# EXPONENTIAL HIGHER DIMENSIONAL ISOPERIMETRIC INEQUALITIES FOR SOME ARITHMETIC GROUPS

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Abstract. We show that arithmetic subgroups of semisimple groups of relative $\mathbb{Q}$-type $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, or $E_7$ have an exponential lower bound to their isoperimetric inequality in the dimension that is 1 less than the real rank of the semisimple group.

Let $\mathbf{G}$ be a connected, semisimple, $\mathbb{Q}$-group that is almost simple over $\mathbb{Q}$. Let $X$ be the symmetric space of noncompact type associated with $\mathbf{G}(\mathbb{R})$ and let $X_\mathbb{Z}$ be a contractible subspace of $X$ that is a finite Hausdorff distance from some $\mathbf{G}(\mathbb{Z})$-orbit in $X$; Raghunathan proved that such a space exists [Ra 1]. We denote the $\mathbb{R}$-rank of $\mathbf{G}$ by $\mathrm{rk}_\mathbb{R}\mathbf{G}$.

Given a homology $n$-cycle $Y \subseteq X_\mathbb{Z}$ we let $v_X(Y)$ be the infimum of the volumes of all $(n+1)$-chains $B \subseteq X$ such that $\partial B = Y$. Similarly, we let $v_\mathbb{Z}(Y)$ be the infimum of the volumes of all $(n+1)$-chains $B \subseteq X_\mathbb{Z}$ such that $\partial B = Y$. We define the ratio

$$R_n(Y) = \frac{v_\mathbb{Z}(Y)}{v_X(Y)}$$

and we let $R_n(\mathbf{G}(\mathbb{Z})) : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 1}$ be the function

$$R_n(\mathbf{G}(\mathbb{Z}))(L) = \sup\{\, R_n(Y) \mid \mathrm{vol}(Y) \leq L \,\}$$

These functions measure a contrast between the geometries of $\mathbf{G}(\mathbb{Z})$ and $X$.

Clearly if $\mathbf{G}$ is $\mathbb{Q}$-anisotropic (or equivalently, if $\mathbf{G}(\mathbb{Z})$ is cocompact in $\mathbf{G}(\mathbb{R})$) then we may take $X_\mathbb{Z} = X$ so that $R_n(\mathbf{G}(\mathbb{Z})) = 1$ for all $n$.

The case is different when $\mathbf{G}$ is $\mathbb{Q}$-isotropic, or equivalently, if $\mathbf{G}(\mathbb{Z})$ is non-cocompact in $\mathbf{G}(\mathbb{R})$.

Leuzinger-Pittet conjectured that $R_{\mathrm{rk}_\mathbb{R}\mathbf{G}-1}(\mathbf{G}(\mathbb{Z}))$ is bounded below by an exponential when $\mathbf{G}$ is $\mathbb{Q}$-isotropic [L-P]. The conjecture in the case $\mathrm{rk}_\mathbb{R}\mathbf{G} = 1$ is equivalent to the well-known observation that the word metric for non-cocompact lattices in rank one real simple Lie groups is exponentially distorted in its corresponding symmetric space. Prior to [L-P], the conjecture was evidenced by other authors in some cases. It was proved by Epstein-Thurston when $\mathbf{G}(\mathbb{Z}) = \mathbf{SL_k}(\mathbb{Z})$

[Ep et al.], by Pittet when $\mathbf{G}(\mathbb{Z}) = \mathbf{SL_2}(\mathcal{O})$ and $\mathcal{O}$ is a ring of integers in a totally real number field [Pi], by Hattori when $\mathbf{G}(\mathbb{Z}) = \mathbf{SL_k}(\mathcal{O})$ and $\mathcal{O}$ is a ring of integers in a totally real number field [Ha 1], and by Leuzinger-Pittet when $\mathrm{rk}_{\mathbb{R}}\mathbf{G} = 2$ [L-P].

This paper contributes to the verification of the Leuzinger-Pittet conjecture by proving

**Theorem 1.** *Let* $\mathbf{G}$ *be as in the introductory paragraph and assume that* $\mathbf{G}$ *is* $\mathbb{Q}$*-isotropic. Furthermore, suppose the* $\mathbb{Q}$*-relative root system of* $\mathbf{G}$ *is of type* $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, *or* $E_7$. *Then there exist constants* $C > 0$ *and* $L_0 > 0$ *such that*

$$R_{\ \mathrm{rk}_{\mathbb{R}}\mathbf{G}-1}(\mathbf{G}(\mathbb{Z}))(L) \geq e^{CL}$$

*for any* $L > L_0$.

0.1. **Example.** Let $\mathcal{O}$ be the ring of integers in a number field $K$, and let $\mathbf{G} = \mathbf{R}_{K/\mathbb{Q}}\mathbf{SL_k}$ where $\mathbf{R}_{K/\mathbb{Q}}$ is the restriction of scalars functor. Then $\mathbf{G}(\mathbb{Z}) = \mathbf{SL_k}(\mathcal{O})$, $\mathbf{G}$ is $\mathbb{Q}$-isotropic, $\mathbf{G}$ has a $\mathbb{Q}$-relative root system of type $A_{k-1}$, and $\mathrm{rk}_{\mathbb{R}}\mathbf{G} = (k-1)S$ where $S$ is the number of inequivalent archimedean valuations on $K$. Therefore, $R_{(k-1)S-1}(\mathbf{SL_k}(\mathcal{O}))$ is bounded below by an exponential.

0.2. **Non-nonpositive curvature of arithmetic groups.** If $\mathbf{G}(\mathbb{Z})$ satisfied a reasonable notion of nonpositive curvature (including CAT(0) or combable, for example), we would expect polynomial bounds on isoperimetric inequalities for $\mathbf{G}(\mathbb{Z})$. Thus, not only does Theorem 1 provide a measure of the difference between $\mathbf{G}(\mathbb{Z})$ and $X$, it also exhibits *non*-nonpositive curvature tendencies for $\mathbf{G}(\mathbb{Z})$ when $\mathbf{G}$ is $\mathbb{Q}$-isotropic and $\mathrm{rk}_{\mathbb{R}}\mathbf{G} > 1$.

0.3. **Type restriction.** Our proof of Theorem 1 excludes the remaining types – $G_2$, $F_4$, $E_8$, and $BC_n$ – because groups of these types do not contain proper parabolic subgroups whose unipotent radicals are abelian. Our techniques require an abelian unipotent radical of a maximal $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}$ to construct cycles in $X_{\mathbb{Z}}$.

0.4. **Related results.** It is an open question whether $R_n(\mathbf{G}(\mathbb{Z}))$ is bounded above by a constant when $n < \mathrm{rk}_{\mathbb{R}}\mathbf{G} - 1$. When $n = 0$, it is; this is a theorem of Lubotzky-Mozes-Raghunathan [L-M-R].

Druţu showed that if the $\mathbb{Q}$-relative root system of $\mathbf{G}$ is of type $A_1$ or $BC_1$, then for any $\varepsilon > 0$, $\mathbf{G}(\mathbb{Z})$ has a Dehn function that is bounded above by $L^{2+\varepsilon}$ for $L$ sufficiently large [Dr].

Young proved that $\mathbf{SL}_k(\mathbb{Z})$ has a quadratic Dehn function if $k \geq 5$.

Gromov proved that all of the functions $R_n(\mathbf{G}(\mathbb{Z}))$ are bounded above by an exponential function, and Leuzinger later provided a more detailed proof of this fact ($5.A_7$ [Gr] and Corollary 5.4 [Le]).

## 1. Choice of parabolic

For $\mathbf{G}$ as in Theorem 1, let $\mathbf{T} \leq \mathbf{G}$ be a maximal $\mathbb{Q}$-split torus in $\mathbf{G}$. We let $\Phi_{\mathbb{Q}}$ be the roots of $\mathbf{G}$ with respect to $\mathbf{T}$. Choose an ordering on $\Phi_{\mathbb{Q}}$. We denote the corresponding sets of simple and positive roots by $\Delta_{\mathbb{Q}}$ and $\Phi_{\mathbb{Q}}^+$ respectively.

If $I \subseteq \Delta_{\mathbb{Q}}$, we let $[I] \subseteq \Phi_{\mathbb{Q}}$ be the set of roots that are linear combinations of elements in $I$, and we let $\Phi_{\mathbb{Q}}(I)^+ = \Phi_{\mathbb{Q}}^+ - [I]$.

For each $\alpha \in \Phi_{\mathbb{Q}}$, we let $\mathbf{U}_\alpha \leq \mathbf{G}$ be the root subgroup associated with $\alpha$. For $J \subseteq \Phi_{\mathbb{Q}}$, we let $\mathbf{U}_J = \prod_{\alpha \in J} \mathbf{U}_\alpha$.

We define $\mathbf{T}_I = \cap_{\alpha \in I} \mathrm{Ker}(\alpha)^\circ$ where the superscript $\circ$ denotes the connected component of the identity, and we label the centralizer of $\mathbf{T}_I$ in $\mathbf{G}$ by $\mathbf{Z}_{\mathbf{G}}(\mathbf{T}_I)$.

### 1.1. Maximal parabolics with abelian unipotent radicals.

For any $\alpha_0 \in \Delta_{\mathbb{Q}}$, we let $\mathbf{P}_{\alpha_0}$ be the maximal proper parabolic subgroup of $\mathbf{G}$ given by $\mathbf{U}_{\Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}}-\alpha_0)^+} \mathbf{Z}_{\mathbf{G}}(\mathbf{T}_{\Delta_{\mathbb{Q}}-\alpha_0})$. The unipotent radical of $\mathbf{P}_{\alpha_0}$ is $\mathbf{U}_{\Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}}-\alpha_0)^+}$.

**Lemma 2.** *There is some $\alpha_0 \in \Delta_{\mathbb{Q}}$ such that $\mathbf{U}_{\Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}}-\alpha_0)^+}$ is abelian.*

*Proof.* Suppose $\Delta_{\mathbb{Q}} = \{\alpha_1, \alpha_2, ..., \alpha_k\}$. The set of positive roots $\Phi_{\mathbb{Q}}^+$ contains a "highest root" $\sum_i n_i \alpha_i$ for positive integers $n_i$ such that if $\sum_i m_i \alpha_i \in \Phi_{\mathbb{Q}}^+$, then $m_i \leq n_i$ ([Bou], VI 1 8).

Given that $\Phi_{\mathbb{Q}}$ is a root system of type $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, or $E_7$, there is some $\alpha_0 \in \{\alpha_1, \alpha_2, ..., \alpha_k\}$ such that $n_0 = 1$; consult the list of root systems in the appendix of [Bou].

Since any $\sum_i m_i \alpha_i \in \Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}} - \alpha_0)^+$ has $m_0 > 0$, it follows that any $\sum_i m_i \alpha_i \in \Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}} - \alpha_0)^+$ has $m_0 = 1$, and thus the sum of two elements in $\Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}} - \alpha_0)^+$ is not a root.

Therefore, given $\tau_1, \tau_2 \in \Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}} - \alpha_0)^+$, we have

$$[\mathbf{U}_{\tau_1}, \mathbf{U}_{\tau_2}] \subseteq \mathbf{U}_{\tau_1+\tau_2} = 1$$

□

In what remains, we let $\mathbf{P} = \mathbf{P}_{\alpha_0}$, we let $U_P$ be the real points of $\mathbf{U}_{\Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}}-\alpha_0)^+}$. Thus, we can rephrase Lemma 2 as

**Lemma 3.** *$U_P$ is abelian.*

1.2. **A contracting ray.** Recall that $\mathbf{T}_{\Delta_{\mathbb{Q}}-\alpha_0} \leq \mathbf{Z}_{\mathbf{G}}(\mathbf{T}_{\Delta_{\mathbb{Q}}-\alpha_0}) \leq \mathbf{P}$ is a 1-dimensional $\mathbb{Q}$-split torus. Choose $a_+ \in \mathbf{T}_{\Delta_{\mathbb{Q}}-\alpha_0}(\mathbb{R})$ such that $\alpha_0(a_+) > 1$ and such that the distance in $\mathbf{T}_{\Delta_{\mathbb{Q}}-\alpha_0}(\mathbb{R})$ between 1 and $a_+$ equals 1.

We denote the Lie algebra of $U_P$ by $\mathfrak{u}$.

**Lemma 4.** *There is some $s > 0$ such that for any $v \in \mathfrak{u}$*

$$Ad(a_+^t)v = e^{st}v$$

*Proof.* Recall that

$$\mathfrak{u} = \prod_{\beta \in \Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}}-\alpha_0)^+} \mathfrak{u}_\beta$$

where

$$\mathfrak{u}_\beta = \{\, v \in \mathfrak{u} \mid Ad(x)v = \beta(x)v \text{ for all } x \in \mathbf{T}\,\}$$

If $\beta \in \Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}} - \alpha_0)^+$, then $\beta = \alpha_0 + \sum_{\alpha_i \in \Delta_{\mathbb{Q}}-\alpha_0} n_i \alpha_i$. Since $a_+ \in \cap_{\alpha_i \in \Delta_{\mathbb{Q}}-\alpha_0} \mathrm{Ker}(\alpha_i)^\circ$, we have $\beta(a_+) = \alpha_0(a_+)$ and thus for $v \in \mathfrak{u}$, it follows that $Ad(a_+)v = \alpha_0(a_+)v$.

Let $s = \log\big(\alpha_0(a_+)\big)$.

□

## 2. A horoball in the symmetric space, disjoint from $X_{\mathbb{Z}}$

**Lemma 5.** *There is a maximal $\mathbb{Q}$-torus $\mathbf{A} \leq \mathbf{G}$ such that the maximal $\mathbb{Q}$-split torus of $\mathbf{A}$ is $\mathbf{T}_{\Delta_{\mathbb{Q}}-\alpha_0}$ and such that $\mathbf{A}$ contains a maximal $\mathbb{R}$-split torus of $\mathbf{G}$.*

*Proof.* See Proposition 3.3 in [B-W] where $K = \mathbb{Q}$, $\mathbf{H} = \mathbf{G}$, $\mathbf{T_1} = \mathbf{T}_{\Delta_{\mathbb{Q}}-\alpha_0}$, $S = \{v\}$, and $K_v = \mathbb{R}$. □

Let $\mathbf{Q}$ be a minimal parabolic that contains $\mathbf{A}$ and is contained in $\mathbf{P}$. We let $\Phi_{\mathbb{R}}$ be the roots of $\mathbf{G}$ with respect to the maximal $\mathbb{R}$-split subtorus of $\mathbf{A}$, $\Delta_{\mathbb{R}}$ be the collection of simple roots given by $\mathbf{Q}$, and $\Phi_{\mathbb{R}}^+$ be the corresponding positive roots.

2.1. **Alternate descriptions of the symmetric space.** Let $G = \mathbf{G}(\mathbb{R})$ and let $A \leq G$ be the $\mathbb{R}$-points of the maximal $\mathbb{R}$-split subtorus of $\mathbf{A}$. Recall that $\mathbf{A}(\mathbb{R}) = AB$ for some compact group $B \leq \mathbf{A}(\mathbb{R})$.

Choose a maximal compact subgroup $K \leq G$ that contains $B$. Then $G/K$ is a symmetric space that $G$ acts on by isometries. We name this symmetric space $X$.

Let $U_Q$ be the group of real points of the unipotent radical of $\mathbf{Q}$. By the Iwasawa decomposition, $U_Q A$ acts simply transitively on $X$ and we identify $X$ with $U_Q A$. In this description of $X$, $A$ is a flat.

2.2. **Integral translations in a flat.** By the Dirichlet unit theorem (see e.g. Cor. 1 in Section 4.5 of [P-R]), $\mathbf{A}(\mathbb{Z})$ contains a finite index free abelian subgroup of rank $\mathrm{rk}_{\mathbb{R}}(\mathbf{G}) - 1 = \dim(A) - 1$. Thus, if $A_{\mathbb{Z}}$ is the convex hull in $X$ of the $\mathbf{A}(\mathbb{Z})$-orbit of the point $1 \in U_Q A = X$, then $A_{\mathbb{Z}}$ is a codimension-1 Euclidean subspace of the flat $A$, and $\mathbf{A}(\mathbb{Z})$ acts cocompactly on $A_{\mathbb{Z}}$. We may assume $A_{\mathbb{Z}} \subseteq X_{\mathbb{Z}}$.

2.3. **Horoballs.** Notice that $\{a_+^t\}_{t>0}$ defines a unit-speed geodesic ray that limits to a point in $A^\infty$ which we denote $a_+^\infty$. We let $b_{a_+^t} : U_Q A \to \mathbb{R}$ be the Busemann function corresponding to the geodesic ray $\{a_+^t\}_{t>0}$. That is,

$$b_{a_+^t}(x) = \lim_{t\to\infty} [d(x, a_+^t) - t]$$

We let $A_0 \leq A$ be the codimension-1 subspace of $A$ consisting of those $a \in A$ for which $b_{a_+^t}(a) = 0$. Thus, $A_0$ is orthogonal to $a_+^{\mathbb{R}}$.

**Lemma 6.** *For $T \in \mathbb{R}$, $(b_{a_+^t})^{-1}(-T) = U_Q A_0 a_+^T$.*

*Proof.* We first show that for $u \in U_Q$ and $x \in X$, $b_{a_+^t}(x) = b_{ua_+^t}(x)$ Where $b_{ua_+^t}$ is the Busemann function for the ray $\{ua_+^t\}_{t>0}$ .

Notice that $U_Q = U_P U_a$ where $U_a \leq \mathbf{Z_G}(\mathbf{T}_{\Delta_{\mathbb{Q}} - \alpha_0})(\mathbb{R})$ is a unipotent group whose elements commute with $a_+$.

If $u \in U_P$, then Lemma 4 implies

$$d(a_+^t, ua_+^t) = d(1, a_+^{-t} u a_+^t) \to 0$$

Therefore,

$$b_{a_+^t}(x) = \lim_{t\to\infty} [d(x, a_+^t) - t] = \lim_{t\to\infty} [d(x, ua_+^t) - t] = b_{ua_+^t}(x)$$

The quotient map of a Lie group by a normal subgroup is distance nonincreasing. Because $U_P$ is normal in $U_Q A$, and because $a_+^{\mathbb{R}}$ is normal in $U_a A$, the following composition is distance nonincreasing

$$U_Q A \to U_a A \to U_a A_0$$

We denote the geodesic between points $z, w \in X$ by $\overline{z, w}$. Orthogonality of $A_0$ and $a_+^{\mathbb{R}}$ and the conclusion of the above paragraph show that for any $u \in U_a$, $\overline{1, u}$ is orthogonal to $a_+^{\mathbb{R}}$ at 1 and to $ua_+^{\mathbb{R}}$ at $u$ and thus that $\overline{a_+^t, ua_+^t}$ is orthogonal to $a_+^{\mathbb{R}}$ at $a_+^t$ and to $ua_+^{\mathbb{R}}$ at $ua_+^t$. Furthermore, the length of $\overline{a_+^t, ua_+^t}$ is independent of $t$ since $u$ commutes with $a_+^t$.

Notice that the angle between $\overline{a_+^t, x}$ and $\overline{a_+^t, 1}$ limits to 0 as $t \to \infty$. Similarly, the angle between $\overline{ua_+^t, x}$ and $\overline{ua_+^t, u}$ limits to 0. Hence, the triangle in $X$ with vertices $a_+^t$, $ua_+^t$, and $x$ approaches a triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, and 0. That is

$$d(x, a_+^t) - d(x, ua_+^t) \to 0$$

Consequently, for $u \in U_a$ we have

$$b_{a_+^t}(x) = \lim_{t\to\infty}[d(x, a_+^t) - t] = \lim_{t\to\infty}[d(x, ua_+^t) - t] = b_{ua_+^t}(x)$$

Therefore, for $u \in U_Q$, $b_{a_+^t}(u^{-1}x) = b_{ua_+^t}(x) = b_{a_+^t}(x)$, and it follows that $U_Q(b_{a_+^t})^{-1}(-T) = (b_{a_+^t})^{-1}(-T)$. The lemma is a combination of this last fact together with $A_0 a_+^T \subseteq (b_{a_+^t})^{-1}(-T)$.

□

**Lemma 7.** *For some $T > 0$, $X_{\mathbb{Z}} \subseteq U_Q A_0 a_+^{(-\infty, T]}$.*

*Proof.* If $u \in U_P \cap \mathbf{G}(\mathbb{Z})$, then Lemma 4 implies $a_+^{-n} u a_+^n \to 1$. By Theorem 1.12 of [Ra 2], $\{a_+^t\}_{t>0}$ is not contained in any compact subset of $\mathbf{G}(\mathbb{Z})\backslash\mathbf{G}(\mathbb{R})$.

Theorem A of [Ha 2] states that $X_{\mathbb{Z}} \subseteq (b_{a_+^t})^{-1}[-T, \infty)$ for some $T > 0$, and $(b_{a_+^t})^{-1}[-T, \infty) = U_Q A_0 a_+^{(-\infty, T]}$ by Lemma 6.

□

2.4. **Projecting onto a horosphere.** Let $\pi : U_Q A_0 a_+^{\mathbb{R}} \to U_Q A_0$ be the obvious projection of $X$ onto the horosphere $(b_{a_+^t})^{-1}(0)$.

**Lemma 8.** *There is some $M > 0$ such that for any $x_1, x_2 \in X_{\mathbb{Z}}$, we have $d(x_1, x_2) + M \geq d(\pi(x_1), \pi(x_2))$.*

*Proof.* Recall that $U_Q = U_P U_a$ where elements of $U_a \leq \mathbf{P}$, and elements of $A_0$, commute with $a_+$. Similar to Lemma 4, we have that for any $t > 0$ and any $v$ in the Lie algebra of $U_Q A$ that

$$||Ad(a_+^{-t})v|| \leq ||v||$$

Let $T$ be as in Lemma 7 and define $\pi_T : U_Q A_0 a_+^{(-\infty, T]} \to U_Q A_0 a_+^T$ by $\pi_T = R_{a_+^T} \circ \pi$ where $R_{a_+^T}$ is right multiplication by $a_+^T$.

We claim that $\pi_T$ is distance nonincreasing. To see this, first let $v$ be a tangent vector to $X$ at the point $a_+^t$ for some $t \leq T$. With $|| \cdot ||_x$ as the norm at $x$, and $f_*$ as the differential of $f$, we have

$$\begin{aligned} ||(\pi_T)_* v||_{\pi_T(a_+^t)} &= ||(R_{a_+^{T-t}})_* v||_{a_+^T} \\ &= ||(L_{a_+^{t-T}})_*(R_{a_+^{T-t}})_* v||_{a_+^t} \\ &= ||Ad(a_+^{t-T})v||_{a_+^t} \\ &\leq ||v||_{a_+^t} \end{aligned}$$

Left-translations by $U_Q A_0$ show that for any $x \in U_Q A_0 a_+^{(-\infty,T]}$, and any $v \in T_x X$,

$$||(\pi_T)_* v||_{\pi_T(x)} \leq ||v||_x$$

For any path $c : [0,1] \to U_Q A_0 a_+^{(-\infty,T]}$, apply $\pi_T$ to those segments contained in $U_Q A_0 a_+^{(-\infty,T]}$ to define a path between $\pi_T(c(0))$ and $\pi_T(c(1))$. This new path will have its length bounded above by the length of $c$ as is easily verified from the inequality on norms of vectors from above. This confirms our claim that $\pi_T$ is distance nonincreasing.

To confirm the lemma, notice that similarly, the map $R_{a_+^{-T}} : U_Q A_0 a_+^T \to U_Q A_0$ translates all point in $X$ a distance of

$$d(x, R_{a_+^{-T}}(x)) = d(1, a_+^{-T})$$

Therefore,

$$d(R_{a_+^{-T}}(x_1), R_{a_+^{-T}}(x_2)) \leq d(x_1, x_2) + 2d(1, a_+^{-T})$$

The lemma follows as $\pi = R_{a_+^{-T}} \circ \pi_T$. □

## 3. Choice of a cell in $X_{\mathbb{Z}}$

We want to construct a cycle $Y \subseteq X_{\mathbb{Z}}$. In this section we begin by constructing a cell $F \subseteq A_0$ that will be used in the construction of $Y$.

**Lemma 9.** *$A_0 \subseteq X_{\mathbb{Z}}$.*

*Proof.* Both $A_0$ and the convex hull of $A_{\mathbb{Z}}$ are codimension 1 subspaces of $A$. Since $A_{\mathbb{Z}} \subseteq X_{\mathbb{Z}} \subseteq U_Q A_0 a_+^{(-\infty,T)}$ we have that $A_{\mathbb{Z}} \subseteq A_0 a_+^{(-\infty,T)}$. Therefore $A_{\mathbb{Z}}$ and $A_0$ are parallel hyperplanes. Since the both contain 1, they are equal.

□

Let $X^\infty$ be the spherical Tits building for $X = U_Q A$, and let $A^\infty \subseteq X^\infty$ be the apartment given by $A$. Let $\Pi^\infty \subseteq X^\infty$ be the simplex given by $\mathbf{P}$ and let $\Pi_-^\infty \subseteq X^\infty$ be the simplex opposite of $\Pi^\infty$ in $A^\infty$, or equivalently, $\Pi_-^\infty$ is the simplex given by the parabolic group $\mathbf{P}^- = \mathbf{U}_{\Phi_{\mathbb{Q}}(\Delta_{\mathbb{Q}} - \alpha_0)^-} \mathbf{Z_G}(\mathbf{T}_{\Delta_{\mathbb{Q}} - \alpha_0})$.

Denote the star of $\Pi_-^\infty$ in $A^\infty$ by $\Sigma \subseteq A^\infty$. Note that $\Sigma$ is homeomorphic to a $\mathrm{rk}_{\mathbb{R}}(\mathbf{G}) - 1$ ball. We denote the codimension 1 faces of $\Sigma$ as $\Sigma_1, ..., \Sigma_n$.

3.1. **$A_0^\infty$ and $\Sigma$ are disjoint.** Let $\Psi \subseteq \Phi_{\mathbb{R}}$ be such that $\mathbf{U}_\Psi = \mathbf{R_u}(\mathbf{P}^-)$. Given $b \in A_0$ we define the following sets of roots:

$$C(b) = \{\, \beta \in \Psi \mid \beta(b) > 1 \,\}$$

$$Z(b) = \{\, \beta \in \Psi \mid \beta(b) = 1 \,\}$$

$$E(b) = \{\, \beta \in \Psi \mid \beta(b) < 1 \,\}$$

Thus, if $U_{C(b)}$ are the real points of $\mathbf{U}_{C(b)}$ etc., then $\mathbf{R_u}(\mathbf{P}^-)(\mathbb{R}) = U_{C(b)}U_{Z(b)}U_{E(b)}$.

**Lemma 10.** *There is a sequence $\gamma_n \in \mathbf{R_u}(\mathbf{P}^-)(\mathbb{Z}) - 1$ such that $d(\gamma_n, U_{C(b)}) \to 0$.*

*Proof.* There is a $\mathbb{Q}$-isomorphism of the variety $\mathbf{R_u}(\mathbf{P}^-)$ with affine space that maps $\mathbf{U}_{C(b)}$ onto an affine subspace. Therefore, the problem reduces to showing that the distance between $\mathbb{Z}^n - 1$ and a line in $\mathbb{R}^n$ that passes through the origin is bounded above by any positive number, and this is well known.

□

**Lemma 11.** $A_0^\infty \cap \Sigma = \emptyset$

*Proof.* Suppose $A_0^\infty \cap \Sigma \neq \emptyset$. Then there is some $b \in A_0$ such that $b^\infty \in \Sigma$ where $b^\infty = \lim_{t\to\infty} b^t$.

If $\mathfrak{C} \subseteq \Sigma$ is a chamber, then $\Pi_-^\infty \subseteq \mathfrak{C}$. Hence, the minimal $\mathbb{R}$-parabolic subgroup corresponding to $\mathfrak{C}$ contains $\mathbf{R_u}(\mathbf{P}^-)$ and thus elements of $\mathbf{R_u}(\mathbf{P}^-)(\mathbb{R})$ fix $\mathfrak{C}$ pointwise. That is, elements of $\mathbf{R_u}(\mathbf{P}^-)(\mathbb{R})$ fix $\Sigma$ pointwise, so they fix $b^\infty$.

Let $u \in \mathbf{R_u}(\mathbf{P}^-)(\mathbb{R})$. Then $ub^\infty = b^\infty$, so $d(ub^t, b^t)$ is bounded, so $\{b^{-t}ub^t\}_{t>0}$ is bounded. It follows that $\beta(b^{-1}) \leq 1$ for all $\beta \in \Psi$, or equivalently that $\beta(b) \geq 1$. Hence, $E(b) = \emptyset$ and $\mathbf{R_u}(\mathbf{P}^-)(\mathbb{R}) = U_{C(b)}U_{Z(b)}$.

Now we use Lemma 10. For any $n \in \mathbb{N}$, there exists $\gamma_n \in \mathbf{R_u}(\mathbf{P}^-)(\mathbb{Z}) - 1$ with $d(\gamma_n, U_{C(b)}) < 1/n$. Let $\gamma_n = c_n z_n$ where $c_n \in U_{C(b)}$, and $z_n \in U_{Z(b)}$. Notice that $z_n \to 1$, $bz_n = z_n b$, and that $b^{-t}c_n b^t \to 1$ as $t \to \infty$.

Choose $t_n > 0$ such that $d(b^{-t_n}c_n b^{t_n}, 1) < 1/n$. Then

$$b^{-t_n}\gamma_n b^{t_n} = (b^{-t_n}c_n b^{t_n})z_n \to 1$$

By Theorem 1.12 of [Ra 2], $\{b^{-t}\}_{t>0}$ is not contained in any compact subset of $\mathbf{G}(\mathbb{Z})\backslash\mathbf{G}(\mathbb{R})$, which contradicts that $b^{-t} \in A_0 \subseteq X_{\mathbb{Z}}$ (Lemma 9).

□

3.2. $L > 0$ **and choice of cell in** $A_0$**.** At this point, we fix $L > 0$ to be sufficiently large. We will use this fixed $L$ for our proof of the Theorem 1.

There is a unique great sphere in $A_\infty$ that contains $\Sigma_i$, and any great sphere defines two hemispheres in $A_\infty$, each of which corresponds to at least one root in $\Phi_\mathbb{R}$. The hemisphere containing the simplex associated with $\mathbf{Q}$ corresponds to a positive root $\beta_i \in \Phi_\mathbb{R}^+$ whose kernel $W_i \subseteq A$ has a visual image in $A^\infty$ that is the great sphere containing $\Sigma_i$.

We let $F$ be the component of $A_0 - \cup_i a_+^L W_i$ that contains 1.

**Lemma 12.** *$F$ is compact Euclidean polyhedron with volume $O(L^{rk_\mathbb{R}\mathbf{G-1}})$.*

*Proof.* The visual cone of $\Sigma$ in $A$ based at $a_+^L$ is a connected component of $A - \cup_i a_+^L W_i$.

The lemma follows if $\Sigma$ and $a_+^\infty$ are contained in distinct components of $A^\infty - A_0^\infty$, and if $a_+^{-\infty} = \lim_{t\to\infty} a_+^{-t} \in \Sigma$. That is indeed the case: $\alpha(a_+) > 1$ for all $\alpha \in \Phi_\mathbb{Q}(\Delta_\mathbb{Q} - \alpha_0)^+$ so $\mathbf{P} = \mathbf{U}_{\Phi_\mathbb{Q}(\Delta_\mathbb{Q}-\alpha_0)^+}\mathbf{Z_G}(\mathbf{T}_{\Delta_\mathbb{Q}-\alpha_0})$ fixes $a_+^\infty$. Hence, $a_+^\infty \in \Pi^\infty$. The antipodal map on $A^\infty$ stabilizes $A_0^\infty$, transposes $a_+^\infty$ and $a_+^{-\infty}$, and maps $\Pi^\infty$ onto $\Pi_-^\infty \subseteq \Sigma$.

□

We denote the face of $F$ given by $a_+^L W_i \cap F$ as $F_i$, so that the topological boundary of $F$ equals $\cup_{i=1}^n F_i$.

## 4. Other cells in $X_\mathbb{Z}$ and their homological boundaries

We denote the real points of the root group $\mathbf{U}_{(\beta_i)}$ as $U_i$, and $\langle U_i \rangle_i$ is the group generated by the $U_i$ for $i \in \{1, 2, ..., n\}$.

**Lemma 13.** *For each $i \in \{1, 2, ..., n\}$, $U_i \leq U_P$, and thus $\langle U_i \rangle_i \leq U_P$ is abelian.*

*Proof.* Since $\beta_i \in \Phi_\mathbb{R}^+$, we have $U_i \leq U_Q = U_P U_a$. Either $U_i \leq U_P$ or $U_i \leq U_a \leq \mathbf{Z_G}(\mathbf{T}_{\Delta_\mathbb{Q}-\alpha_0})$.

Because $\mathbf{Z_G}(\mathbf{T}_{\Delta_\mathbb{Q}-\alpha_0})$ is contained in both $\mathbf{P}$ and $\mathbf{P}^-$, the latter case implies that $U_i$ fixes the antipodal cells $\Pi^\infty$ and $\Pi_-^\infty$. The fixed point set of $U_i$ is a hemisphere in $A^\infty$ with boundary equal to $W_i^\infty$. Thus, $\Pi^\infty$ and $\Pi_-^\infty$ are contained in $W_i^\infty$, which contradicts that $\Sigma_i = \Sigma \cap W_i^\infty$ does not contain $\Pi_-^\infty$.

Having ruled out the latter case, $U_i \leq U_P$ and the lemma follows from Lemma 3.

□

4.1. **A space for making cycles in** $X_\mathbb{Z}$**.**

**Lemma 14.** $\langle U_i \rangle_i F \subseteq X_\mathbb{Z}$.

*Proof.* Because $\mathbf{R_u}(\mathbf{P})$ is unipotent, $\mathbf{R_u}(\mathbf{P})(\mathbb{Z})$ is a cocompact lattice in $U_P$. We choose a compact fundamental domain $D \subseteq U_P$ for the $\mathbf{R_u}(\mathbf{P})(\mathbb{Z})$-action.

There is also a compact set $C \subseteq A_0 = A_\mathbb{Z}$ such that $\mathbf{A}(\mathbb{Z})C = A_\mathbb{Z} = A_0$. As $DC$ is compact, we may assume that $\mathbf{G}(\mathbb{Z})DC \subseteq X_\mathbb{Z}$.

Recall that $\mathbf{A}$ is contained in $\mathbf{P}$, so $\mathbf{A}$ normalizes $\mathbf{R_u}(\mathbf{P})$. Hence,

$$\begin{aligned}\langle U_i \rangle_i A_0 &\subseteq U_P \mathbf{A}(\mathbb{Z}) C \\ &\subseteq \mathbf{A}(\mathbb{Z}) U_P C \\ &\subseteq \mathbf{A}(\mathbb{Z}) \mathbf{R_u}(\mathbf{P})(\mathbb{Z}) DC \\ &\subseteq \mathbf{G}(\mathbb{Z}) DC \\ &\subseteq X_\mathbb{Z}\end{aligned}$$

□

4.2. **Description of cells used to build our cycle.** Given $i \in \{1, ..., n\}$, let $f_i$ be a point in $F_i$ that minimizes the distance to $1 \in A$, and let $u_i \in U_i$ be such that $d(u_i f_i, f_i) = 1$. Since $F_i \subseteq a_+^L W_i$, any $f \in F_i$ can be expressed as $f = w f_i$ for some $w \in Ker(\beta_i)$. It follows that $Ad(w)$ acts trivially on the Lie algebra of $U_i$, that $u_i$ commutes with $w$, and that

$$d(u_i f, f) = d(u_i w f_i, w f_i) = d(w u_i f_i, w f_i) = d(u_i f_i, f_i) = 1$$

Setting $\overline{u_i} = \{u_i^t\}_{t=0}^1$, the space $\overline{u_i} F_i$ is a metric direct product of volume $O(L^{\dim(F_i)})$.

For $I \subseteq \{1, ..., n\}$, let let $F_I = \cap_{i \in I} F_i$ with $F_\emptyset = F$. And let $u_I = \prod_{i \in I} u_i$ and $\overline{u_I} = \prod_{i \in I} \overline{u_i}$ with $\overline{u_\emptyset} = u_\emptyset = 1$.

Similar to the case when $|I| = 1$, $\overline{u_I} F_I$ is a metric direct product of volume $O(L^{\dim(F_I)})$.

4.3. **Homological boundaries of the cells.** We endow each interval $\overline{u_i} = [0, u_i]$ with the standard orientation on the closed interval, and we orient each $\overline{u_I}$ with the product orientation, where the product is taken over ascending order in $\mathbb{N}$. Given $m \in I$, we let $s_I(m)$ be the ordinal of $m$ assigned by the order on $I$ induced by $\mathbb{N}$. Notice that the standard formula for the homological boundary of a cube then becomes

$$\partial(\overline{u_I}) = \sum_{m \in I} (-1)^{s_I(m)} \big( \overline{u_{I-m}} - u_m \overline{u_{I-m}} \big)$$

We assign an orientation to $F$, and then assign the orientation to each $F_i$ such that

$$\partial(F) = \sum_{i=1}^n F_i$$

In what follows, if we are given a set $I \subseteq \{1, ..., n\}$ with an ordering (which may differ from the standard order on $\mathbb{N}$), and if $m \in \{1, ..., n\}$ with $m \notin I$, then the set $I \cup m$ is ordered such that the original order on $I$ is preserved and $m$ is the "greatest" element of $I \cup m$. For example, $\{1, 7, 5\} \cup 3 = \{1, 7, 5, 3\}$.

If $m \in I$, for some ordered set $I \subseteq \{1, ..., n\}$, then we endow $I - m$ with the order restricted from $I$.

For an ordered $I$ and $m \in I$, let $r_I(m) = 1$ if an even number of transpositions are required to transform the order on $I$ to the order on $(I - m) \cup m$. Let $r_I(m) = -1$ otherwise.

Given an ordering on a set $I \subseteq \{1, ..., n\}$, an orientation on $F_I$, and some $m \in \{1, ..., n\}$ with $m \notin I$, we define the orientation of $F_{I\cup m}$ to be such that $F_{I\cup m}$, and not $-F_{I\cup m}$, is the oriented cell that appears as a summand in $\partial(F_I)$. Therefore

$$\partial(F_I) = \sum_{m \notin I} F_{I\cup m}$$

In what follows, whenever we write the *exact* symbols $F_I$ or $F_{I'}$ – but not necessarily the symbol $F_{I\cup m}$ – the order on $I$ or $I'$ will be the order from $\mathbb{N}$. Thus, the orientation on $F_I$ and $F_{I'}$ can be unambiguously determined from the above paragraph.

It's easy to check that if $I$ is ordered by the standard order on $\mathbb{N}$ and $m \in I$, then $(-1)^{s_I(m)} r_I(m) = (-1)^{|I|}$ and thus

$$-(-1)^{s_I(m)} = (-1)^{|I|-1} r_I(m)$$

Suppose $w_0$ is an outward normal vector for $F_{I\cup m}$ with respect to $F_I$, and $w_1, ...w_k$ is a collection of vectors tangent to $F_{I\cup m}$ such that $\{w_0, w_1, ..., w_k\}$ defines the orientation for $F_I$. Then $\{w_1, ..., w_k\}$ defines the orientation for $F_{I\cup m}$. If $\{v_1, ..., v_{|I|}\}$ is an ordered basis for the tangent space of $\overline{u_I}$ that induces the standard orientation on $\overline{u_I}$, then $|I|$ transpositions are required to arrange the ordered basis

$$\{w_0, v_1, ..., v_{|I|}, w_1, ..., w_k\}$$

into the ordered basis

$$\{v_1, ..., v_{|I|}, w_0, w_1, ..., w_k\}$$

That is, the orientation on $\overline{u_I} F_{I\cup m}$ defined above is a $(-1)^{|I|}$-multiple of the orientation on $\overline{u_I} F_{I\cup m}$ assigned by $\partial(\overline{u_I} F_I)$.

It follows from this fact and our above formulas for $\partial(\overline{u_I})$ and $\partial(F_I)$ that

$$\partial(\overline{u_I} F_I) = \sum_{m \in I} (-1)^{s_I(m)} \big(\overline{u_{I-m}} - u_m \overline{u_{I-m}}\big) F_I + (-1)^{|I|} \sum_{m \notin I} \overline{u_I} F_{I\cup m}$$

## 5. A cycle in $X_{\mathbb{Z}}$

Let

$$Y = \sum_{\substack{K,I\subseteq\{1,\dots,n\}\\ K\cap I=\emptyset}} (-1)^{|K|} u_K \overline{u_I} F_I$$

**Lemma 15.** *$Y$ is a cycle that is contained in $X_{\mathbb{Z}}$ and has volume $O(L^{\mathrm{rk}_{\mathbb{R}}\mathbf{G-1}})$.*

*Proof.* Each cell of $Y$ is contained in $X_{\mathbb{Z}}$ by Lemma 14 and has volume $O(L^k)$ for $k \leq \mathrm{rk}_{\mathbb{R}}\mathbf{G} - 1$, so we have to check that $\partial Y = 0$.

From our formula for $\partial(\overline{u_I}F_I)$ we have that

$$\begin{aligned}
\partial Y = & \sum_{\substack{K,I\subseteq\{1,\dots,n\}\\ K\cap I=\emptyset}} (-1)^{|K|} u_K \Big[ \sum_{m\in I} (-1)^{s_I(m)} \big(\overline{u_{I-m}} - u_m \overline{u_{I-m}}\big) F_I \\
& \qquad + (-1)^{|I|} \sum_{m\notin I} \overline{u_I} F_{I\cup m}\Big] \\
= & \sum_{\substack{K,I\subseteq\{1,\dots,n\}\\ K\cap I=\emptyset}} \sum_{m\in I} (-1)^{s_I(m)} (-1)^{|K|} u_K \big(\overline{u_{I-m}} - u_m \overline{u_{I-m}}\big) F_I \\
& \qquad + \sum_{\substack{K,I\subseteq\{1,\dots,n\}\\ K\cap I=\emptyset}} (-1)^{|I|} \sum_{m\notin I} (-1)^{|K|} u_K \overline{u_I} F_{I\cup m}
\end{aligned}$$

For $K, I\subseteq\{1, \dots, n\}$ with $K \cap I = \emptyset$ we have

$$\begin{aligned}
&\sum_{m\notin I} (-1)^{|K|} u_K \overline{u_I} F_{I\cup m} \\
&\qquad = \sum_{m\notin I\cup K} (-1)^{|K|} u_K \overline{u_I} F_{I\cup m} \\
&\qquad\qquad + \sum_{m\in K} (-1)^{|K|} u_K \overline{u_I} F_{I\cup m} \\
&\qquad = \sum_{m\notin I\cup K} (-1)^{|K|} u_K \overline{u_{(I\cup m)-m}} F_{I\cup m} \\
&\qquad\qquad + \sum_{m\in K} (-1)^{|K|} u_{K-m} u_m \overline{u_{(I\cup m)-m}} F_{I\cup m}
\end{aligned}$$

There is a natural bijection between triples $(I, K, m)$ where $K\cap I = \emptyset$ and $m \notin I \cup K$, and triples $(I', K', m)$ where $K' \cap I' = \emptyset$ and $m \in I'$. To realize the bijection, let $K' = K = K - m$ and $I' = I \cup m$.

There is also a bijection between triples $(I, K, m)$ where $K \cap I = \emptyset$ and $m \in K$, and triples $(I', K', m)$ where $K' \cap I' = \emptyset$ and $m \in I'$. This bijection is also realized by setting $K' = K - m$ and $I' = I \cup m$.

Therefore, if we let $K' = K - m$ and $I' = I \cup m$ then the above equation gives

$$
\begin{aligned}
&\sum_{\substack{K,I \subseteq \{1,\ldots,n\} \\ K \cap I = \emptyset}} (-1)^{|I|} \sum_{m \notin I} (-1)^{|K|} u_K \overline{u_I} F_{I \cup m} \\
&= \sum_{\substack{K',I' \subseteq \{1,\ldots,n\} \\ K' \cap I' = \emptyset}} (-1)^{|I'|-1} \Big[ \sum_{m \in I'} (-1)^{|K'|} r_{I'}(m) u_{K'} \overline{u_{I'-m}} F_{I'} \\
&\qquad + \sum_{m \in I'} (-1)^{|K' \cup m|} r_{I'}(m) u_{K'} u_m \overline{u_{I'-m}} F_{I'} \Big] \\
&= \sum_{\substack{K',I' \subseteq \{1,\ldots,n\} \\ K' \cap I' = \emptyset}} (-1)^{|I'|-1} \Big[ \sum_{m \in I'} (-1)^{|K'|} r_{I'}(m) u_{K'} \overline{u_{I'-m}} F_{I'} \\
&\qquad - \sum_{m \in I'} (-1)^{|K'|} r_{I'}(m) u_{K'} u_m \overline{u_{I'-m}} F_{I'} \Big] \\
&= \sum_{\substack{K,I \subseteq \{1,\ldots,n\} \\ K \cap I = \emptyset}} (-1)^{|I|-1} \Big[ \sum_{m \in I} (-1)^{|K|} r_I(m) u_K \overline{u_{I-m}} F_I \\
&\qquad - \sum_{m \in I} (-1)^{|K|} r_I(m) u_K u_m \overline{u_{I-m}} F_I \Big] \\
&= \sum_{\substack{K,I \subseteq \{1,\ldots,n\} \\ K \cap I = \emptyset}} (-1)^{|I|-1} \sum_{m \in I} (-1)^{|K|} r_I(m) u_K \left( \overline{u_{I-m}} - u_m \overline{u_{I-m}} \right) F_I \\
&= \sum_{\substack{K,I \subseteq \{1,\ldots,n\} \\ K \cap I = \emptyset}} \sum_{m \in I} (-1)^{|I|-1} r_I(m) (-1)^{|K|} u_K \left( \overline{u_{I-m}} - u_m \overline{u_{I-m}} \right) F_I \\
&= - \sum_{\substack{K,I \subseteq \{1,\ldots,n\} \\ K \cap I = \emptyset}} \sum_{m \in I} (-1)^{s_I(m)} (-1)^{|K|} u_K \left( \overline{u_{I-m}} - u_m \overline{u_{I-m}} \right) F_I
\end{aligned}
$$

Substituting the preceding equation into our equation for $\partial Y$ proves

$$\partial Y = 0$$

□

## 6. Fillings of Y

There exists polynomially efficient fillings for $Y$ in the symmetric space $X$.

**Lemma 16.** *There exists a chain $Z$ with volume $O(L^{\mathrm{rk}_{\mathbb{R}}\mathbf{G}})$ and $\partial Z = Y$.*

*Proof.* As $Y \subseteq \overline{u_I}F$, it follows from Lemma 4 that there is some $T = O(L)$ such that $a_+^T Y$ is contained in an $\varepsilon$-neighborhood of $a_+^T F$, which is isometric to $F$. Thus, there is a filling, $Z_0$, of $a_+^T Y$ of volume $O(L^{\mathrm{rk}_{\mathbb{R}}\mathbf{G-1}})$.

Let $Z = Z_0 \cup_{t \in \{1,T\}} a_+^T Y$.

□

6.1. **Fillings of $Y$ in $X_{\mathbb{Z}}$.** In contrast to Lemma 16, the fillings of $Y$ that are contained in $X_{\mathbb{Z}}$ have volumes bounded below by an exponential in $L$. A fact that we will prove after a couple of helpful lemmas.

For $f \in F$, define $d_i(f)$ to be the distance in the flat $A$ between $f$ and $a_+^L W_i$.

**Lemma 17.** *There are $s_i > 1$ and $s_0 > 0$ such that the cube $\overline{u_I}f$ with the path metric is isometric to $\prod_{i \in I}[0, e^{s_i d_i(f)+s_0}]$.*

*Proof.* It suffices to prove that $\overline{u_i}f$ is isometric to $[0, e^{s_i d_i(f)+s_0}]$.

Choose $b_i \in A$ such that $d(b_i, 1) = d(f, a_+^L W_i) = d_i(f)$ and such that there exists some $w_i \in W_i$ with $f = b_i a_+^L w_i$. Notice that $W_i$ separates $b_i$ from $a_+^L$ in $A$. Since $U_i \leq U_P$, Lemma 4 shows that $\beta_i(a_+^L) > 1$. It follows that $\beta_i(b_i) < 1$.

With $d_\Omega$ as the path metric of a subspace $\Omega \subseteq X$,

$$d_{U_i f}(u_i f, f) = d_{U_i f}(u_i b_i a_+^L w_i, b_i a_+^L w_i)$$

As $W_i$ is the kernel of $\beta_i$, $w_i$ commutes with $u_i$ implying

$$\begin{aligned} d_{U_i f}(u_i f, f) &= d_{w_i^{-1} U_i f}(u_i b_i a_+^L, b_i a_+^L) \\ &= d_{U_i}(a_+^{-L} b_i^{-1} u_i b_i a_+^L, 1) \end{aligned}$$

On the Lie algebra of $U_i$, $Ad(a_+^{-L} b_i^{-1})$ scales by $\beta_i(a_+^{-L})\beta_i(b_i)^{-1}$.

□

In the above lemma we may let $f = 1$ and let $I$ be the singleton $i$. It can easily be seen that $d_i(1) = O(L)$ which leaves us

**Lemma 18.** *There is some $C > 0$ such that $d_{U_i}(u_i, 1) \geq e^{CL+s_0}$ for any $i$.*

We conclude our proof of Theorem 1 with the following

**Lemma 19.** *Suppose there is a chain* $B \subseteq X_{\mathbb{Z}}$ *such that* $\partial B = Y$. *Then the volume of* $B$ *is bounded below by* $e^{C_0 L}$ *for some* $C_0 > 0$.

*Proof.* Suppose $B$ has volume $\lambda$. By Lemma 8, $\pi(B) \subseteq U_Q A_0$ has volume $O(\lambda)$.

Recall that $Y \subseteq U_Q A_0$, so $\partial \pi(B) = Y$.

After perturbing $\pi(B)$, we may assume that $\pi(B)$ is transverse to $U_Q$, and that the 1-manifold $\pi(B) \cap U_Q$ has length proportional to the volume of $\pi(B)$. Since

$$\partial(\pi(B) \cap U_Q) = \partial \pi(B) \cap U_Q = Y \cap U_Q = \{u_I\}_{I \subseteq \{1,\ldots,n\}}$$

there is an $I \subseteq \{1, ..., n\}$ and a path $\rho : [0,1] \to \pi(B) \cap U_Q$ such that $\rho(0) = 1$ and $\rho(1) = u_I$ with $\text{length}(\rho) = O(\lambda)$.

Choose $i \in I$. $U_Q$ is nilpotent, so the distortion of the projection $q : U_Q \to U_i$ is at most polynomial. Therefore, $q \circ \rho$ is a path in $U_i$ between 1 and $u_i$ with $\text{length}(q \circ \rho) = O(\lambda^k)$ for some $k \in \mathbb{N}$.

The preceding lemma showed $e^{CL+s_0} \leq \text{length}(q \circ \rho)$. Therefore, $\lambda \geq \kappa e^{\frac{C}{k}L}$ for some $\kappa > 0$.

□

Combining Lemmas 16 and 19 yields Theorem 1.

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