AN INFINITELY GENERATED VIRTUAL COHOMOLOGY GROUP FOR NONCOCOMPACT ARITHMETIC GROUPS OVER FUNCTION FIELDS

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Abstract. Let $\mathbf{G}(\mathcal{O}_S)$ be a noncocompact irreducible arithmetic group over a global function field K of characteristic p, and let Γ be a finite-index, residually p-finite subgroup of $\mathbf{G}(\mathcal{O}_S)$. We show that the cohomology of Γ in the dimension of its associated Euclidean building with coefficients in the field of p elements is infinite.

Let K be a global function field that contains the field with p elements, $\mathbb{F}_p$. We let S be a finite nonempty set of inequivalent valuations of K. The ring $\mathcal{O}_S \subseteq K$ will denote the corresponding ring of S-integers. For any $v \in S$, we let K_v be the completion of K with respect to v so that K_v is a locally compact field.

We denote by $\mathbf{G}$ a connected noncommutative absolutely almost simple K-group, and we let

$$k(\mathbf{G}, S) = \sum_{v \in S} \mathrm{rank}_{K_v} \mathbf{G}$$

so that $k(\mathbf{G}, S)$ is the dimension of the Euclidean building on which the arithmetic group $\mathbf{G}(\mathcal{O}_S)$ acts as a lattice. Thus for example, $k(\mathbf{SL_n}, S) = |S|(n-1)$.

If $\mathbf{G}$ is K-anisotropic, then $\mathbf{G}(\mathcal{O}_S)$ contains a torsion-free finite-index subgroup that acts freely and cocompactly on a Euclidean building of dimension $k(\mathbf{G}, S)$. Determining the finiteness properties of arithmetic groups $\mathbf{G}(\mathcal{O}_S)$ in the case that $\mathbf{G}$ is K-isotropic has been more difficult. The model for the K-isotropic case was provided by the following theorem of Stuhler [14].

Theorem 1. *The arithmetic group $\mathbf{SL_2}(\mathcal{O}_S)$ is of type $F_{k(\mathbf{SL_2},S)-1}$, and if Γ is any finite-index subgroup of $\mathbf{SL_2}(\mathcal{O}_S)$ whose only torsion elements are p-elements, then $\mathrm{H}^{k(\mathbf{SL_2},S)}(\Gamma ; \mathbb{F}_p)$ is infinite.*

Recall that a group π is *of type F_n* if there exists a $K(\pi, 1)$ with finite n-skeleton.

The author gratefully acknowledges the support of the National Science Foundation.

It is well-known, by Selberg's Lemma, that $\mathbf{SL_2}(\mathcal{O}_S)$, or that any arithmetic group over function fields $\mathbf{G}(\mathcal{O}_S)$ as above, contains a finite-index subgroup whose only torsion elements are p-elements.

Bux-Köhl-Witzel [5] completely generalized "half" of Theorem 1 with the following theorem.

Theorem 2. *If* $\mathbf{G}$ *is* K*-isotropic, then* $\mathbf{G}(\mathcal{O}_S)$ *is of type* $F_{k(\mathbf{G},S)-1}$.

Important evidence for the theorem of Bux-Köhl-Witzel was contributed by Behr [3], Abels [1], Abramenko [2], and Bux-Wortman [7].

There are now three proofs that $\mathbf{G}(\mathcal{O}_S)$ as in Theorem 2 is not of type $F_{k(\mathbf{G},S)}$ due to Bux-Wortman [6], Bux-Köhl-Witzel [5], and Kropholler [11] as observed by Gandini [9]. However, outside of the case that $k(\mathbf{G}, S) = 1$, the "second half" of Stuhler's Theorem 1 had not been generalized to include any other arithmetic groups. This paper uses the results of Bux-Köhl-Witzel and Schulz [13] to further generalize the results of Stuhler by proving

Theorem 3. *Suppose* $\mathbf{G}$ *is* K*-isotropic. If* Γ *is a finite-index subgroup of* $\mathbf{G}(\mathcal{O}_S)$ *that is residually p-finite, then* $\mathrm{H}^{k(\mathbf{G},S)}(\,\Gamma\,;\,\mathbb{F}_p\,)$ *is infinite.*

A group Γ is *residually p-finite* if for any nontrivial $\gamma \in \Gamma$, there is a homomorphism of Γ onto a finite p-group that evaluates γ nontrivially. Such finite-index subgroups of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ are well-known to exist, by Platanov's Theorem, and we provide a proof of their existence in Section 7 for completeness.

To compare Theorems 1 and 3, notice that any torsion element of a residually p-finite group has order a power of p. The author does not know of an example of a finite-index subgroup $\Gamma \leq \mathbf{G}(\mathcal{O}_S)$ whose only torsion elements are p-elements, but such that Γ is not residually p-finite.

As an example of Theorem 3, there is a finite-index subgroup of $\mathbf{SL_n}(\mathcal{O}_S)$ whose cohomology in dimension $|S|(n-1)$ with coefficients in $\mathbb{F}_p$ is infinite. In particular, there is a finite-index subgroup Γ of $\mathbf{SL_n}(\mathbb{F}_p[t])$ such that $H^{n-1}(\Gamma; \mathbb{F}_p)$ is infinite.

0.1. **Outline of the proof.** To prove Theorem 1, Stuhler analyzed the cell stabilizers of the $\mathbf{SL_2}(\mathcal{O}_S)$-action on the associated Euclidean building which is a product of regular $(p+1)$-valent trees. The cell stabilizers of Γ as in Theorem 1 are products of the group $\mathbb{F}_p$, but the cell stabilizers of a random arithmetic group acting on its associated Euclidean building are more difficult to describe and to work with, so our proof of Theorem 3 proceeds in a different direction.

The main tool in our proof of Theorem 3 is the work of Bux-Köhl-Witzel, and we spend a good portion of the beginning of our proof recalling their work. Let $k = k(\mathbf{G}, S)$ and let X be the Euclidean building that $\mathbf{G}(\mathcal{O}_S)$ acts on as a lattice. Bux-Köhl-Witzel finds a $\mathbf{G}(\mathcal{O}_S)$-invariant, cocompact, $(k-2)$-connected complex $X_{k-2} \subseteq X$. We attach k-cells and $(k+1)$-cells to X_{k-2} to produce a k-connected complex X_k endowed with a Γ-action and a Γ-equivariant map $\psi : X_k \to X$.

We find an unbounded sequence of points $\Gamma y_n \in \Gamma\backslash X$, and a sequence of normal subgroups Γ_n of Γ with index a power of p such that each $y_n \in X$ is contained in a neighborhood of X that injects into $\Gamma_n\backslash X$, and such that the p-group Γ/Γ_n acts on the homology of the image of the neighborhood in the quotient, with coefficients in $\mathbb{F}_p$. The action of the p-group on the homology group produces a functional that nontrivially, and Γ-invariantly, evaluates the image under ψ of the attached k-cells in X_k. Therefore, for each n, we have an assignment of k-cells in $\Gamma\backslash X_k$ to elements of $\mathbb{F}_p$. This produces an infinite sequence in $\mathrm{H}^k(\,\Gamma\backslash X_k\,;\,\mathbb{F}_p\,)$. The group Γ may not act freely on X_k, but the lack of freeness is confined to a cocompact subspace of X_k, namely X_{k-2}, and that implies that $\mathrm{H}^k(\,\Gamma\,;\,\mathbb{F}_p\,)$ is infinite.

0.2. **Acknowledgements.** Thanks to Dan Margalit for a discussion that lead to the proof of this result and to Kai-Uwe Bux for careful explanations of his work with Köhl and Witzel.

Thanks also to Mladen Bestvina, Morgan Cesa, Brendan Kelly, Amir Mohammadi, and Dave Morris for helpful conversations, and to Stefan Witzel for pointing out some improvements that were made to an earlier version of this paper.

1. Preliminaries on $\mathbf{G}(\mathcal{O}_S)$ and its action on a Euclidean building

This section establishes some conventions for notation.

1.1. **Basic group structure.** Let K, $\mathcal{O}_S$, and $\mathbf{G}$ be as in Theorem 3. Because $\mathbf{G}$ is K-isotropic, it contains a proper minimal K-parabolic subgroup $\mathbf{J}$. Let $\mathbf{A}$ be a maximal K-split torus in $\mathbf{J}$, and let $\mathbf{P}$ be a maximal proper K-parabolic subgroup of $\mathbf{G}$ that contains $\mathbf{J}$.

Recall the Langlands decomposition that

$$\mathbf{P} = \mathbf{UHT}$$

where $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$, $\mathbf{H}$ is a reductive K-group with K-anisotropic center, $\mathbf{T}$ is a 1-dimensional connected subtorus of $\mathbf{A}$, and $\mathbf{T}$ commutes with $\mathbf{H}$.

In the remainder of this paper we denote the product over S of local points of a K-group by "unbolding", so that, for example,

$$G = \prod_{v \in S} \mathbf{G}(K_v)$$

1.2. **Euclidean building.** Let X be the Euclidean building for the semisimple group G. We let $k = k(\mathbf{G}, S)$ so that $k = \dim(X)$.

For each $v \in S$ we choose a maximal K_v-split torus in $\mathbf{J}$ that contains $\mathbf{A}$, and name it $\mathbf{A}_v$. We let $\Sigma \subseteq X$ be the apartment corresponding to the group $\prod_{v \in S} \mathbf{A}_v(K_v)$.

2. Review of Bux-Köhl-Witzel and an unbounded sequence of points $y_n \in X$

Our proof makes use of two results from Bux-Köhl-Witzel [5]: the existence of a $\mathbf{G}(\mathcal{O}_S)$-invariant, $(k-2)$-connected subcomplex $X_{k-2} \subseteq X$ that is cocompact modulo $\mathbf{G}(\mathcal{O}_S)$, and a lemma that will allow us to extend certain "local" k-disks about neighborhoods of points in X to "global" k-disks in X—Lemma 9 and Corollary 10 below. Most of this section is devoted to recalling the work of Bux-Köhl-Witzel. For details omitted from the account in this paper, see [5].

We will use the notation of [5] in our Section 2 except for the following: we will refer to cells in the spherical building for G by the parabolic groups they represent. For example, if $g \in G$ and we write that $g \in P$, then we are treating P as a parabolic group, but if x is a point in the visual boundary of X and we write that $x \in P$, then we are treating P as the simplex in the visual boundary of X that corresponds to P. The correct interpretation should always be clear from context.

2.1. **Busemann function for P.** For each $v \in S$, let X_v be the Euclidean building for $\mathbf{G}(K_v)$, so that $X = \prod_{v \in S} X_v$. If $\mathcal{O}_v \subseteq K_v$ is the ring of integers, then we let x_v be the vertex in X_v stabilized by $\mathbf{G}(\mathcal{O}_v)$.

Let $\mathbb{A}_K$ be the ring of adeles for K, and let $\mathbb{A}_S$ be the subring of S-adeles. The group $\mathbf{G}(\mathbb{A}_S)$ has a natural left action on X. Given a point $y \in X$ we let $\mathbf{G}(\mathbb{A}_S)_y$ be the stabilizer of y in $\mathbf{G}(\mathbb{A}_S)$.

Following Harder ([10]) and [5], for any $y \in \prod_{v \in S} \mathbf{G}(K_v)x_v$ we let

$$\tilde{\beta}_P(y) = \log_q \Big[\mathrm{vol}\big[\mathbf{U}(\mathbb{A}_K) \cap \mathbf{G}(\mathbb{A}_S)_y\big]\Big]$$

where q is the cardinality of the field of constants in K.

We let $\chi_{\mathbf{P}}$ be the canonical character of $\mathbf{P}$. (See Section 1.3 [10] for the definition of $\chi_{\mathbf{P}}$.) The essential feature of $\chi_{\mathbf{P}}$ that will be used below is that the determinant of conjugation by $g \in P$ on U is $\chi_{\mathbf{P}}(g)$.

If $g \in P$, then we have the following transformation rule from Harder [10] Satz 1.3.2:

$$\tilde{\beta}_P(gy) = \tilde{\beta}_P(y) + \log_q(||\chi_{\mathbf{P}}(g)||)$$

where $||\cdot||$ denotes the idele norm. (There is a difference in sign in the line above with [10] and [5] that comes from our convention of using left actions in this paper rather than right actions as in [10] and [5].)

Recall that a Busemann function on the Euclidean building X is given by first choosing a unit speed geodesic $\rho \subseteq X$ and then assigning to any point $x \in X$ the limit as $t \to \infty$ of the difference between the distance between $\rho(t)$ and $\rho(0)$ and the distance between $\rho(t)$ and x.

Proposition 4. *There is some $s > 0$ and a Busemann function $\beta_P : X \to \mathbb{R}$ such that $\beta_P(y) = \tilde{\beta}_P(y)$ for all $y \in \prod_{v \in S} \mathbf{G}(K_v)x_v$, and such that β_P is nonconstant on factors of X.*

Proof. This is Proposition 12.2 of [5]. □

Lemma 5. *The Busemann function β_P is invariant under the actions of U, H, and $\mathbf{T}(\mathcal{O}_S)$ on X, and thus is invariant under the action of $\mathbf{P}(\mathcal{O}_S) \leq UH\mathbf{T}(\mathcal{O}_S)$.*

Proof. Any K-defined character on $\mathbf{P}$, including the canonical character $\chi_{\mathbf{P}}$, evaluates $\mathbf{U}$ trivially since it is unipotent and $\mathbf{H}$ trivially since it is reductive with K-anisotropic center. Thus the result for U and H follows from the transformation rule above.

Similarly, we need to observe that $||\chi_{\mathbf{P}}(t)|| = 1$ for any $t \in \mathbf{T}(\mathcal{O}_S)$. This follows from the product formula (since $\chi_{\mathbf{P}}(t) \in K$) and from the fact that $\mathbf{T}(K_w)$ is bounded if $w \notin S$. □

2.2. **Descending chambers at a vertex.** Given a vertex $x \in X$, we let $\mathrm{St}(x) \subseteq X$ denote the star of x, the union of all chambers in X that contain x. Thus, the boundary of the star – denoted as $\partial\mathrm{St}(x)$ – is the link of x.

We let $\mathrm{St}^{\downarrow}(x)$ denote the union of chambers $\mathfrak{C} \subseteq X$ containing x with the property that $\beta_P(z) < \beta_P(x)$ for all $z \in \mathfrak{C}$ with $z \neq x$. We let $B\mathrm{St}^{\downarrow}(x) = \mathrm{St}^{\downarrow}(x) \cap \partial\mathrm{St}(x)$.

Recall that a special vertex $x \in \Sigma$ is a vertex that is contained in a representative from each parallel family of walls in the Coxeter complex Σ. Thus, the Coxeter complex of an apartment in the spherical building $\partial\mathrm{St}(x)$ is isomorphic to the Coxeter complex of an apartment in the boundary of X when x is special.

The following result is due to Schulz [13].

Lemma 6. *If $x \in X$ is a special vertex, then $B\mathrm{St}^{\downarrow}(x)$ is homotopy equivalent to a noncontractible wedge of $(k-1)$-spheres.*

Proof. Recall that the Busemann function β_P is nonconstant on the factors of X. Since x is a special vertex, the join factors of $\partial\mathrm{St}(x)$ correspond to the factors of X. Therefore, β_P is nonconstant on the join factors of $\partial\mathrm{St}(x)$. That is to say, in the terminology used in [5], the "vertical part" of $\partial\mathrm{St}(x)$ is $\partial\mathrm{St}(x)$ in its entirety.

Notice that $B\mathrm{St}^{\downarrow}(x)$ is exactly the maximal subcomplex of $\partial\mathrm{St}(x)$ that is supported on the complement of the closed ball of radius $\frac{\pi}{2}$ around the gradient direction of β_P in $\partial\mathrm{St}(x)$. Thus, by Theorem B of [13] – restated in Theorem 4.6 of [5] – $B\mathrm{St}^{\downarrow}(x)$ is $(k-1)$-dimensional, $(k-2)$-connected, and noncontractible. □

See also Theorem A.2 of Dymara-Osajda [8].

2.3. **Reduction datum.** If $\mathbf{M_a}$ is a maximal proper K-parabolic subgroup of $\mathbf{G}$, then we can define a Busemann function β_{M_a} with respect to $\mathbf{M_a}$ similarly to how we defined β_P with respect to $\mathbf{P}$.

In [5], and following [10], there are real constants $r < R$ such that the collection of Busemann functions β_{M_a} forms what is called a *uniform* $\mathbf{G}(\mathcal{O}_S)$*-invariant and cocompact reduction datum.* (See Theorem 1.9 of [5].) The remainder of Section 2.3 is a recollection of what this sort of datum entails. In Section 2.3 we will use $\mathbf{M_a}$ to denote a maximal proper K-parabolic subgroup of $\mathbf{G}$. We will use $\mathbf{M_i}$ to denote a minimal K-parabolic subgroup of $\mathbf{G}$.

For $x \in X$ and a K-parabolic subgroup $\mathbf{Q} \leq \mathbf{G}$, we let $\beta_Q(x)$ be the maximum of all $\beta_{M_a}(x)$ with $\mathbf{Q} \leq \mathbf{M_a}$.

Given an apartment $\Sigma' \subseteq X$ that contains Q as a cell in its boundary, and given $t \in \mathbb{R}$, we let

$$Y_{\Sigma',Q}(t) = \{\, x \in \Sigma' \mid \beta_Q(x) \leq t \,\}$$

This set is convex in Σ' as it is the intersection of the convex sets $\Sigma' \cap \beta_{M_a}^{-1}(\mathbb{R}_{\leq t})$ for $\mathbf{M_a}$ containing $\mathbf{Q}$. Thus, there is a closest point projection

$$\mathrm{pr}^t_{\Sigma',Q} : \Sigma' \to Y_{\Sigma',Q}(t)$$

The group $\sigma_t(x, Q)$ is defined to be the group of $\prod_{v \in S} K_v$-points of the intersection of all $\mathbf{M_a}$ that contain $\mathbf{Q}$ and such that $\beta_{M_a}(\mathrm{pr}^t_{\Sigma',Q}(x)) = t$. We have that $\sigma_t(x, Q) \geq Q$ (as groups, not as cells in the boundary) and we say that Q *t-reduces* $x \in X$ if $\sigma_t(x, Q) = Q$.

To say that the collection of β_{M_a} is an (r, R) *reduction datum* for $r < R$ means that if $\mathbf{M_i}$ is a minimal K-parabolic subgroup of $\mathbf{G}$ that r-reduces $x \in X$, then $\mathbf{M_i} \leq \sigma_R(x, M_i)$.

To say that the reduction datum is *uniform* means that there exists a constant d such that any point in a subset of X whose diameter is less than d can be r-reduced by a common minimal K-parabolic. We can assume, as in [5], by perhaps choosing a lesser r, that d is greater than the diameter of closed stars of cells in X.

The reduction datum is $\mathbf{G}(\mathcal{O}_S)$-*invariant* since

$$\beta_{{}^{\gamma}M_a}(\gamma x) = \beta_{M_a}(x)$$

for all $x \in X$, $\gamma \in \mathbf{G}(\mathcal{O}_S)$, and maximal proper K-parabolic $\mathbf{M_a}$. (Here ${}^{\gamma}M_a = \gamma M_a \gamma^{-1}$.)

That the reduction datum is *cocompact* means that for any real number $t \geq R$, the set of $x \in X$ for which $\beta_{M_i}(x) \leq t$ for all minimal K-parabolics $\mathbf{M_i}$ that r-reduce x is cocompact with respect to the action of $\mathbf{G}(\mathcal{O}_S)$.

2.4. **Definition of height.** In [5], the reduction datum is used to define a height function $h : X \to \mathbb{R}_{\geq 0}$. In Section 2.4, we recall this definition.

Choose a special vertex $z \in \Sigma$, and let W_z be the spherical Coxeter group that fixes z in Σ.

The affine space Σ may be realized as a vector space with origin z. Let V_z be the set of all differences of vertices in Σ whose closed stars intersect, where we regard vertices in this context as vectors in Σ. Notice that V_z is finite.

We let $D = W_z V_z$. Again, realizing points of D as vectors of the vector space Σ with origin z, we let

$$Z(D) = \Big\{ \sum_{d \in D} a_d d \mid 0 \leq a_d \leq 1 \text{ for all } d \in D \Big\}$$

The set $Z(D) \subseteq \Sigma$ depended on the choice of vertex z, but modulo isometric translations of Σ, $Z(D)$ is defined intrinsically in terms of the geometry of Σ. Furthermore, if $\Sigma' \subseteq X$ is any apartment in X, then Σ' is isometric to Σ as Coxeter complexes, and thus $x + Z(D)$ is a well-defined subset of Σ' for any $x \in \Sigma'$.

To define a height function, a suitably large $R^* > R$ is chosen. For any apartment $\Sigma' \subseteq X$, any $x \in \Sigma'$, and any minimal K-parabolic $\mathbf{M_i}$ such that M_i represents a cell in the boundary of Σ' that r-reduces x; the point x^*_{Σ', M_i} is defined to be the closest point to x in $Y_{\Sigma', M_i}(R^*) - Z(D)$. Then $h(x)$ is defined as the distance between x and x^*_{Σ', M_i}, and it is shown in Proposition 5.2 of [5] to be independent of Σ' or $\mathbf{M_i}$.

If $h(x) > 0$, then $e(x)$ is defined as the point in the visual boundary of Σ' that is determined as the limit point of the geodesic ray in Σ'

from x^*_{Σ',M_i} through x. The point $e(x)$ is also shown to be independent of Σ' or $\mathbf{M_i}$ in Proposition 5.2 of [5]. If we let $\sigma(x)$ denote the group of $\prod_{v\in S} K_v$-points of the K-parabolic subgroup of $\mathbf{G}$ that is minimal with respect to the property that $\sigma(x)$ contains every $\sigma_R(x, M_i)$ for which M_i r-reduces x, then $e(x) \in \sigma(x)$.

As the reduction datum used in this section is $\mathbf{G}(\mathcal{O}_S)$-invariant, we have that $h(\gamma x) = h(x)$ for any $\gamma \in \mathbf{G}(\mathcal{O}_S)$. And if $h(x) > 0$, then $e(\gamma x) = \gamma e(x)$ and $\sigma(\gamma x) = {}^{\gamma}\sigma(x)$.

The subsets of X whose values under h are bounded from above are shown to have bounded quotient on $\mathbf{G}(\mathcal{O}_S)\backslash X$ (See Proposition 2.4 and Observation 5.5 of [5]).

2.5. **Choice of y_n.** We still have more to discuss about the results of [5], but we take a short break from our account of [5] to establish a sequence of points in X that will be used throughout our proof in this paper.

Lemma 7. *Let $N^* > 0$ be twice the maximum diameter of stars in X. We can choose $R^* \gg 0$ as above to satisfy the following: There is a constant $C^* \in \mathbb{R}$, and a geodesic ray $\ell_Y \subseteq \Sigma$ that limits to a point $\ell_Y(\infty)$ in the simplex P and is orthogonal to level sets of β_P in Σ, such that every point z in the N^*-neighborhood of $U\ell_Y$ in X is r-reduced by J, has $h(z) = \beta_P(z) + C^* > 0$, and has $e(z) = \ell_Y(\infty) \in P$.*

Furthermore, there is a sequence of special vertices $y_n \in \Sigma$ that are contained in chambers of Σ that intersect ℓ_Y, such that $\beta_P(y_n)$ is a strictly increasing sequence of numbers, and such that the set of all $(y_n)^*_{\Sigma,P}$ is a bounded set.

Proof. There are $\mathrm{rank}_K\mathbf{G} \leq \dim(\Sigma)$ maximal proper K-parabolic subgroups that contain $\mathbf{J}$. The space $Y_{\Sigma,J}(R^*) \subseteq \Sigma$ is the intersection of one half-apartment of Σ for every maximal proper K-parabolic subgroup that contains $\mathbf{J}$, and the set $\beta_P^{-1}(R^*) \cap Y_{\Sigma,J}(R^*)$ is an unbounded face of the boundary of $Y_{\Sigma,J}(R^*)$. We call this face F_{P,R^*}. It has dimension equal to $\dim(\Sigma) - 1$.

We let

$$\Omega(r, R^*, J, P) = \{\, x \in \Sigma \mid \sigma_r(x, J) = J \text{ and } \sigma_{R^*}(x, J) = P \,\}$$

For $x \in \Sigma$, we let $B_\Sigma(x; N^*) \subseteq \Sigma$ be the ball in Σ centered at x with radius N^*. Notice that by replacing R^* with a greater constant, we may assume that there is some $x \in F_{P,R^*} \cap \Omega(r, R^*, J, P)$ such that

$$F_{P,R^*} \cap [B_\Sigma(x; N^*) + Z(D)] \subseteq F_{P,R^*} \cap \Omega(r, R^*, J, P)$$

Furthermore, if y is contained in the geodesic ray $\ell_Y \subseteq \Sigma$ that begins at x, is orthogonal to F_{P,R^*}, and is contained in $\Omega(r, R^*, J, P)$, then

$B_\Sigma(y;N^*)+Z(D)\subseteq\Omega(r,R^*,J,P)$ as long as the distance between y and x is sufficiently large. We replace ℓ_Y with a subray so that $B_\Sigma(y;N^*)+Z(D)\subseteq\Omega(r,R^*,J,P)$ for any $y\in\ell_Y$.

If z is contained in the interior of $\Omega(r,R^*,J,P)$, then $e(z)$ is given by the direction of the gradient of β_P restricted to Σ — which is the direction of $\ell_Y(\infty)$. Thus by Lemma 5, $UHe(z)=UH\ell_Y(\infty)=\ell_Y(\infty)=e(z)$. And $T\leq A$ acts trivially on the boundary of Σ, so we have $Pe(z)=UHTe(z)=e(z)$ which implies that $e(z)\in P$.

We let d_0 be the constant difference of the distance between $y\in\ell_Y$ and $\Sigma\cap\beta_P^{-1}(R^*)$ and the distance between $y+Z(D)$ and $\Sigma\cap\beta_P^{-1}(R^*)$. (Note that the latter of the two distances is $h(y)$.) Then for $z\in B_\Sigma(y;N^*)$, $h(z)=\beta_P(z)-R^*-d_0$. Thus we let $C^*=-R^*-d_0$.

Again let $y\in\ell_Y$ and now let $z\in B_X(y;N^*)$, where $B_X(y;N^*)$ is the ball in X of radius N that is centered at y. We will show that z is r-reduced by J, has $h(z)=\beta_P(z)+C^*>0$, and has $e(z)=\ell_Y(\infty)\in P$.

For every $v\in S$, let $\mathbf{J}_v\leq\mathbf{G}$ be a minimal K_v-parabolic subgroup of $\mathbf{G}$ such that $\mathbf{A}_v\leq\mathbf{J}_v\leq\mathbf{J}$. We let $\mathbf{U}_v$ be the unipotent radical of $\mathbf{J}_v$, so that $\mathbf{U}_v\leq\mathbf{J}\leq\mathbf{P}$ and $\mathbf{U}_v\leq\mathbf{UH}$.

If X_v is the Euclidean building for $\mathbf{G}(K_v)$, and Σ_v is the apartment that $\mathbf{A}_v(K_v)$ acts on, then because any point in X_v is contained in a $\mathbf{J}_v(K_v)$ translate of Σ_v

$$X_v=\mathbf{J}_v(K_v)\Sigma_v=\mathbf{U}_v(K_v)\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_v)(K_v)\Sigma_v=\mathbf{U}_v(K_v)\Sigma_v$$

where $\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_v)$ is the centralizer of $\mathbf{A}_v$ in $\mathbf{G}$, and thus is a Levi subgroup of $\mathbf{J}_v$. Therefore,

$$X=\prod_{v\in S}\mathbf{U}_v(K_v)\Sigma$$

and there is a distance nonincreasing retraction

$$\varrho:X\to\Sigma$$

defined on each $u\Sigma$ for $u\in\prod_{v\in S}\mathbf{U}_v(K_v)$ as the map $u^{-1}:u\Sigma\to\Sigma$.

So for $z\in B_X(y;N^*)$ we choose $u\in\prod_{v\in S}\mathbf{U}_v(K_v)$ such that $u^{-1}z\in\Sigma$. Because ϱ is distance nonincreasing and $\varrho(y)=y$, we have that $u^{-1}z\in B_\Sigma(y;N^*)$. By Lemma 5

$$\beta_P(z)+C^*=\beta_P(u^{-1}z)+C^*=h(u^{-1}z)>0$$

If $\mathbf{Q}$ is a proper K-parabolic subgroup of $\mathbf{G}$ containing $\mathbf{J}$, then $\mathbf{Q}$ contains $\mathbf{U}_v$ and thus $u^{-1}\mathbf{Q}u=\mathbf{Q}$, so applying the clear analogue of Lemma 5 to each maximal proper K-parabolic group containing $\mathbf{J}$ yields $uY_{\Sigma,J}(R^*)=Y_{u\Sigma,J}(R^*)$ and that $z\in u\Omega(r,R^*,J,P)$ since $u^{-1}z\in B_\Sigma(y;N^*)\subseteq\Omega(r,R^*,J,P)$. Thus, z is r-reduced by $uJu^{-1}=J$

and $u^{-1}(z^*_{u\Sigma,J}) = (u^{-1}z)^*_{\Sigma,J}$ and

$$h(z) = h(u^{-1}z) = \beta_P(z) + C^*$$

Furthermore, as the set $\Omega(r, R^*, J, P)$ limits to the cell P and $u \in P$, the set $u\Omega(r, R^*, J, P)$ also limits to P and thus

$$e(z) = e(u^{-1}z) = \ell_Y(\infty) \in P$$

To review, we have shown that for any z in the N^*-neighborhood of ℓ_Y in X that z is r-reduced by J, has $h(z) = \beta_P(z) + C^* > 0$, and has $e(z) = \ell_Y(\infty) \in P$. We still need to show the same results apply to the weaker condition that z is contained in the N^*-neighborhood of $U\ell_Y$ in X. For that, recall that $\mathbf{U}$ is unipotent, so $\mathbf{U}(\mathcal{O}_S)$ is a cocompact lattice in U. That is, there is a compact set $B \subseteq U$ such that $\mathbf{U}(\mathcal{O}_S)B = U$. Since ℓ_Y limits to P and $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$, any element of U fixes pointwise a subray of ℓ_Y. Therefore, there is a common subray of ℓ_Y that is fixed pointwise by every element of B. Thus, by replacing ℓ_Y with a subray we may assume that B fixes ℓ_Y and thus that

$$U\ell_Y = \mathbf{U}(\mathcal{O}_S)B\ell_Y = \mathbf{U}(\mathcal{O}_S)\ell_Y$$

Hence, if $z \in UB_X(\ell_Y; N^*) = \mathbf{U}(\mathcal{O}_S)B_X(\ell_Y; N^*)$ then $uz \in B_X(\ell_Y; N^*)$ for some $u \in \mathbf{U}(\mathcal{O}_S)$, and since h is $\mathbf{G}(\mathcal{O}_S)$-invariant and β_P is U-invariant,

$$h(z) = h(uz) = \beta_P(uz) + C^* = \beta_P(z) + C^*$$

Since the reduction datum is $\mathbf{G}(\mathcal{O}_S)$-invariant and uz is r-reduced by J, we see that z is r-reduced by $u^{-1}Ju = J$. Last, since $u \in \mathbf{U}(\mathcal{O}_S) \leq P$ and $e(uz) \in P$ we have $e(z) = u^{-1}e(uz) = e(uz) = \ell_Y(\infty)$.

To find the sequence of y_n, just choose an unbounded sequence of chambers in Σ that intersect ℓ_Y. Any chamber in X contains a special vertex, and this produces the sequence of y_n. Because each of the $y_n \in \Sigma$ are a uniformly bounded distance from ℓ_Y, each $(y_n)^*_{\Sigma,P} \in F_{P,R^*}$ is a uniformly bounded distance from the point $x \in F_{P,R^*}$. □

In the remainder of this paper, we shall abbreviate $\mathrm{St}(y_n)$ as S_n. Similarly, we shall abbreviate $\mathrm{St}^{\downarrow}(y_n)$ and $B\mathrm{St}^{\downarrow}(y_n)$ as $S_n^{\downarrow}$ and $BS_n^{\downarrow}$ respectively.

2.6. **Morse function.** Section 2.6 is the final section in which we recount the work of Bux-Köhl-Witzel. In this section we recall the definition of a combinatorial Morse function from [5] that is defined on the vertices of the barycentric subdivision of X and used to deduce connectivity properties of subsets of X.

For any cell $\tau \in X$ we let $\dim(\tau)$ be its dimension. There is also a number defined in [5] as $\mathrm{dp}(\tau)$ which refers to the "depth" of a cell.

We refer the reader to Section 8 of [5] for the definition of the depth of a cell.

We let $\mathring{X}$ be the barycentric subdivision of the Euclidean building X. For any cell $\tau \subseteq X$, we let $\mathring{\tau}$ be its barycenter. Bux-Köhl-Witzel assigned to $\mathring{\tau}$ the triple of real numbers

$$f_{BKW}(\mathring{\tau}) = \big(\max_{x \in \tau}(h(x)), \mathrm{dp}(\tau), \dim(\tau)\big)$$

The function f_{BKW} is a combinatorial Morse function when triples of real numbers are ordered lexicographically.

For any triple of real numbers s that is greater than or equal to the triple $s_0 = (1, 0, 0)$, we let $\mathring{X}(s)$ be the subcomplex of $\mathring{X}$ spanned by the $\mathring{\tau}$ for which $f_{BKW}(\mathring{\tau}) \leq s$. Since f_{BKW} is $\mathbf{G}(\mathcal{O}_S)$-invariant, so to is $\mathring{X}(s)$. Since $\mathring{X}(s)$ is a closed subset of $\mathring{X}$ whose height is bounded, it is cocompact modulo $\mathbf{G}(\mathcal{O}_S)$. The values of f_{BKW} are finite below any given bound, and we let $s+1$ denote the least value of f_{BKW} that is greater than s.

We let $\mathrm{Lk}(\mathring{\tau})$ be the link of $\mathring{\tau}$ in $\mathring{X}$, and we define the *Morse descending link* of $\mathring{\tau}$ with respect to the Morse function f_{BKW} to be the complex of simplices $\sigma \subseteq \mathrm{Lk}(\mathring{\tau})$ such that $f_{BKW}(v) < f_{BKW}(\mathring{\tau})$ for every vertex $v \in \sigma$. To obtain $\mathring{X}(s+1)$ we attach to $\mathring{X}(s)$ the descending links of cells $\mathring{\tau} \subseteq \mathring{X}$ with $f_{BKW}(\mathring{\tau}) = s + 1$. The work of Bux-Köhl-Witzel is to have defined f_{BKW} in such a way as to utilize the work of Schulz [13] in showing that the Morse descending links of vertices in $\mathring{X}$ are either contractible or spherical of dimension $(k-1)$. Thus, up to homotopy equivalence, $\mathring{X}(s+1)$ is obtained by attaching k-cells to $\mathring{X}(s)$. This process induces an isomorphism of homotopy groups $\pi_i(\mathring{X}(s)) \cong \pi_i(\mathring{X}(s+1))$ for $i \leq k-2$. Since X is contractible and the union of the $\mathring{X}(s)$, we have that $\mathring{X}(s)$ is $(k-2)$-connected for any $s \geq s_0$. It is the existence of a $\mathbf{G}(\mathcal{O}_S)$-cocompact $(k-2)$-connected space that can be viewed as the main result of [5] as it immediately implies that $\mathbf{G}(\mathcal{O}_S)$ is of type F_{k-1}.

In what remains, we will let $X_{k-2} = X(s_0)$. In particular, X_{k-2} is a $(k-2)$-connected subcomplex of X that is invariant and cocompact under the action of $\mathbf{G}(\mathcal{O}_S)$. We will also pass to a subsequence of the y_n to assume that $S_n \cap X_{k-2} = \emptyset$ for all n.

The following lemma demonstrates the compatibility of β_P and f_{BKW} on S_n.

Lemma 8. *The Morse descending link of y_n with respect to f_{BKW} equals $BS_n^{\downarrow}$.*

Proof. As in Section 6 of [5], the height function h forces a decomposition of the link of $y_n \in X$ into a join of a "horizontal link" of y_n and a "vertical link" of y_n where the horizontal link of y_n is the join of all factors of the link of y_n whose points are evaluated by h as $h(y_n)$.

By Lemma 7, the restriction of β_P to the horizontal link of y_n is constant. But y_n is a special vertex, so Proposition 4 implies that the horizontal link of y_n is trivial, and therefore, that the vertical link of y_n equals the link of y_n.

Now by Proposition 9.6 of [5], the Morse descending link of y_n is the subcomplex of the link of y_n in X that is spanned by all vertices v in the link of y_n such that $h(v) < h(y_n)$. (Keep in mind that any vertex of X is "significant".) Again, by Lemma 7, this complex is equal to $BS_n^{\downarrow}$. □

2.7. **Extending local disks near y_n.** In addition to the existence of X_{k-2}, we shall utilize the results of [5] to extend "local" disks near y_n to "global" disks in X. More precisely, we have

Lemma 9. *Let $\sigma : S^{k-1} \to X$ be a continuous map of a $(k-1)$-sphere into X. Suppose there is some triple $s > s_0$ such that $\sigma(S^{k-1}) \subseteq \mathring{X}(s)$. Then there is a homotopy $F : S^{k-1} \times [0,1] \to X$ such that for all $x \in S^{k-1}$ we have $F(x,t) \in \mathring{X}(s)$, $F(x,0) = \sigma(x)$, and $F(x,1) \in \mathring{X}(s_0) = X_{k-2}$.*

Proof. Let $c_1^0, \ldots, c_m^0 \subseteq X$ be the image under σ of the 0-cells of S^{k-1}. Let $c_{i,F}^0 \subseteq \mathring{X}(s)$ be paths from c_i^0 to X_{k-2}. The boundary of each $c_{i,F}^0$ is c_i^0 and b_i^0 for some $b_i^0 \in X_{k-2}$.

If $k = 1$, then $m = 2$, and $c_{1,F}^0 \cup c_{2,F}^0$ is the image of the homotopy F.

If $k \geq 2$, then let $c_i^1 \subseteq \sigma(S^{k-1})$ be the image of the 1-cell with boundary c_ℓ^0 and c_j^0. Since $\mathring{X}(s)$ is obtained from X_{k-2} by attaching k-cells, there is a homotopy relative b_ℓ^0 and b_j^0 between $c_i^1 \cup c_{\ell,F}^0 \cup c_{j,F}^0$ and a 1-cell $b_i^1 \subseteq X_{k-2}$. We name the image of this homotopy $c_{i,F}^1$.

If $k = 2$, then the union of the $c_{i,F}^1$ defines the homotopy F.

If $k \geq 3$, then we proceed as above by induction on the skeleta of S^{k-1}. □

We let $I_n = S_n - \partial S_n$ be the interior of S_n. As a consequence of the above lemma, we have

Corollary 10. *For $n \gg 0$, there is a k-disk $D_n^k \subseteq S_n^{\downarrow} \cup (X - \mathbf{G}(\mathcal{O}_S)I_n)$ with $\partial D_n^k \subseteq X_{k-2}$ and such that $D_n^k \cap S_n^{\downarrow}$ is a k-disk that represents a noncontractible k-sphere in the quotient space $S_n^{\downarrow}/BS_n^{\downarrow}$.*

Proof. Let s_n be the triple such that $f_{BKW}(y_n) = s_n$. By Lemma 7, and the definition of the Morse function f_{BKW}, we have for any cell $\tau \subseteq S_n$ that is not contained in ∂S_n that $f_{BKW}(\mathbf{G}(\mathcal{O}_S)\tau) = f_{BKW}(\tau) \geq s_n$ since $y_n \in \tau$. That is, $\mathbf{G}(\mathcal{O}_S)I_n \cap \mathring{X}(s_n - 1) = \emptyset$.

By Lemmas 6 and 8, there is a noncontractible $(k-1)$-sphere $\sigma_n^{k-1} \subseteq BS_n^{\downarrow}$. We let $d_n^k \subseteq S_n^{\downarrow}$ be the cone at $y_n \in S_n^{\downarrow}$ on

$$\sigma_n^{k-1} \subseteq BS_n^{\downarrow} \subseteq \mathring{X}(s_n - 1)$$

By Lemma 9, there is a homotopy F between ∂d_n^k and a $(k-1)$-sphere in X_{k-2} whose image is contained in $\mathring{X}(s_n - 1)$. We let D_n^k be the union of d_n^k and F. Then

$$D_n^k \subseteq S_n^{\downarrow} \cup \mathring{X}(s_n - 1) \subseteq S_n^{\downarrow} \cup (X - \mathbf{G}(\mathcal{O}_S)I_n)$$

That $D_n^k \cap S_n^{\downarrow} = d_n^k$ represents a noncontractible k-sphere in $S_n^{\downarrow}/BS_n^{\downarrow}$ follows from the natural identification of $d_n^k/\partial d_n^k$ and $S_n^{\downarrow}/BS_n^{\downarrow}$ with the suspensions of σ_n^{k-1} and $BS_n^{\downarrow}$ respectively. □

Lemma 11. *Suppose that $\mathfrak{C}_a, \mathfrak{C}_b \subseteq S_n^{\downarrow}$ are chambers in X, and that there is some $\gamma \in \mathbf{G}(\mathcal{O}_S)$ such that $\gamma\mathfrak{C}_a = \mathfrak{C}_b$. Then $\gamma y_n = y_n$.*

Proof. The vertex y_n is the only vertex of any chamber in $S_n^{\downarrow}$ with $f_{BKW}(v) = f_{BKW}(y_n)$. Since f_{BKW} is $\mathbf{G}(\mathcal{O}_S)$ invariant, we have for $\gamma y_n \in \mathfrak{C}_b$ that $f_{BKW}(\gamma y_n) = f_{BKW}(y_n)$ so that $\gamma y_n = y_n$. □

3. Construction of a k-connected $\mathbf{G}(\mathcal{O}_S)$-complex

Bux-Köhl-Witzel gives us a $(k-2)$-connected complex that $\mathbf{G}(\mathcal{O}_S)$ acts on properly and cocompactly, namely X_{k-2}. In order to determine the cohomology of finite-index subgroups of $\mathbf{G}(\mathcal{O}_S)$ in dimension k, we will create a k-connected space that $\mathbf{G}(\mathcal{O}_S)$ acts on. In this section we will construct such a space by attaching k-cells to X_{k-2} and then attaching $(k+1)$-cells after that.

3.1. **Construction of X_k.** We let $\psi : X_{k-2} \to X$ be the inclusion. In the process of our construction of a k-connected space that contains X_{k-2}, we will be extending ψ to a map from that k-connected space into X.

Let $\sigma : S^{k-1} \to X_{k-2}$ be a continuous map of a $(k-1)$-sphere into the $(k-1)$-skeleton of X_{k-2}. We regard σ as an attaching map for a k-cell that we name $D_{1,\sigma}^k$.

For each nontrivial $\gamma \in \mathbf{G}(\mathcal{O}_S)$, we attach another k-cell $D_{\gamma,\sigma}^k$ to X_{k-2} using the attaching map $\gamma \circ \sigma$. We assign a homeomorphism

$\gamma : D^k_{1,\sigma} \to D^k_{\gamma,\sigma}$ that restricts to the γ-action on $\partial D^k_{1,\sigma}, \partial D^k_{\gamma,\sigma} \subseteq X_{k-2}$. Then for any $\lambda \in \mathbf{G}(\mathcal{O}_S)$, we let

$$\lambda : D^k_{\gamma,\sigma} \to D^k_{\lambda\gamma,\sigma}$$

be the homeomorphism defined by $\lambda = (\lambda\gamma)\gamma^{-1}$. In this way, we have defined a $\mathbf{G}(\mathcal{O}_S)$-action on the complex

$$X_{k-2} \cup \bigcup_{\gamma \in \mathbf{G}(\mathcal{O}_S)} D^k_{\gamma,\sigma}$$

We repeat the process above for every continuous $\sigma : S^{k-1} \to X_{k-2}$ with image in the $(k-1)$-skeleton of X_{k-2}. The resulting union of X_{k-2} with the union of every $D^k_{\gamma,\sigma}$ for every pair of γ and σ is a k-complex that we will denote by X_{k-1}. Notice that X_{k-1} is a $(k-1)$-connected, $\mathbf{G}(\mathcal{O}_S)$-complex. The group $\mathbf{G}(\mathcal{O}_S)$ will not in general act freely on X_{k-1}, but any nontrivial point stabilizers correspond to points in X_{k-2} since the interiors of each of the $D^k_{\gamma,\sigma}$ are disjoint.

We extend ψ to each $D^k_{\gamma,\sigma}$ — and thus to all of X_{k-1} — by assigning arbitrary continuous maps $\psi : D^k_{1,\sigma} \to X$ that agree with ψ on $\partial D^k_{1,\sigma} \subseteq X_{k-2}$ and then by defining $\psi : D^k_{\gamma,\sigma} \to X$ as $\gamma \circ \psi \circ \gamma^{-1}$. Notice that $\gamma \circ \psi = \psi \circ \gamma$ so that ψ is $\mathbf{G}(\mathcal{O}_S)$-equivariant.

Now repeat the above process, this time attaching $(k+1)$-cells $D^{k+1}_{\gamma,\sigma}$ to X_{k-1} with attaching maps $\sigma : S^k \to X_{k-1}$ to obtain a k-connected complex X_k that $\mathbf{G}(\mathcal{O}_S)$ acts on with a $\mathbf{G}(\mathcal{O}_S)$-equivariant map $\psi : X_k \to X$ that restricts to $X_{k-2} \subseteq X$ as the inclusion map. The action of $\mathbf{G}(\mathcal{O}_S)$ on $X_k - X_{k-2}$ is free.

4. Assigning attaching disks to cycles in a finite complex

In this section we will begin to focus some attention on a given finite-index subgroup Γ of $\mathbf{G}(\mathcal{O}_S)$ from the statement of our main result, Theorem 3. That is, we let Γ be any finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ that is residually p-finite.

Our goal in proving our main result is to show that $H^k(\Gamma\backslash X_k; \mathbb{F}_p)$ is infinite. In the penultimate section of this paper we explain why this implies that $H^k(\Gamma; \mathbb{F}_p)$ is infinite.

4.1. **Definition of Γ_n.** Our proof of our main result relies on forming a sequence of finite quotients of the group Γ. These quotients are described in the following

Lemma 12. *For any $n \geq 0$, there is a normal subgroup $\Gamma_n \trianglelefteq \Gamma$ such that Γ/Γ_n is a finite p-group and Γ_n acts cocompactly and freely on ΓS_n.*

Proof. The group Γ acts cocompactly on ΓS_n.

For any cell $\tau \subseteq S_n$, let Γ_τ be the finite stabilizer of τ in Γ, and let $Z_n \subseteq \Gamma$ be the finite set of the union of Γ_τ over the finite set of cells $\tau \subseteq S_n$.

Since Γ is residually p-finite, there is for each nontrivial $\gamma \in Z_n$ a finite p-group, G_γ, and a homomorphism $\phi_\gamma : \Gamma \to G_\gamma$ such that $\phi_\gamma(\gamma) \neq 1$. Now let $\phi : \Gamma \to \prod_\gamma G_\gamma$ be the product of the ϕ_γ, and let Γ_n be the kernel of ϕ. Then $\Gamma_n \trianglelefteq \Gamma$, Γ/Γ_n is a finite p-group, and $Z_n \cap \Gamma_n = \{1\}$.

Since Γ_n is finite-index in Γ, it acts cocompactly on ΓS_n. Furthermore, if $\gamma \in \Gamma_n$ and $\gamma g \tau = g\tau$ for some $g \in \Gamma$ and some cell $\tau \subseteq S_n$, then $g^{-1}\gamma g \in \Gamma_n$ is contained in $\Gamma_\tau \subseteq Z_n$, and thus $g^{-1}\gamma g$, and hence γ, is trivial. □

4.2. **Definition of θ_n.** We define

$$\theta_n : X \to \Gamma_n \backslash X$$

to be the quotient map. Notice that Γ acts on $\Gamma_n \backslash X$ since Γ_n is normal in Γ. Furthermore, θ_n is Γ-equivariant.

Also note that Γ acts on the pair $(X, X - \Gamma I_n)$ and thus on the pair $(\theta_n(X), \theta_n(X - \Gamma I_n))$, and therefore on the homologies of these pairs. (All homologies of complexes in this paper are cellular.)

4.3. **Definition of $\Theta_n(D^k_{\gamma,\sigma})$.** Given a k-cell $D^k_{\gamma,\sigma}$ attached to X_{k-2} in the construction of X_k, we have that $\psi(\partial D^k_{\gamma,\sigma}) \subseteq X_{k-2}$.

By Lemma 7, the sequence of $h(y_n)$, and hence of $f_{BKW}(\Gamma y_n)$ is unbounded. Thus we may assume that X_{k-2} intersects each ΓS_n trivially, which implies $\partial\psi(D^k_{\gamma,\sigma}) \subseteq X - \Gamma I_n$ and thus that $\psi(D^k_{\gamma,\sigma})$ represents a class in the homology group $H_k(X, X - \Gamma I_n; \mathbb{F}_p)$, and further, that $\theta_n \circ \psi(D^k_{\gamma,\sigma})$ represents a class in the homology group $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$. In the remainder we shall let

$$\Theta_n(D^k_{\gamma,\sigma}) = [\theta_n \circ \psi(D^k_{\gamma,\sigma})] \in H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$$

Recall that ψ is Γ-equivariant, and that θ_n is Γ-equivariant. Therefore, the group Γ acts on the set of all $\Theta_n(D^k_{\gamma,\sigma})$ by the rule that if $g \in \Gamma$, then

$$\begin{aligned} g\Theta_n(D^k_{\gamma,\sigma}) &= g[\theta_n \circ \psi(D^k_{\gamma,\sigma})] \\ &= [\theta_n \circ \psi(gD^k_{\gamma,\sigma})] \\ &= [\theta_n \circ \psi(D^k_{g\gamma,\sigma})] \\ &= \Theta_n(D^k_{g\gamma,\sigma}) \end{aligned}$$

4.4. **Definition of W_n.** We let W_n be the vector subspace of $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$ generated by the classes $\Theta_n(D^k_{\gamma,\sigma})$ for every pair γ and σ.

By the above, the Γ-action on $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$ restricts to a Γ-action on W_n. Since Γ_n acts trivially on $\theta_n(X)$, the action of Γ on W_n factors through the finite p-group Γ/Γ_n.

Lemma 13. *The vector space W_n is finite-dimensional and nonzero.*

Proof. The space X is the union of ΓS_n and $X - \Gamma I_n$, so ΓS_n surjects via θ_n onto the quotient $\theta_n(X)/\theta_n(X - \Gamma I_n)$. Lemma 12 gives us that $\theta_n(\Gamma S_n)$ is a finite complex, and thus, $\theta_n(X)/\theta_n(X - \Gamma I_n)$ is finite. The finite dimensionality of W_n now follows from the finite dimensionality of $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$.

Let $D^k_n \subseteq X$ be as in Corollary 10. We claim that $\theta_n(D^k_n)$ represents a nonzero class in $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$. Indeed, $BS^{\downarrow}_n \subseteq X - \Gamma I_n$ and it suffices to prove that

$$(\theta_n)_* : H_k(S^{\downarrow}_n, BS^{\downarrow}_n; \mathbb{F}_p) \longrightarrow H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$$

is injective. As $\theta_n(X)$ is a k-dimensional complex, this reduces to showing that $\theta_n(\mathfrak{C}_a) \neq \theta_n(\mathfrak{C}_b)$ for distinct chambers $\mathfrak{C}_a, \mathfrak{C}_b \subseteq S^{\downarrow}_n$. In other words, we want to show that $\gamma\mathfrak{C}_a = \mathfrak{C}_b$ for any $\gamma \in \Gamma_n$ and any pair of chambers $\mathfrak{C}_a, \mathfrak{C}_b \subseteq S^{\downarrow}_n$ implies that $\mathfrak{C}_a = \mathfrak{C}_b$. By Lemma 11, any such $\gamma \in \Gamma_n$ fixes $y_n \in \Gamma S_n$, and by Lemma 12, γ is trivial so that $\mathfrak{C}_a = \mathfrak{C}_b$.

Now let $\sigma_n : S^{k-1} \to X_{k-2}$ represent ∂D^k_n, and let D^k_{1,σ_n} be the k-disk attached to X_{k-2} by σ_n in the construction of X_k. Since X is contractible and k-dimensional, and since D^k_n and $\psi(D^k_{1,\sigma_n})$ share a common boundary, they represent the same k-chain in the homology of X. Therefore, by the above paragraph,

$$\Theta_n(D^k_{1,\sigma_n}) = [\theta_n \circ \psi(D^k_{1,\sigma_n})] = [\theta_n(D^k_n)]$$

is a nonzero class in $W_n \leq H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$. □

5. A sequence of cycles and cocycles for $\Gamma \backslash X_k$

The action of Γ on W_n induces an action of Γ on the dual vector space W^*_n by $\gamma\phi(x) = \phi(\gamma^{-1}x)$ for $\gamma \in \Gamma$, $\phi \in W^*_n$, and $x \in W_n$.

Lemma 14. *For each n, there is a Γ-invariant $\varphi_n \in W^*_n$ and some $\lambda_n \in$ $\mathbf{G}(\mathcal{O}_S)$ and $\tau_n : S^{k-1} \to X_{k-2}$ such that $\varphi_n(\Theta_n(D^k_{\lambda_n,\tau_n})) \neq 0$. Furthermore, after passing to a subsequence, if $m > n$ then $\varphi_m(\Theta_m(D^k_{\lambda_n,\tau_n})) = 0$.*

Proof. A linear transformation of a finite-dimensional nonzero vector space of characteristic p is unipotent if and only if it has order p^k for some k. Since the action of Γ on W_n^* factors through the p-group Γ/Γ_n, the elements of Γ act on W_n^* as unipotent transformations. By Kolchin's Theorem, any group of unipotent transformations on a finite-dimensional nonzero vector space fixes a nonzero vector. That is, there is some Γ-invariant $\varphi_n \in W_n^*$ and some k-disk $D^k_{\lambda_n,\tau_n}$ from the construction of X_k such that $\varphi_n(\Theta_n(D^k_{\lambda_n,\tau_n})) \neq 0$.

Given the disk $D^k_{\lambda_n,\tau_n}$ above, we may assume that the f_{BKW}-values of the cells in S_{n+1}, and hence of those in ΓS_{n+1} exceed the f_{BKW}-values of the finitely many cells in $\psi(D^k_{\lambda_n,\tau_n})$. Thus, if $m > n$ we have that $\psi(D^k_{\lambda_n,\tau_n}) \subseteq X - \Gamma I_m$ and thus $\Theta_m(D^k_{\lambda_n,\tau_n}) = 0$ in W_m. □

5.1. **Cocycles.** Let $D^k_{\gamma,\sigma}$ be a k-cell that was attached to X_{k-2} in the construction of X_k. Recall that $\Theta_n(D^k_{\gamma,\sigma})$ represents a class in W_n and that φ_n is a Γ-invariant functional on W_n.

Lemma 15. *For any $n \geq 0$, $\gamma \in \mathbf{G}(\mathcal{O}_S)$, $g \in \Gamma$, and $D^k_{\gamma,\sigma}$, we have $\varphi_n(\Theta_n(D^k_{\gamma,\sigma})) = \varphi_n(\Theta_n(gD^k_{\gamma,\sigma}))$.*

Proof. This is immediate since ψ is Γ-equivariant, θ_n is Γ-equivariant, and φ_n is Γ-invariant. □

Let $q : X_k \to \Gamma\backslash X_k$ be the quotient map. Note that any k-cell in $\Gamma\backslash X_k$ is contained in $\Gamma\backslash X_{k-2}$ or else is of the form $q(D^k_{\gamma,\sigma})$ for some $D^k_{\gamma,\sigma} \subseteq X_k$. We define the k-cochain Φ_n on k-chains in $\Gamma\backslash X_k$ with values in $\mathbb{F}_p$ as 0 on $\Gamma\backslash X_{k-2}$ and

$$\Phi_n(q(D^k_{\gamma,\sigma})) = \varphi_n(\Theta_n(D^k_{\gamma,\sigma}))$$

for any $q(D^k_{\gamma,\sigma})$, and then we extend linearly. The previous lemma tells us that Φ_n is well-defined.

Lemma 16. *Φ_n is a cocycle.*

Proof. The $(k+1)$ cells of $\Gamma\backslash X_k$ are of the form $q(D^{k+1}_{\gamma,\sigma})$, so we must check that Φ_n evaluates the boundary of any $q(D^{k+1}_{\gamma,\sigma})$ trivially.

Let $\mathfrak{C}_1, \ldots, \mathfrak{C}_m$ be a collection of k-cells in X_{k-2} such that the chain $\partial D^{k+1}_{\gamma,\sigma}$ equals $\sum_j \mathfrak{C}_j + \sum_i D^k_{\gamma_i,\sigma_i}$ for some $D^k_{\gamma_i,\sigma_i}$ where we suppress in this notation the orientation of k-cells. Then $\partial q(D^{k+1}_{\gamma,\sigma}) = \sum_j q(\mathfrak{C}_j) + \sum_i q(D^k_{\gamma_i,\sigma_i})$.

Note that $\psi(\partial D^{k+1}_{\gamma,\sigma})$ is a k-sphere in the k-dimensional and contractible X, and hence it represents the 0-chain. That is, the chain

$\psi(\sum_j \mathfrak{C}_j + \sum_i D^k_{\gamma_i,\sigma_i}) \cap \Gamma S_n$, and hence $\psi(\sum_i D^k_{\gamma_i,\sigma_i}) \cap \Gamma S_n$, is the 0-chain. Therefore, $\Theta_n(\sum_i D^k_{\gamma_i,\sigma_i})$ is the 0-chain, which implies

$$\begin{aligned}\Phi_n\big(\partial q(D^{k+1}_{\gamma,\sigma})\big) &= \Phi_n\Big(\sum_j q(\mathfrak{C}_j) + \sum_i q(D^k_{\gamma_i,\sigma_i})\Big)\\ &= \Phi_n\Big(\sum_i q(D^k_{\gamma_i,\sigma_i})\Big)\\ &= \varphi_n \circ \Theta_n\Big(\sum_i D^k_{\gamma_i,f_i}\Big)\\ &= \varphi_n(0)\\ &= 0\end{aligned}$$

□

5.2. **Cycles.** Given $D^k_{\lambda_n,\tau_n}$ as in Lemma 14, the k-chain $D^k_{\lambda_n,\tau_n} - D^k_{\lambda_0,\tau_0}$ is the difference of two k-disks in X_k. We let

$$C_n = q\big(D^k_{\lambda_n,\tau_n}) - q(D^k_{\lambda_0,\tau_0}\big)$$

which is a k-chain in $\Gamma\backslash X_k$.

Lemma 17. *After passing to a subsequence in n, each C_n is a k-cycle over $\mathbb{F}_p$ in $\Gamma\backslash X_k$.*

Proof. Notice that $q(\partial D^k_{\gamma_n,\sigma_n})$ is a $(k-1)$-cycle in $\Gamma\backslash X_{k-2}$. Since $\Gamma\backslash X_{k-2}$ is compact, there are only finitely many cellular $(k-1)$-chains in $\Gamma\backslash X_{k-2}$ with coefficients in $\mathbb{F}_p$. Therefore, we may pass to a subsequence and assume that $q(\partial D^k_{\lambda_n,\tau_n})$ is a constant $\mathbb{F}_p$-cycle for $n \geq 0$. □

We can now prove

Proposition 18. *$H^k(\Gamma\backslash X_k; \mathbb{F}_p)$ and $H_k(\Gamma\backslash X_k; \mathbb{F}_p)$ are infinite.*

Proof. Let $m \geq n > 0$. By the definitions of Φ_n and C_n, and by Lemma 14,

$$\begin{aligned}\Phi_m(C_n) &= \Phi_m\big(q(D^k_{\lambda_n,\tau_n})\big) - \Phi_m\big(q(D^k_{\lambda_0,\tau_0})\big)\\ &= \varphi_m\big(\Theta_m(D^k_{\lambda_n,\tau_n})\big) - \varphi_m\big(\Theta_m(D^k_{\lambda_0,\tau_0})\big)\\ &= \varphi_m(\Theta_m\big(D^k_{\lambda_n,\tau_n})\big)\end{aligned}$$

does not equal 0 if $m = n$, but does equal 0 if $m > n$. Thus, each of the terms in the sequences $[\Phi_n] \in H^k(\Gamma\backslash X_k; \mathbb{F}_p)$ and $[C_n] \in H_k(\Gamma\backslash X_k; \mathbb{F}_p)$ are distinct. □

6. Proof of Theorem 3

If Γ acts freely on X_k, then Theorem 3 is immediate from Proposition 18. And one can always choose a finite-index, residually p-finite subgroup of $\mathbf{G}(\mathcal{O}_S)$ that acts freely on X_k (see the following section). However, to show Theorem 3 holds for any, and not just some, finite-index, residually p-finite subgroup of $\mathbf{G}(\mathcal{O}_S)$, we need to apply one more technique. That is the goal of this section.

By our construction of X_k, the group Γ acts freely on $X_k - X_{k-2}$, and while it may not be true that Γ acts freely on X_{k-2}, it does act cocompactly on X_{k-2}. That is, there are only finitely many k-cells in the quotient $\Gamma\backslash X_{k-2}$. This will imply Theorem 3 after the application of a spectral sequence.

The material from this section is taken from Chapter VII of Brown's text on Cohomology of Groups [4].

We begin by subdividing X_k such that individual cells in X_k inject into $\Gamma\backslash X_k$.

We let $H_k^\Gamma(X_k;\mathbb{F}_p)$ be the k-th equivariant homology group of Γ and X_k with coefficients in $\mathbb{F}_p$. That is, if $C_*(X_k;\mathbb{F}_p)$ is the chain complex for the homology of X_k with coefficients in $\mathbb{F}_p$, and if F_* is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, then

$$H_k^\Gamma(X_k;\mathbb{F}_p) = H_k(F_* \otimes_\Gamma C_*(X_k;\mathbb{F}_p))$$

Lemma 19. $H_k^\Gamma(X_k;\mathbb{F}_p) = H_k(\Gamma;\mathbb{F}_p)$

Proof. The complex $F_* \otimes_\Gamma C_*(X_k;\mathbb{F}_p)$ is a double complex with an associated spectral sequence

$$E^1_{\ell,q} = H_q(F_\ell \otimes_\Gamma C_*(X_k;\mathbb{F}_p)) = F_\ell \otimes_\Gamma H_q(X_k;\mathbb{F}_p)$$

and

$$E^2_{\ell,q} = H_\ell(\Gamma; H_q(X_k;\mathbb{F}_p))$$

Notice that if $0 < q \leq k$ then $E^2_{\ell,q} = H_\ell(\Gamma;0) = 0$ since X_k is k-connected. It follows that $E^r_{\ell,q} = 0$ when $r \geq 2$ and $0 < q \leq k$. Hence,

$$H_k(\Gamma;\mathbb{F}_p) = E^2_{k,0} = E^\infty_{k,0} = \bigoplus_{\ell+q=k} E^\infty_{\ell,q}$$

The lemma follows since the spectral sequence converges to $H_*^\Gamma(X_k;\mathbb{F}_p)$. □

The complex $F_* \otimes_\Gamma C_*(X_k;\mathbb{F}_p)$ is also a double complex with an associated spectral sequence where $E^1_{\ell,q} = H_q(F_* \otimes_\Gamma C_\ell(X_k;\mathbb{F}_p))$. The spectral sequence converges to $H^\Gamma_*(X_k;\mathbb{F}_p)$, and in particular,

$$H_k(\Gamma;\mathbb{F}_p)=H^\Gamma_k(X_k;\mathbb{F}_p)=\bigoplus_{\ell+q=k}E^\infty_{\ell,q}$$

As in VII.7.7 of [4],

$$E^1_{\ell,q}=\bigoplus_{c\in Y_\ell}H_q(\Gamma_c;\mathbb{F}_p)$$

where Y_ℓ is a set of representatives of ℓ-cells in X_k modulo Γ, and Γ_c is the stabilizer in Γ of c.

Lemma 20. *If $r,q\geq 1$, then $E^r_{\ell,q}$ is finite.*

Proof. Since Γ acts cocompactly on X_{k-2} and freely on X_k-X_{k-2}, there are only finitely many $c\in Y_\ell$ such that $\Gamma_c\neq 1$. Thus, $E^1_{\ell,q}$ is finite as it is a finite sum of homology groups of finite groups with coefficients in a finite field. The lemma follows since the dimension of $E^r_{\ell,q}$ is bounded by that of $E^1_{\ell,q}$. □

Lemma 21. $E^2_{\ell,0}=H_\ell(\Gamma\backslash X_k;\mathbb{F}_p)$. *In particular, by Proposition 18, $E^2_{k,0}$ is infinite.*

Proof. Let ∂' be the boundary operator for $C_*(X_k;\mathbb{F}_p)$, and for any $(\ell-1)$-cell $d\subseteq X_k$, let π_d be the projection of $C_{\ell-1}(X_k;\mathbb{F}_p)$ onto the coordinate represented by d.

We let ∂ be the boundary operator for the chain complex of $\Gamma\backslash X_k$, denoted as $C_*(\Gamma\backslash X_k;\mathbb{F}_p)$.

Notice that $E^2_{*,0}$ is the homology of the complex $(E^1_{k,0},d^1)$ where $d^1:E^1_{\ell,0}\to E^1_{\ell-1,0}$. There is a natural identification of

$$E^1_{\ell,0}=\bigoplus_{c\in Y_\ell}H_0(\Gamma_c;\mathbb{F}_p)=\bigoplus_{c\in Y_\ell}\mathbb{F}_p$$

with

$$C_\ell(\Gamma\backslash X_k;\mathbb{F}_p)$$

given by

$$(a_c)_{c\in Y_\ell}\mapsto\sum_{\Gamma c\subseteq\Gamma\backslash X_k}a_c(\Gamma c)$$

where $a_c\in\mathbb{F}_p$. Below we apply this identification liberally.

Our goal is to show that d^1 can be identified with ∂. For this, if $c\in Y_\ell$ then we let $\mathcal{D}_c$ be the set of $(\ell-1)$-cells in X_k contained in c. Then VII.8.1 of [4] tells us that if $a_c\in\mathbb{F}_p=H_0(\Gamma_c;\mathbb{F}_p)$ then, up to sign,

$$d^1(a_c)=\sum_{d\in\mathcal{D}_c}v_d\circ u_{cd}\circ t_c(a_c)$$

where $t_c : H_0(\Gamma_c;\mathbb{F}_p) \to H_0(\Gamma_c;\mathbb{F}_p)$ is transfer — and thus is the identity — and where $v_d : H_0(\Gamma_d;\mathbb{F}_p) \to H_0(\Gamma_{d_0};\mathbb{F}_p)$ for $d_0 \in Y_{\ell-1}$ is such that $\Gamma d = \Gamma d_0$ and v_d is induced by conjugation in Γ — and thus is the identity — and where $u_{cd} : H_0(\Gamma_c;\mathbb{F}_p) \to H_0(\Gamma_d;\mathbb{F}_p)$ is induced by $\Gamma_c \hookrightarrow \Gamma_d$ and $\pi_d \circ \partial'|_c$ — and thus is identified with

$$\pi_d \circ \partial'|_c : \{\, a_c c \mid a_c \in \mathbb{F}_p \,\} \to \{\, a_d d \mid a_d \in \mathbb{F}_p \,\}$$

Therefore,

$$\begin{aligned} d^1(a_c) &= \sum_{d\in\mathcal{D}_c} u_{cd}(a_c) \\ &= \sum_{d\in\mathcal{D}_c} \pi_d \circ \partial'(a_c) \\ &= \partial(a_c(\Gamma c)) \end{aligned}$$

□

6.1. **Proof of Theorem 3.** By the two preceding lemmas, we have for each $r \geq 2$ that the kernel of $d^r : E^r_{k,0} \to E^r_{k-r,r-1}$ is infinite, which implies the infiniteness of

$$E^\infty_{k,0} \leq \bigoplus_{\ell+q=k} E^\infty_{\ell,q} = H_k(\Gamma;\mathbb{F}_p) \cong H^k(\Gamma;\mathbb{F}_p)$$

7. Existence of finite-index, residually p-finite subgroups of $\mathbf{G}(\mathcal{O}_S)$

In this section we give a sketch of the well-known existence statement from the title of this section. The existence essentially follows from Platonov's Theorem on finitely-generated matrix groups. We took our account below from Nica [12].

Let w be a valuation of K that is not contained in S, and let $\mathfrak{m} \subseteq \mathcal{O}_S$ be the ideal $\{\, x \in \mathcal{O}_S \mid |x|_w < 1 \,\}$. Note that $\cap_k \mathfrak{m}^k = 0$. Furthermore, $\mathcal{O}_S/\mathfrak{m}$ is identified with the values of elements of $\mathcal{O}_S$ at w, and hence is finite. Similarly, $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is finite for any $k \geq 1$, so that $\mathcal{O}_S/\mathfrak{m}^k$ is a finite ring.

For $k \geq 1$, let Λ_k be the kernel of

$$\alpha_k : \mathbf{GL_n}(\mathcal{O}_S) \to \mathbf{GL_n}(\mathcal{O}_S/\mathfrak{m}^k)$$

Since $\mathcal{O}_S/\mathfrak{m}^k$ is a finite ring, Λ_k is a finite-index normal subgroup of $\mathbf{GL_n}(\mathcal{O}_S)$. Also note that if $m > k$ then Λ_m is a normal subgroup of Λ_k since Λ_m is the kernel of α_m restricted to Λ_k.

We claim that Λ_k/Λ_{k+1} is a p-group. Indeed, if $g \in \Lambda_k$ then the matrix entries of $g - 1$ are contained in $\mathfrak{m}^k$. Thus, the matrix entries of $(g-1)^p$ are contained in $\mathfrak{m}^{k+1}$. Since $\mathcal{O}_S \subseteq K$ has characteristic p, $g^p - 1 = (g-1)^p$ so that $g^p \in \Lambda_{k+1}$, establishing our claim.

Note that $\cap_k \mathfrak{m}^k = 0$ implies $\cap_k \Lambda_k = 1$. Thus, if $Z \subseteq \Lambda_1$ is finite we can choose $k \gg 0$ such that $Z \cap \Lambda_k \subseteq \{1\}$, and

$$[\Lambda_1 : \Lambda_k] = \prod_{i=1}^{k-1} [\Lambda_i : \Lambda_{i+1}]$$

is a power of p. Therefore, Λ_1 is a finite-index, residually p-finite subgroup of $\mathbf{GL_n}(\mathcal{O}_S)$.

For general $\mathbf{G}(\mathcal{O}_S)$ we have an embedding of K-groups $\mathbf{G} \leq \mathbf{GL_n}$ and we replace Λ_k in the above with $\Lambda_k \cap \mathbf{G}(\mathcal{O}_S)$.

References

[1] Abels, H., *Finiteness properties of certain arithmetic groups in the function field case.* Israel J. Math., **76** (1991), 113-128.

[2] Abramenko, P., *Finiteness properties of Chevalley groups over $\mathbb{F}_q[t]$.* Israel J. Math., **87** (1994), 203-223.

[3] Behr, H., *Arithmetic groups over function fields. I. A complete characterization of finitely generated and finitely presented arithmetic subgroups of reductive algebraic groups.* J. Reine Angew. Math. **495** (1998), 79-118.

[4] Brown, K., *Cohomology of groups.* Springer. Graduate text in mathematics **87** (1982).

[5] Bux, K.-U., Köhl, R., and Witzel, S., *Higher finiteness properties of reductive arithmetic groups in positive characteristic: the rank theorem.* Ann. Math. **177** (2013), 311-366.

[6] Bux, K.-U., and Wortman, K., *Finiteness properties of arithmetic groups over function fields.* Invent. Math. **167** (2007), 355-378.

[7] Bux, K.-U., and Wortman, K., *Connectivity properties of horospheres in Euclidean buildings and applications to finiteness properties of discrete groups.* Invent. Math. **185** (2011), 395-419.

[8] Dymara, J., and Osajda, D., *Boundaries of right-angled hyperbolic buildings.* Fund. Math. **197** (2007), 123-165.

[9] Gandini, G., *Bounding the homological finiteness length.* Bull. Lond. Math. Soc. **44** (2012), 1209-1214.

[10] Harder, G., *Minkowskische Reduktionstheorie über Funtionenkörpern.* Invent. Math. **7** (1969), 33-54.

[11] Kropholler, P., *On groups of type $(FP)_\infty$.* J. Pure Appl. Algebra **90** (1993), 55-67.

[12] Nica, B., *Linear groups - Malcev's Theorem and Selberg's Lemma.* Preprint.

[13] Schulz, B., *Spherical subcomplexes of spherical buildings.* Geom. Topol. **17** (2013), 531-562.

[14] Stuhler, U. *Homological properties of certain arithmetic groups in the function field case.* Invent. Math. **57** (1980), 263-281.