp \} \text{ is bounded} \}\), so \(P_v^\xi\) is a parabolic \(Q_v\)-subgroup of \(\Gamma(\xi)\) that fixes \(\xi\). The Iwasawa decomposition [12, §3.3.2] allows us to choose a maximal horospherical subgroup \(N_v\) of \(\Gamma(\xi)\) that is contained in \(P_v^\xi\) and is normalized by \(A_v\), such that \(F_vN_v = X_v\).
- (\(P_v, M_v, T_v, M^*\)) By applying the \(S\)-arithmetic generalization of Ratner’s Theorem that was proved independently by Margulis-Tomanov [7] and Ratner [11] (or, if \(\text{char } \mathbb{Q} \neq 0\), by applying a theorem of Mohammadi [8, Cor. 4.2]), we obtain an \(S\)-arithmetic analogue of [11, Cor. 2.13]. Namely, for some parabolic \(Q\)-subgroup \(P\) of \(\Gamma\), if we let \(P_v = P(Q_v)\) for each \(v \in S\), and let \(P_v = M_vT_vU_v\) be the Langlands decomposition over \(Q_v\) (so
\( T_v \) is the maximal \( Q_v \)-split torus in the center of the reductive group \( M_v T_v \), and \( U_v \) is the unipotent radical), then we have

\[
N_S \subseteq M_S^* U_S \quad \text{and} \quad M_S^* U_S \Gamma \subseteq \overline{N_S} \Gamma,
\]

where \( M_S^* \) is the product of all the isotropic almost-simple factors of \( M_v \).

Since \( N_v \subseteq P_v \) for every \( v \) (and \( P_S \) is parabolic), we have \( U_S \subseteq N_S \) and \( A_S \subseteq P_S \) (cf. proof of \cite[Thm. 20.9(ii), p. 228]{[1]}). Therefore, since all maximal \( Q_v \)-split tori of \( P_v \) are conjugate \cite[Thm. 20.9(ii), p. 228]{[2]}, and \( M_S^* T_v \) contains a maximal \( Q_v \)-split torus, there is no harm in assuming \( A_S \subseteq M_S^* T_S \), by replacing \( M_S^* T_S \) with a conjugate. Let \( A_S^M = A_S \cap M_S = A_S \cap M_S^* \).

Note that \( N_v \) is in the kernel of every continuous homomorphism from \( P_S^\xi \) to \( \mathbb{R} \). Since \( P_S^\xi \) acts continuously on the set of horospheres based at \( \xi \), and these horospheres are parametrized by \( \mathbb{R} \), this implies that \( N_v \) fixes every horosphere based at \( \xi \). Then, since \( F_S N_S = X_S \), we see that, for each \( a \in A_S \), the set \( F_{1a} N_S \) is the horosphere based at \( \xi \) through the point \( xa \). By the definition of \( A_{1a} \), this implies that the horosphere is at bounded Hausdorff distance from

\[
\mathcal{H}_a = x a A_{1a} N_S.
\]

(Also note that every horosphere is at bounded Hausdorff distance from some \( \mathcal{H}_a \), since \( A_S \) acts cocompactly on \( F_S \).) We have

\[
(2.2) \quad a A_{1a} N_S \Gamma \supseteq a A_{1a} \cdot \overline{N_S} \Gamma \supseteq a A_{1a} \cdot M_S^* U_S \Gamma.
\]

We claim that \( F_{1a} A_S^M \) is not coarsely dense in \( F_S \). Indeed, suppose, for the sake of a contradiction, that the set is coarsely dense. Then \( A_{1a} A_S^M \) is coarsely dense in \( A_S \), which means there is a compact subset \( K_1 \) of \( A_S \), such that \( A_S = K_1 A_{1a} A_S^M \). Also, the Iwasawa decomposition \cite[§3.3.2]{[12]} of each \( G(Q_v) \) implies there is a compact subset \( K_S \) of \( G_S \), such that \( K_S A_S N_S = G_S \). Then, for every \( a \in A_\gamma \), we have

\[
K_{S} K_{1} \cdot a A_{1a} M_S^* U_S = K_{Sa}(K_{1} A_{1a} M_S^*) U_S \supseteq K_{Sa} A_S M_S^* U_S \supseteq K_{S} A_S N_S = G_S.
\]

Since the compact set \( K_{S} K_{1} \) is independent of \( a \), this (together with \( (2.2) \)) implies that the sets \( \mathcal{H}_a \) are uniformly coarsely dense in \( X/\Gamma \). This contradicts the fact that the horospheres based at \( \xi \) are not uniformly coarsely dense.

Since \( F_{1a} \) is a hyperplane of codimension one in \( F_S \) (and \( A_S^M \) is a group that acts by translations), the claim proved in the preceding paragraph implies \( F_{1a} = F_{1a} A_S^M \supseteq x A_S^M \). This means that \( \gamma \) is orthogonal to the convex hull of \( x A_S^M \).

On the other hand, we know that \( M_S \) centralizes \( T_S \). Therefore, \( M_S \) fixes the endpoint \( \xi_T \) of any geodesic ray \( \gamma_T \) in the convex hull of \( x T_S \). So \( M_S \) acts (continuously) on the set of horospheres based at \( \xi_T \). However, \( M_S \) is the almost-direct product of compact groups and semisimple groups over local fields, so it has no has no nontrivial homomorphism to \( \mathbb{R} \).

(For the semisimple groups, this follows from the truth of the Kneser-Tits Conjecture \cite[Thm. 7.6]{[10]}.) Since the horospheres are parametrized by \( \mathbb{R} \), we conclude that \( M_S \) fixes every horosphere based at \( \xi_T \). Hence \( A_S^M \) also fixes these horospheres. So \( x A_S^M \) is contained in the horosphere through \( x \), which means the convex hull of \( x A_S^M \) must be perpendicular to the convex hull of \( x T_S \). Since \( A_S^M T_S \) has finite index in \( A_S \), the conclusion of the preceding paragraph now implies that \( \gamma \) is contained in the convex hull of \( x T_S \), so \( C_{G_S}(T_S) \) fixes \( \xi \).

We also have

\[
P_S = M_S T_S U_S = C_{G_S}(T_S) U_S \subseteq C_{G_S}(T_S) N_S.
\]
Since \( C_{G_S}(T_S) \) and \( N_S \) each fix the point \( \xi \), we conclude that \( P_S \) fixes \( \xi \). This completes the proof of (1).

From here, the proof of (2) is almost identical to the proof of [1, Thm. 4.3(2)]. \( \square \)

### 3. Proof of \((2 \Rightarrow 3)\)

(2 \( \Rightarrow \) 3) of Theorem 1.5 is the contrapositive of Proposition 3.4 below.

**Notation 3.1.** Suppose \( T \) is a torus that is defined over \( Q \). Let

1. \( X^*_Q(T) \) be the set of \( Q \)-characters of \( T \), and
2. \( T^{(1)}_S = \{ g \in T_S \mid \prod_{v \in S} \| \chi(g_v) \|_v = 1, \forall \chi \in X_Q(T) \} \).

**Definition 3.2.** Suppose \( F \) is a flat in \( X_S \) (not necessarily maximal). We say \( F \) is \( Q \)-good if there exists a \( Q \)-torus \( T \), such that

- \( T \) contains a maximal \( Q \)-split torus of \( G \),
- \( T \) contains a maximal \( Q_v \)-split torus \( A_v \) of \( G_v \) for every \( v \in S \),
- \( F \) is contained in the maximal flat \( F_S \) that is fixed by \( A_S \), and
- \( F \) is orthogonal to the convex hull of an orbit of \( T^{(1)}_S \) in \( F_S \).

**Remark 3.3.** \( Q \)-good flats are a natural generalization of \( Q \)-split flats. Indeed, the two notions coincide in the setting of arithmetic groups. Namely, suppose

- \( Q \) is an algebraic number field,
- \( S \) is the set of all archimedean places of \( Q \),
- \( T \) is a maximal \( Q \)-split torus in \( G \), and
- \( H = \text{Res}_{Q/Q} G \) is the \( Q \)-group obtained from \( G \) by restriction of scalars.

Then \( T_S \) can be viewed as the real points of a \( Q \)-torus in \( H(R) \), and \( T^{(1)}_S \) is the group of real points of the \( Q \)-anisotropic part of \( T_S \). Thus, in this setting, the \( Q \)-good flats in the symmetric space of \( G_S \) are naturally identified with the \( Q \)-split flats in the symmetric space of \( H(R) \).

**Proposition 3.4** (cf. [1, Prop. 4.4]). If there is a parabolic \( Q \)-subgroup \( P \) of \( G \), such that \( P_S \) fixes \( \xi \), and \( P(Z_S) \) fixes every horosphere based at \( \xi \), then \( \xi \) is on the boundary of a \( Q \)-good flat in \( X_S \).

**Proof.** Choose a maximal \( Q \)-split torus \( R \) of \( P \). The centralizer of \( R \) in \( G \) is an almost direct product \( R M \) for some reductive \( Q \)-subgroup \( M \) of \( P \).

Choose a \( Q \)-torus \( L \) of \( M \), such that \( L(Q_v) \) contains a maximal \( Q_v \)-split torus \( B_v \) of \( M(Q_v) \) for each \( v \in S \). (This is possible when \( \text{char } Q = 0 \) by [10 Cor. 3 of §7.1, p. 405], and the same proof works in positive characteristic, because a theorem of A. Grothendieck tells us that the variety of maximal tori is rational [5 Exp. XIV, Thm. 6.1, p. 334], [3 Thm. 7.9].) Let \( T = RL \) and \( A_v = R(Q_v)B_v \), so that \( T \) is a \( Q \)-torus that contains the maximal \( Q \)-split torus \( R \) as well as the maximal \( Q_v \)-split tori \( A_v \) for all \( v \in S \).

Let \( F_S \) be the maximal flat corresponding to \( A_S \), and choose some \( x \in F_S \). Since \( P_S \) fixes \( \xi \), there is a geodesic \( \gamma = \{ \gamma_t \} \) in \( F \), such that \( \lim_{t \to \infty} \gamma_t = \xi \) (and \( \gamma_0 = x \)).

Now \( T(Z_S) \) is a cocompact lattice in \( T^{(1)}_S \) (because the “Tamagawa number” of \( T \) is finite: see [10 Thm. 5.6, p. 264] if \( \text{char } Q = 0 \); or see [3 Thm. IV.1.3] for the general case), and, by assumption, \( T(Z_S) \) fixes the horosphere through \( x \). This implies that all of \( T^{(1)}_S \) fixes this...
horosphere, so $xT_S^{(1)}$ is contained in the horosphere. Therefore, the convex hull of $xT_S^{(1)}$ is perpendicular to the geodesic $\gamma$, so $\gamma$ is a $Q$-good flat. \hfill \Box

4. Proof of $(1 \Rightarrow 2)$

$(1 \Rightarrow 2)$ of Theorem 1.5 is the contrapositive of the following result.

**Proposition 4.1** (cf. [1, Prop. 3.1] or [6, Thm. A]). If $\xi$ is on the boundary of a $Q$-good flat, then $\xi$ is not a horospherical limit point for $G(Z_S)$.

**Proof.** Let:

- $\mathcal{F}$ be a $Q$-good flat, such that $\xi$ is on the boundary of $\mathcal{F}$.
- $\gamma$ be a geodesic in $\mathcal{F}$, such that $\lim_{t \to \infty} \gamma(t) = \xi$.
- $T$, $A_S$, and $F_S$ be as in Definition 3.2.
- $x = \gamma(0) \in F_S$.
- $F_S$ be considered as a real vector space with Euclidean inner product, by specifying that the point $x$ is the zero vector.
- $C_x$ be a compact set, such that $C_x A_S = F_S$ (and $x \in C_x$).
- $\gamma^\perp$ be the orthogonal complement of the 1-dimensional subspace $\gamma$ in the vector space $F_S$.
- $\gamma_A^\perp = \{ a \in A_S \mid C_x a \cap \gamma^\perp \neq \emptyset \}$.
- $\gamma_A(t) \in A_S$, such that $\gamma(t) \in C_x \gamma_A(t)$, for each $t \in \mathbb{R}$.
- $R$ be a maximal $Q$-split torus of $G$ that is contained in $T$.
- $\Phi$ be the system of roots of $G$ with respect to $R$.
- $\alpha^S: T_S \to \mathbb{R}^+$ be defined by $\alpha^S(g) = \prod_{v \in S} \|\alpha(g_v)\|_v$ for $\alpha \in \Phi$ (where $\|\cdot\|_v \circ \alpha$ is extended to be defined on all of $T(Q_v)$ by making it trivial on the $Q$-anisotropic part).
- $\hat{\alpha}^S: F_S \to \mathbb{R}$ be the linear map satisfying $\hat{\alpha}^S(xa) = \log \alpha^S(a)$ for all $a \in A_S$.
- $\alpha^F \in F_S$, such that $\langle \alpha^F \mid y \rangle = \hat{\alpha}^S(y)$ for all $y \in F_S$.
- $\Phi^+ = \{ \alpha \in \Phi \mid \hat{\alpha}^S(\gamma(t)) > 0 \text{ for } t > 0 \}$.
- $\Delta$ be a base of $\Phi$, such that $\Phi^+$ contains $\Phi^++$.
- $\Delta^{++} = \Delta \cap \Phi^++$.
- $P_\alpha = R_\alpha M_\alpha N_\alpha$ be the parabolic $Q$-subgroup corresponding to $\alpha$, for $\alpha \in \Delta$, where
  - $R_\alpha$ is the one-dimensional subtorus of $R$ on which all roots in $\Delta \setminus \{\alpha\}$ are trivial,
  - $M_\alpha$ is reductive with $Q$-anisotropic center, and
  - the unipotent radical $N_\alpha$ is generated by the roots in $\Phi^+$ that are not trivial on $R_\alpha$.

Given any large $t \in \mathbb{R}^+$, we know $\hat{\alpha}^S(\gamma(t))$ is large for all $\alpha \in \Delta^{++}$. By definition, we have $T_S^{(1)} = \bigcap_{\alpha \in \Delta^{++}} \ker \alpha^S$. Since $\gamma$ is perpendicular to the convex hull of $x \cdot T_S^{(1)}$, this implies that $\gamma(t)$ is in the span of $\{\alpha^F\}_{\alpha \in \Delta}$. Also, for $\alpha \in \Delta$, we have

$$\langle \alpha^F \mid \gamma(t) \rangle = \hat{\alpha}^S(\gamma(t)) \geq 0.$$ 

There is no harm in renormalizing the metric on $X_S$ by a positive scalar on each irreducible factor (cf. [1, Rem. 5.4]). This allows us to assume $\langle \alpha^F \mid \beta^F \rangle \leq 0$ whenever $\alpha \neq \beta$ (see Lemma 4.2 below). Therefore, for any $b \in \gamma_A^\perp$, there is some $\alpha \in \Delta$, such that $\hat{\alpha}^S(x\gamma_A(t)b)$ is large (see Lemma 4.3 below). This means $\alpha^S(\gamma_A(t)b)$ is large.
Since conjugation by the inverse of \( \gamma_A(t) b \) contracts the Haar measure on \((N_\alpha)_S\) by a factor of \( \alpha^k (\gamma_A(t) b)^k \) for some \( k \in \mathbb{Z}^+ \), and the action of \( N_S \) on \((N_\alpha)_S\) is volume-preserving, this implies that, for any \( g \in \gamma_A(t) b N_S \), conjugation by the inverse of \( g \) contracts the Haar measure on \((N_\alpha)_S\) by a large factor. Since \( N_\alpha(Z_S) \) is a cocompact lattice in \((N_\alpha)_S\) (because the “Tamagawa number” of \( N_\alpha \) is finite: see [10, Thm. 5.6, p. 264] if \( \text{char } Q = 0 \); or see [9, Thm. IV.1.3] for the general case), this implies there is some nontrivial \( h \in N_\alpha(Z_S) \), such that \( \|ghg^{-1} - e\| \) is small. We conclude that \( \xi \) is not a horospherical limit point for \( G(Z_S) \) (cf. [1] Lem. 2.5(2))).

**Lemma 4.2.** Assume the notation of the proof of Proposition [4, 1]. The metric on \( X_S \) can be renormalized so that we have \( \langle \alpha^F | \beta^F \rangle \leq 0 \) for all \( \alpha, \beta \in \Delta \) with \( \alpha \neq \beta \).

**Proof.** When \( v \) is archimedean, the Killing form provides a metric on \( X_v \). We now construct an analogous metric when \( v \) is nonarchimedean. To do this, let \( \Phi_v \) be the root system of \( G \) with respect to the maximal \( Q_v \)-split torus \( A_v \), let \( t \oplus \bigoplus_{\alpha \in \Phi_v} g_\alpha \) be the corresponding weight-space decomposition of the Lie algebra of \( G_v \), choose a uniformizer \( \pi_v \) of \( Q_v \), let \( \mathcal{X}_v(A_v) \) be the group of co-characters of \( A_v \), and define a \( \mathbb{Z} \)-bilinear form \( \langle | \rangle_v : \mathcal{X}_v(A_v) \times \mathcal{X}_v(A_v) \to \mathbb{R} \) by

\[
\langle \varphi_1 | \varphi_2 \rangle_v = \sum_{\alpha \in \Phi_v} \left( \frac{\alpha(\varphi_1(\pi_v))}{\dim g_\alpha} \right) \left( \frac{\alpha(\varphi_2(\pi_v))}{\dim g_\alpha} \right) \dim g_\alpha.
\]

This extends to a positive-definite inner product on \( \mathcal{X}_v(A_v) \otimes \mathbb{R} \) (and the extension is also denoted by \( \langle | \rangle_v \)). It is clear that this inner product is invariant under the Weyl group, so it determines a metric on \( X_v \) [12, §2.3]. By renormalizing, we may assume that the given metric on \( X_v \) coincides with this one.

Let \( E \) be the \( Q \)-anisotropic part of \( T \). Then it is not difficult to see that \( \mathcal{X}_v(R) \otimes \mathbb{R} \) is the orthogonal complement of \( \mathcal{X}_v(E(Q_v)) \otimes \mathbb{R} \), with respect to the inner product \( \langle | \rangle_v \) (cf. [1] Lem. 2.8]). Since every \( Q \)-root annihilates \( E(Q_v) \), this implies that the \( F_v \)-component \( \alpha_v^F \) of \( \alpha^F \) belongs to the convex hull of \( x R(E(Q_v)) \), for every \( \alpha \in \Phi \).

From [4, Cor. 5.5], we know that the Weyl group over \( Q \) is the restriction to \( R \) of a subgroup of the Weyl group over \( Q_v \). So the restriction of \( \langle | \rangle_v \) to \( \mathcal{X}_v(R) \otimes \mathbb{R} \) is invariant under the \( Q \)-Weyl group. Assume, for simplicity, that \( G \) is \( Q \)-simple, so the invariant inner product on \( \mathcal{X}_v(R) \otimes \mathbb{R} \) is unique (up to a positive scalar). (The general case is obtained by considering the simple factors individually.) This means that, after passing to the dual space \( \mathcal{X}^*(R) \otimes \mathbb{R} \), the inner product \( \langle | \rangle_v \) must be a positive scalar multiple \( c_v \) of the usual inner product (for which the reflections of the root system \( \Phi \) are isometries), so \( \langle \alpha_v^F | \beta_v^F \rangle_v = c_v \langle \alpha | \beta \rangle \) for all \( \alpha, \beta \in \Delta \). Since it is a basic property of bases in a root system that \( \langle \alpha \ | \beta \rangle \leq 0 \) whenever \( \alpha \neq \beta \), we therefore have

\[
\langle \alpha^F | \beta^F \rangle = \sum_{v \in S} \langle \alpha_v^F | \beta_v^F \rangle_v = \sum_{v \in S} c_v \langle \alpha | \beta \rangle = \sum_{v \in S} (\geq 0)(\leq 0) \leq 0.
\]

**Lemma 4.3 ([1] Lem. 2.6]).** Suppose

1. \( v, v_1, \ldots, v_n \in \mathbb{R}^k \), with \( v \neq 0 \),
2. \( v \) is in the span of \( \{v_1, \ldots, v_n\} \),
3. \( \langle v | v_i \rangle \geq 0 \) for all \( i \),
4. \( \langle v_i | v_j \rangle \leq 0 \) for \( i \neq j \), and
5. \( T \in \mathbb{R}^+ \).

Then, for all sufficiently large \( t \in \mathbb{R}^+ \) and all \( w \perp v \), there is some \( i \), such that \( \langle tv + w | v_i \rangle > T \).
References


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