Finiteness properties of arithmetic groups over function fields

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Abstract. We determine when an arithmetic subgroup of a reductive group defined over a global function field is of type FP_∞ by comparing its large-scale geometry to the large-scale geometry of lattices in real semisimple Lie groups.

1. Introduction

Throughout this paper, K is a global function field, and S is a finite nonempty set of pairwise inequivalent valuations on K. We let $\mathcal{O}_S \leq K$ be the corresponding ring of S-integers. We denote a reductive K-group by $\mathbf{G}$.

In 1971 Serre proved that $\mathbf{G}(\mathcal{O}_S)$ is of type WFL if and only if the semisimple K-rank of $\mathbf{G}$ equals 0; see Théorème 4 of [Se 1] and the following Compléments.

As type FP_∞ is a weaker property than type WFL, an immediate consequence is that $\mathbf{G}(\mathcal{O}_S)$ is of type FP_∞ if the semisimple K-rank of $\mathbf{G}$ equals 0. The converse of this statement had been believed since the late 1970's[1] and evidence had been collected to support it as a conjecture. However, it remained unresolved in general.

Our main result confirms this conjecture:

Theorem 1.1. *The arithmetic group $\mathbf{G}(\mathcal{O}_S)$ is of type FP_∞ if and only if the semisimple K-rank of $\mathbf{G}$ equals 0.*

As a special case of our main result, $\mathbf{SL_n}(\mathbb{F}_q[t])$ is not of type FP_∞. Even this basic example was previously unknown in full generality;

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[1] see e.g. the final introductory paragraph of [St 2]

see Example below. Here, $\mathbb{F}_q[t]$ is the ring of polynomials with one indeterminate t and coefficients in the finite field with q elements, $\mathbb{F}_q$.

We will also give a more precise statement about the finiteness lengths of arithmetic groups; see Theorem B. As a special case of that result, $\mathbf{SL_n}(\mathbb{F}_q[t])$ is not even of type FP_{n-1}.

Historical remarks. Interest in the finiteness properties of arithmetic groups over function fields was sparked in 1959 by Nagao's proof that $\mathbf{SL_2}(\mathbb{F}_q[t])$ is not finitely generated [Na].

Activities of the next 33 years completely determined which arithmetic subgroups of reductive groups over function fields are finitely generated, and which are finitely presented (the answers fit the form of Conjecture 1.3 below). Work on these results was carried out by Behr, Hurrelbrink, Keller, Kneser, Lubotzky, McHardy, Nagao, O'Meara, Rehmann-Soulé, Serre, Splitthoff, and Stuhler. See [Be 1], [Be 2], [Be 3], [Hu], [Ke], [Lu], [McH], [OM], [Re-So], [Se 2], [Spl], and [St 1].

Less understood are the higher finiteness properties for these groups, such as type FP_n for $n \geq 3$. Aside from the result of Serre mentioned earlier, all of the work in this direction has been carried out with heavy restrictions on $\mathbf{G}$ and $\mathcal{O}_S$; see the papers of Abels, Abramenko, Behr, and Stuhler ([Abl], [Abr 1], [Abr 2], [Abr 3], [Be 4], and [St 2]).

Theorem 1.1 follows as a corollary of Theorem 1.2 below. Before presenting the statement of Theorem 1.2, we introduce some notation.

Type $\mathbf{FP_m}$. Recall that for a commutative ring R, we say a group Γ is *of type FP_m over R* if there exists a projective resolution

$$P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to R \to 0$$

of finitely generated $R\Gamma$ modules, where the action of $R\Gamma$ on R is trivial. If Γ is of type FP_m over R for all nonnegative integers m, we say that Γ is *of type FP_∞ over R*. If Γ is of type FP_m (resp. FP_∞) over $\mathbb{Z}$, we simply write that Γ is *of type FP_m* (resp. FP_∞).

Remark. Every group is of type FP_0. Type FP_1 is equivalent to the property of finite generation. Every finitely presented group is of type FP_2, but the converse does not hold in general; see Bestvina-Brady's Example 6.3(3) in [Be-Br].

Finiteness length. The *homological finiteness length of Γ over R* is defined to be

$$\phi(\Gamma; R) = \sup\{m \mid \Gamma \text{ is of type } FP_m \text{ over } R\}$$

For short, we write $\phi(\Gamma)$ in place of $\phi(\Gamma; \mathbb{Z})$.

Rank. For any field extension L/K, the *L-rank* of $\mathbf{G}$, denoted $\operatorname{rank}_L \mathbf{G}$, is the dimension of a maximal L-split torus of $\mathbf{G}$. The *semisimple L-rank* of $\mathbf{G}$ is the L-rank of the derived subgroup of $\mathbf{G}$.

If $\mathrm{rank}_L(\mathbf{G}) = 0$, we say $\mathbf{G}$ is L-anisotropic. Otherwise, $\mathbf{G}$ is L-isotropic.

Sum of local ranks. For a valuation v of K, let K_v be the completion of K with respect to v. For any K-group $\mathbf{G}$, we define the nonnegative integer

$$k(\mathbf{G}, S) = \sum_{v \in S} \mathrm{rank}_{K_v} \mathbf{G}$$

We are now prepared to state

Theorem 1.2. *If* $\mathbf{H}$ *is a connected noncommutative absolutely almost simple* K*-isotropic* K*-group, then*

$$\phi\big(\mathbf{H}(\mathcal{O}_S)\big) \leq k(\mathbf{H}, S) - 1$$

That Theorem 1.1 follows from Theorem 1.2 is routine; see e.g. 2.6(c) of [Be 3].

Example. A special case of Theorem 1.2 is the inequality

$$\phi\Big(\mathbf{SL_n}(\mathbb{F}_q[t])\Big) \leq n - 2$$

or more generally,

$$\phi\Big(\mathbf{SL_n}(\mathcal{O}_S)\Big) \leq |S|(n-1) - 1$$

Indeed, for any field L, the number $\mathrm{rank}_L \mathbf{SL_n}$ equals the dimension of the diagonal subgroup in $\mathbf{SL_n}$. Hence, for any K and any S, we have

$$k(\mathbf{SL_n}, S) = \sum_{v \in S} \mathrm{rank}_{K_v} \mathbf{SL_n} = |S|(n-1)$$

This inequality is known to be sharp in some cases. For example, Stuhler showed that $\phi(\mathbf{SL_2}(\mathcal{O}_S)) = |S| - 1$ [St 2], and Abels and Abramenko independently showed that $\phi(\mathbf{SL_n}(\mathbb{F}_q[t])) = n - 2$ as long as $q \geq 2^{n-2}$ or $q \geq \binom{n-2}{\lfloor \frac{n-2}{2} \rfloor}$ respectively [Abl], [Abr 1].

Is the inequality sharp in general? Theorem 1.2 provides evidence for the following long-standing conjecture, which offers a striking relation between the two functions ϕ and k.

Conjecture 1.3. If $\mathbf{H}$ is a connected noncommutative absolutely almost simple K-isotropic K-group, then

$$\phi\big(\mathbf{H}(\mathcal{O}_S)\big) = k(\mathbf{H}, S) - 1$$

See [Be 3] for other evidence.

Type $\mathbf{F_m}$. Recall that a group Γ is of type F_m if there exists an Eilenberg-Mac Lane complex $K(\Gamma, 1)$ with finite m-skeleton. For $m \geq 2$, a group is of type F_m if and only if it is finitely presented and of type FP_m. It then follows from [Be 3] that FP_m and F_m are equivalent conditions for the arithmetic groups considered in this paper. Thus, Theorems 1.1 and 1.2, and Conjecture 1.3, may be equivalently stated by substituting F_m for FP_m.

Type WFL. Although we will make no further use of it, we recall the definition of type WFL for completeness with respect to comments in the initial portion of the introduction: A group Γ is *of type* WFL if there exists a torsion-free finite-index subgroup of Γ, and if for any such subgroup Γ', the ring $\mathbb{Z}$ admits a finite length resolution by finitely generated free $\mathbb{Z}\Gamma'$-modules.

Contrast with number fields. Our theorems are particular to the case of global fields of positive characteristic. In characteristic zero, we have the following

Theorem 1.4 (Raghunathan, Borel-Serre)). *Any S-arithmetic subgroup of a reductive group defined over a global number field contains a finite-index torsion-free subgroup Γ that allows for a finite $K(\Gamma, 1)$.*

In particular, any S-arithmetic subgroup of a reductive group defined over a number field is finitely presented and of type FP_∞. Examples of groups for which the above theorem applies include $\mathbf{SL_n}(\mathbb{Z})$ and $\mathbf{SL_n}(\mathbb{Z}[1/p])$. See [Ra 1] for the case of arithmetic groups, and [Bo-Se] for the case of S-arithmetic groups.

Idea behind the proof. Although Theorem 1.1 shows a difference between arithmetic groups in positive characteristic and those in characteristic zero, it is through the deep-rooted similarity of these two families that we shall find a proof of Theorem 1.1.

Indeed, our motivating example for proving Theorem 1.2 was the proof of Epstein-Thurston that $\mathbf{SL_3}(\mathbb{Z})$ is not combable; see Chapter 10 Section 4 of [Ep et al.]. Recall that their proof proceeds by creating an exponential Dehn function for $\mathbf{SL_3}(\mathbb{Z})$ as follows. A family of closed curves with increasing lengths are constructed in a portion of the symmetric space $\mathbf{SL_3}(\mathbb{R})/\mathbf{SO_3}(\mathbb{R})$ that is a bounded distance from the subset $\mathbf{SL_3}(\mathbb{Z})\mathbf{SO_3}(\mathbb{R}) \subseteq \mathbf{SL_3}(\mathbb{R})/\mathbf{SO_3}(\mathbb{R})$. The discs in $\mathbf{SL_3}(\mathbb{R})/\mathbf{SO_3}(\mathbb{R})$ that fill these loops in the most metrically efficient manner have areas that are quadratic in the length of the loops that they bound. These efficient discs are not so useful in studying the large-scale geometry of $\mathbf{SL_3}(\mathbb{Z})$ though, since the discs travel farther away from the subspace $\mathbf{SL_3}(\mathbb{Z})\mathbf{SO_3}(\mathbb{R})$ as the length of their boundary curves increase.

To better understand the geometry of $\mathbf{SL_3}(\mathbb{Z})$, we only consider filling discs of the constructed loops that are contained in the original bounded neighborhood of $\mathbf{SL_3}(\mathbb{Z})\mathbf{SO_3}(\mathbb{R})$. What is shown in [Ep et al.] is that any such family of discs would be metrically inefficient in the sense that the discs would have areas that are at least exponential in the length of their boundary curves. The result is an exponential Dehn function for $\mathbf{SL_3}(\mathbb{Z})$, which implies that $\mathbf{SL_3}(\mathbb{Z})$ is not combable.

Our proof of Theorem 1.2 in the special case when $\mathbf{H}(\mathcal{O}_S) = \mathbf{SL_3}(\mathbb{F}_q[t])$ proceeds by constructing an analogous family of loops in a bounded neighborhood of a given $\mathbf{SL_3}(\mathbb{F}_q[t])$-orbit in the Euclidean building, X, associated to $\mathbf{SL_3}\big(\mathbb{F}_q((t^{-1}))\big)$ where $\mathbb{F}_q((t^{-1}))$ is a field of formal Laurent series. As with the case for $\mathbf{SL_3}(\mathbb{Z})$, the closed curves have metrically efficient filling discs in X whose areas are quadratic in the length of their boundary curves. Also like the case for $\mathbf{SL_3}(\mathbb{Z})$, these filling discs travel farther away from the given orbit as the length of their boundary curves increase, so they are not helpful in learning about the large-scale geometry of $\mathbf{SL_3}(\mathbb{F}_q[t])$. However, in this case, there does not exist a filling disc for any of our constructed loops that is contained in the fixed bounded neighborhood of the $\mathbf{SL_3}(\mathbb{F}_q[t])$-orbit. Indeed, X is a contractible 2-dimensional simplicial complex, so filling discs are essentially unique. We then apply K. Brown's filtration criterion to conclude that $\mathbf{SL_3}(\mathbb{F}_q[t])$ is not finitely presented.

Distortion dimension. The contrast between arithmetic groups over function fields with arithmetic groups over number fields diminishes if we consider a metric analogue of finiteness length.

Let us direct our attention for the moment to an irreducible lattice Γ in a semisimple group over arbitrary nondiscrete locally compact fields; we can even allow for Γ to be nonarithmetic. We let X_Γ be the natural product of irreducible symmetric spaces and Euclidean buildings that Γ acts on. Given a point $x \in X_\Gamma$ and a real number r, we define the space

$$X_\Gamma(r) = \{\, y \in X_\Gamma \mid d(y\,,\,\Gamma x) \leq r \,\}$$

Using the Hurewicz theorem—as in Abels-Tiemeyer's Theorem 1.1.4 of [A-T]—and recalling that type FP_m and type F_m are equivalent conditions for Γ allows us to state K. Brown's filtration criterion for Γ to be of type FP_m in terms of homotopy groups. Precisely, Γ is of type FP_m if and only if for any real number $r \geq 0$ there exists a real number $r' \geq r$ such that for any $k < m$ the homomorphism induced by inclusion

$$\pi_k\big(X_\Gamma(r)\,,\,x\big) \longrightarrow \pi_k\big(X_\Gamma(r')\,,\,x\big)$$

is trivial; see Theorems 2.2 and 3.2 [Br 1].

As a special case we recover the easy to prove fact that Γ is of type FP_∞ if it acts cocompactly on X_Γ. If Γ does not act cocompactly, then Theorems 1.1 and 1.4 (along with Corollary 7.3 of [Lu]) characterize those Γ contained in semisimple groups over function fields as precisely those which fail to be of type FP_∞. (Recall that an arithmetic lattice $\mathbf{H}(\mathcal{O}_S)$ acts cocompactly on $X_{\mathbf{H}(\mathcal{O}_S)}$ if and only if the absolutely almost simple K-group $\mathbf{H}$ is K-anisotropic.)

To include metric properties of the large-scale geometry of lattices, we define Γ as being *undistorted up to dimension* m if: given any $r \geq 0$, there exist real numbers $r' \geq r$, $\lambda \geq 1$, and $C \geq 0$ such that for any $k < m$ and any Lipschitz k-sphere $s \subseteq X_\Gamma(r)$, there exists a Lipschitz $(k+1)$-ball $B_\Gamma \subseteq X_\Gamma(r')$ with $\partial B_\Gamma = s$ and

$$\text{volume}(B_\Gamma) \leq \lambda[\text{volume}(B_X)] + C$$

for all Lipschitz $(k+1)$-balls $B_X \subseteq X$ with $\partial B_X = s$. We adopt the convention that Γ is always undistorted up to dimension 0.

Now we define the *distortion dimension of* Γ to be the nonnegative integer

$$\psi(\Gamma) = \sup\{m \mid \Gamma \text{ is undistorted up to dimension } m\}$$

Conjecture 1.5. If Γ is an irreducible lattice in a semisimple group over nondiscrete locally compact fields, then $\psi\big(\Gamma\big) = \infty$ if and only if Γ acts cocompactly on X_Γ.

That $\psi(\Gamma) = \infty$ when Γ acts cocompactly is clear. The converse had been conjectured for lattices in real semisimple Lie groups following the Epstein-Thurston proof that $\psi(\mathbf{SL_n}(\mathbb{Z})) \leq n-2$, and a general proof seems approachable. (See 10.4 [Ep et al.] for $\mathbf{SL_n}(\mathbb{Z})$.)

Less attention has been given to S-arithmetic lattices in characteristic zero, but the conjecture should not change in this setting. In positive characteristic, Conjecture 1.5 follows from Theorem 1.1.

As Conjecture 1.5 extends Theorem 1.1 for absolutely almost simple K-groups $\mathbf{G}$ into the context of arbitrary global fields, we are naturally led to speculate how Conjecture 1.3 might be broadened to include fields of characteristic zero. Thus, we define $\tau(X_\Gamma)$ to be the Euclidean rank of X_Γ, and we note that for $\Gamma = \mathbf{H}(\mathcal{O}_S)$ as in Conjecture 1.3, $\tau(X_\Gamma) = k(\mathbf{H}, S)$. We ask

Question 1.6. Let Γ be a noncocompact irreducible lattice in a semisimple group over nondiscrete locally compact fields. Is it true that

$$\psi\big(\Gamma\big) = \tau(X_\Gamma) - 1$$

If not, then can the definition of ψ be reasonably modified so that the above formula is true?

This problem is daunting. For example, an affirmative answer to the first question implies Thurston's claim that $\mathbf{SL_n}(\mathbb{Z})$ has a quadratic Dehn function for $n \geq 4$. See also remarks from Gromov's book (5.D.(5).(c) [Gr]).

We will just mention a few pieces of evidence for a positive answer. We note that $\psi(\Gamma) = 0$ if and only if either Γ is not finitely generated or the word metric on Γ is not quasi-isometric to the metric induced from its action on X_Γ; see the example in Section 2. Hence, it follows from work of Lubotzky (Corollary 7.3 of [Lu]) and Lubotzky-Mozes-Raghunathan [L-M-R] that $\psi(\Gamma) = 0$ if and only if $\tau(X_\Gamma) = 1$. Note also that Leuzinger-Pittet [Le-Pi 1], Behr [Be 3], and (a generalization of) Taback (Lemma 4.2 of [Ta]), show that $\tau(X_\Gamma) = 2$ implies $\psi(\Gamma) = 1$.

For related material, see the papers of Druţu, Hattori, Leuzinger-Pittet, Noskov, and Pittet: [Dr 1], [Dr 2], [Hat], [Le-Pi 2], [No], and [Pi].

Possible generalizations to other rings of functions. In [Bu-Wo] we use techniques from this paper to give a geometric proof that $\mathbf{SL_2}(\mathbb{Z}[t, t^{-1}])$ is not finitely presented—a fact first proved by Krstić-McCool [Kr-Mc].

It is likely that the ideas below can be used to do more in this direction of generalizing Theorem 1.2 to apply to a class of fields and rings that properly includes global function fields and their rings of S-integers; see the question in the introduction of [Bu-Wo].

Outline of the paper. We begin in Section 2 with a special case of our proof to motivate what follows. The proof of Theorem 1.2 is contained in Section 3.

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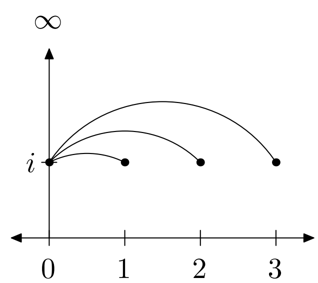
2. An example

The first piece of evidence for Theorem 1.1 was:

Theorem 2.1 (Nagao). *The group* $\mathbf{SL_2}(\mathbb{F}_q[t])$ *is not finitely generated.*

In this section we will see how our proof of Theorem 1.2 applies to this special case. For motivation, we will first review some of the geometry of $\mathbf{SL_2}(\mathbb{Z})$, a mathematical cousin of $\mathbf{SL_2}(\mathbb{F}_q[t])$.



Consider the action of $\mathbf{SL_2}(\mathbb{Z})$ on the hyperbolic plane $\mathbb{H}^2$. The diagram shows the upper half-plane model. There is a distinguished point ∞ at the top of the diagram that no $\mathbf{SL_2}(\mathbb{Z})$-orbit accumulates on. Specifically, it is well-known that the orbit of the complex number i avoids the open horoball B that is centered at ∞ and consists of all complex numbers with imaginary parts greater than 1. The boundary of this horoball is approximated by the points $n+i$ for $n \in \mathbb{Z}$. (Notice that $n+i = \left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) i$.)

The geodesic joining i and $1+i$ travels into the horoball B. The geodesic between i and $2+i$ travels farther into the horoball, the geodesic between i and $3+i$ farther still, and so on. Continuing this process, we see that no metric neighborhood of the orbit $\mathbf{SL_2}(\mathbb{Z})i \subseteq \mathbb{H}^2$ is convex in $\mathbb{H}^2$. Sufficiently large metric neighborhoods of $\mathbf{SL_2}(\mathbb{Z})i$ are however connected, as $\mathbf{SL_2}(\mathbb{Z})$ is finitely generated:

Lemma 2.2. *Suppose a finitely generated group Γ acts on a geodesic metric space X. Then, for any point $x \in X$, there is a number $R > 0$ such that the R-neighborhood*

$$\mathrm{Nbhd}_R(\Gamma x) \subseteq X$$

is connected.

Proof. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a finite generating set for Γ. Choose R such that the ball $B_R(x)$ contains all translates $\gamma_i x$. Then $\Gamma B_R(x) = \mathrm{Nbhd}_R(\Gamma x)$ is connected. $\square$

Now let's look at a proof of Nagao's theorem along the lines of our proof for Theorem 1.2. This is not Nagao's original proof, rather it is a simplified form of Stuhler's argument [St 2].

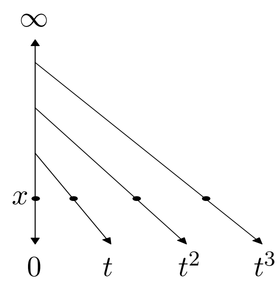
The locally compact field $\mathbb{F}_q((t^{-1}))$ contains $\mathbb{F}_q[t]$ as a discrete subring. Thus, $\mathbf{SL_2}(\mathbb{F}_q[t])$ is a discrete subgroup of $\mathbf{SL_2}\big(\mathbb{F}_q((t^{-1}))\big)$.

There is a natural nonpositively curved space that $\mathbf{SL_2}\big(\mathbb{F}_q((t^{-1}))\big)$ acts on: the regular $(q+1)$-valent tree T. If $\mathbb{F}_q[[t^{-1}]] < \mathbb{F}_q((t^{-1}))$ is the ring of Taylor series, then this well-known action is obtained by identifying the vertices of T with homothety classes of spanning $\mathbb{F}_q[[t^{-1}]]$-submodules of a 2-dimensional vector space over $\mathbb{F}_q((t^{-1}))$

that are free and of rank 2. This is in analogy to the identification of the unit tangent bundle of $\mathbb{H}^2$ with the unit tangent bundle of the Teichmüller space of 2-dimensional Euclidean tori with volume equal to 1, or equivalently, with homothety classes of spanning $\mathbb{Z}$-submodules of $\mathbb{R}^2$ that are free and of rank 2.

Just as the boundary of $\mathbb{H}^2$ is a circle, or $\mathbb{P}^1(\mathbb{R})$, the boundary of T can be identified with $\mathbb{P}^1\big(\mathbb{F}_q((t^{-1}))\big)$. We use the standard identification of $\mathbb{P}^1\big(\mathbb{F}_q((t^{-1}))\big)$ with $\mathbb{F}_q((t^{-1}))\cup\{\infty\}$. The group $\mathbf{SL_2}\big(\mathbb{F}_q((t^{-1}))\big)$ has two induced actions on $\mathbb{P}^1\big(\mathbb{F}_q((t^{-1}))\big)$: one from its action on T, and one from its action on the 2-dimensional vector space $\mathbb{F}_q((t^{-1}))^2$. These actions coincide.

In another analogy with the situation for $\mathbf{SL_2}(\mathbb{Z})$, any ray from any point $x \in T$ towards ∞ escapes every metric neighborhood of the orbit $\mathbf{SL_2}(\mathbb{F}_q[t])x$ (one can see this using Mahler's compactness criterion). The diagram on the right has x contained in the geodesic joining the two boundary points 0 and ∞. We write $f * x \in T$ as shorthand for the point $\left(\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right)x$. The geodesic segment between x and $t^n * x$ is the portion of the geodesic joining the boundary points 0 and t^n that lies at or above the level of x in the diagram. These segments contain increasing subsets of the geodesic ray from x to ∞ as $n \in \mathbb{N}$ grows. Hence, no metric neighborhood of the orbit $\mathbf{SL_2}(\mathbb{F}_q[t])x \subseteq T$ is convex.



The comparison with $\mathbf{SL_2}(\mathbb{Z})$ stops here since, in T, convexity is equivalent to connectedness. Using Lemma 2.2, we see that $\mathbf{SL_2}(\mathbb{F}_q[t])$ is not finitely generated. Our proof is complete.

3. Proof of Theorem B

In what follows, we let p equal the characteristic of K. Rather than proving Theorem 1.2 directly, our goal will be to prove a slightly stronger claim:

Proposition 3.1. *If* $\mathbf{H}$ *is a connected noncommutative absolutely almost simple* K*-isotropic* K*-group, then*

$$\phi\big(\mathbf{H}(\mathcal{O}_S);\mathbb{F}_p\big) \leq k(\mathbf{H}, S) - 1$$

Theorem 1.2 follows since a group Γ is of type FP_m over $\mathbb{F}_p$ if it is of type FP_m over $\mathbb{Z}$: just tensor a projective resolution for $\mathbb{Z}$ by finitely generated $\mathbb{Z}\Gamma$-modules with $\mathbb{F}_p$ to obtain a projective resolution for $\mathbb{F}_p$ by finitely generated $\mathbb{F}_p\Gamma$-modules.

3.1. Method of proof for Proposition 3.1

We define the ring

$$K_S = \prod_{v \in S} K_v$$

so that

$$\mathbf{H}(K_S) = \prod_{v \in S} \mathbf{H}(K_v)$$

Let X be the Euclidean building corresponding to $\mathbf{H}(K_S)$, that is the product of the irreducible Euclidean buildings for $\mathbf{H}(K_v)$. Recall that X has dimension $k(\mathbf{H}, S)$.

We fix a base point $e \in X$ (to be specified later) and consider closed metric neighborhoods of the orbit $\mathbf{H}(\mathcal{O}_S)e$. That is, for each number $r \geq 0$, we set

$$X(r) = \big\{ x \in X \mid d\big(x\,,\, \mathbf{H}(\mathcal{O}_S)e\big) \leq r \big\}$$

We will find a number $r_0 > 0$ and construct, for each $r \geq r_0$, a cycle in $X(r_0)$ that represents a nontrivial element in the reduced homology group with coefficients in $\mathbb{F}_p$

$$\widetilde{\mathrm{H}}_{k(\mathbf{H},S)-1}\left(X(r)\,;\,\mathbb{F}_p\right)$$

This shows that the inclusions $X(r_0) \subseteq X(r)$ induce nontrivial homomorphisms

$$\widetilde{\mathrm{H}}_{k(\mathbf{H},S)-1}\left(X(r_0)\,;\,\mathbb{F}_p\right) \longrightarrow \widetilde{\mathrm{H}}_{k(\mathbf{H},S)-1}\left(X(r)\,;\,\mathbb{F}_p\right)$$

In view of K. Brown's filtration criterion (see Theorem 2.2 and the following remark in [Br 1]), the existence of this family of nontrivial homomorphisms together with the following standard facts about the action of $\mathbf{H}(\mathcal{O}_S)$ on X implies Proposition 3.1: (i) X is contractible; (ii) $\mathbf{H}(\mathcal{O}_S)$ acts on X with finite cell stabilizers; and (iii) the subspaces $X(r)$ are $\mathbf{H}(\mathcal{O}_S)$-invariant and compact modulo $\mathbf{H}(\mathcal{O}_S)$.

Excluding a tree. For the remainder of this paper, we will assume that $k(\mathbf{H}, S) > 1$. That is, we assume that X is not a tree. This assumption is made only to avoid complications in our exposition; the philosophy of the proof still applies to the case when $k(\mathbf{H}, S) = 1$ as is shown in Section 2.

3.2. An apartment coarsely separated by $\mathbf{H}(\mathcal{O}_S)$

We will find an apartment in X that "coarsely intersects" an $\mathbf{H}(\mathcal{O}_S)$-orbit in a hyperplane. (Later, we will use this $(k(\mathbf{H},S)-1)$-dimensional hyperplane and its translates to construct the $(k(\mathbf{H},S)-1)$-cycles mentioned above.) Since apartments in X correspond to products of maximal K_v-split tori in $\mathbf{H}$, this problem reduces to algebra.

We begin by choosing a parabolic group that will accompany us throughout our proof. In what follows, we are assuming that the reader has a basic knowledge of the structure of parabolic subgroups of reductive groups relative to fields that are not algebraically closed, as can be found for example in 21.11 and 21.12 of [Bo].

Since $\mathbf{H}$ is K-isotropic, there exists a nontrivial maximal K-split torus of $\mathbf{H}$. We let Φ_K be the roots of $\mathbf{H}$ with respect to this torus. Choose an ordering on Φ_K, and let $\Delta_K \subseteq \Phi_K$ denote the corresponding collection of simple roots.

Choose, and fix throughout, a root $\alpha_0 \in \Delta_K$. We define the 1-dimensional K-split torus

$$\mathbf{T_1} = \Big(\bigcap_{\alpha \in \Delta_K - \alpha_0} \ker(\alpha) \Big)^{\circ}$$

The above superscript $\circ$ denotes the connected component of the identity. We let $\mathbf{Z_H}(\mathbf{T_1})$ be the centralizer of $\mathbf{T_1}$ in $\mathbf{H}$.

There exists a maximal proper K-parabolic subgroup of $\mathbf{H}$, denoted $\mathbf{P}^+$, with the following Levi decomposition:

$$\mathbf{P}^+ = \mathbf{R_u}(\mathbf{P}^+) \rtimes \mathbf{Z_H}(\mathbf{T_1})$$

In the above, $\mathbf{R_u}(\mathbf{P}^+)$ is the unipotent radical of $\mathbf{P}^+$.

We can expand the Levi decomposition to a Langlands decomposition by noting that $\mathbf{Z_H}(\mathbf{T_1})$ is an almost direct product of $\mathbf{T_1}$, the derived group $\mathbf{Z_H}(\mathbf{T_1})_{\mathrm{der}}$, and $\mathbf{D_a}$ for some K-anisotropic diagonalizable group $\mathbf{D_a}$. Thus:

$$\mathbf{P}^+ = \mathbf{R_u}(\mathbf{P}^+) \rtimes \mathbf{T_1}\mathbf{D_a}\mathbf{Z_H}(\mathbf{T_1})_{\mathrm{der}}$$

Before proceeding with the existence of the torus and the apartment that is our goal in this section, we record the following well-known result.

Proposition 3.2. *Let* $\mathbf{G}$ *be a reductive* K*-group. Then for any finite nonempty set* S' *of pairwise inequivalent valuations and any family* $\{\mathbf{A}_v\}_{v \in S'}$ *of maximal* K_v*-tori of* $\mathbf{G}$*, there is a maximal* K*-torus* $\mathbf{A}_\pi$ *of* $\mathbf{G}$ *and group elements* $g_v \in \mathbf{G}(K_v)$ *such that*

$$\mathbf{A}_\pi = {}^{g_v}\mathbf{A}_v$$

for all $v \in S'$*, where* ${}^{g_v}\mathbf{A}_v$ *denotes* $\mathbf{A}_v$ *conjugated by* g_v*.*

Proof. There is a proof of this proposition in Section 7.1 Corollary 3 of [Pl-Ra] for the case when K is a global number field. The proof also applies for global function fields after replacing the argument for the K-rationality of the variety of maximal tori in $\mathbf{G}$ with the proof of Theorem 7.9 in [Bo-Sp]. □

We will make use of the above proposition in the proof of the proposition below.

Proposition 3.3. *There exists a maximal K-torus $\mathbf{A} \leq \mathbf{H}$ such that:*

(i) The maximal K-split torus of $\mathbf{A}$ is $\mathbf{T_1}$, and
(ii) $\mathbf{A}$ contains a maximal K_v-split torus of $\mathbf{H}$ for all $v \in S$.

Proof. For each $v \in S$, let $\mathbf{A}_v$ be a maximal K_v-torus of $\mathbf{Z_H}(\mathbf{T_1})_{\mathrm{der}}$ such that $\mathbf{A}_v$ contains a maximal K_v-split torus of $\mathbf{Z_H}(\mathbf{T_1})_{\mathrm{der}}$.

Then choose a valuation of K, call it w, that is inequivalent to any of the valuations of S, and let $\mathbf{A}_w$ be a maximal K_w-torus in $\mathbf{Z_H}(\mathbf{T_1})_{\mathrm{der}}$ that is K_w-anisotropic. The existence of such a torus is well-known; see e.g. Section 2.4 of [De].

Now apply Proposition 3.2 to $\mathbf{G} = \mathbf{Z_H}(\mathbf{T_1})_{\mathrm{der}}$ and $S' = S \cup \{w\}$. Since $\mathbf{A}_\pi$ is K_w-anisotropic, it is necessarily K-anisotropic. Therefore, part (i) is satisfied by

$$\mathbf{A} = \mathbf{T_1}\mathbf{D_a}^\circ \mathbf{A}_\pi$$

To verify part (ii), note that $\mathbf{T_1}$ is contained in a maximal K_v-split torus of $\mathbf{H}$. Hence,

$$\begin{aligned}
\mathrm{rank}_{K_v}(\mathbf{H}) &= \mathrm{rank}_{K_v}(\mathbf{Z_H}(\mathbf{T_1})) \\
&= \mathrm{rank}_{K_v}(\mathbf{T_1}\mathbf{D_a}) + \mathrm{rank}_{K_v}(\mathbf{Z_H}(\mathbf{T_1})_{\mathrm{der}}) \\
&= \mathrm{rank}_{K_v}(\mathbf{T_1}\mathbf{D_a}) + \mathrm{rank}_{K_v}(\mathbf{A}_\pi) \\
&= \mathrm{rank}_{K_v}(\mathbf{A})
\end{aligned}$$

□

Since $\mathbf{A}$ contains a maximal K_v-split torus for all $v \in S$, there is an apartment $\Sigma \subseteq X$ that $\mathbf{A}(K_S)$ acts on properly and cocompactly as a translation group of maximal rank, $\dim(\Sigma) = k(\mathbf{H}, S)$. By Dirichlet's units theorem (see Theorem 5.12 [Pl-Ra]) and the preceding proposition, the arithmetic group $\mathbf{A}(\mathcal{O}_S)$ is a finitely generated abelian group of rank

$$\Big(\sum_{v \in S} \mathrm{rank}_{K_v}(\mathbf{A})\Big) - \mathrm{rank}_K(\mathbf{A}) = k(\mathbf{H}, S) - 1$$

Choose a point $e \in \Sigma$. Since $\mathbf{A}(\mathcal{O}_S) \leq \mathbf{A}(K_S)$ acts properly on Σ, the base point e is contained in an affine hyperplane $V \subseteq \Sigma$, of dimension $k(\mathbf{H}, S) - 1$, that $\mathbf{A}(\mathcal{O}_S)$ acts on cocompactly. This point

$e \in \Sigma$ is the point we specify for our definition in Section 3.1 of the spaces $X(r) \subseteq X$.

Example. In the case when $K = \mathbb{F}_q(t)$, $\mathcal{O}_S = \mathbb{F}_q[t]$, and $\mathbf{H} = \mathbf{SL_3}$, the torus $\mathbf{T_1}$ can be taken as the group of matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix}$$

Then the parabolic group $\mathbf{P}^+$ can be taken to be the determinate 1 matrices of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

The resulting group $\mathbf{R_u}(\mathbf{P}^+)$ would be the 2-dimensional commutative group

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

This would leave the semisimple group $\mathbf{Z_H}(\mathbf{T_1})_{\mathrm{der}}$ to be the copy of $\mathbf{SL_2}$ that sits in the upper left corner of $\mathbf{SL_3}$, and $\mathbf{D_a}$ would be trivial.

The group $\mathbf{A}(\mathcal{O}_S)$ in this example can be taken to be the group generated by the matrix

$$\begin{pmatrix} t^2+1 & t & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

With the notation from the proof of Proposition 3.3, the Zariski closure of $\mathbf{A}(\mathcal{O}_S)$ would equal $\mathbf{A}_\pi \leq \mathbf{Z_H}(\mathbf{T_1})_{\mathrm{der}}$.

3.3. A space for the manufacture of cycles: choosing r_0

Let $D \subseteq V$ be a fundamental domain for the $\mathbf{A}(\mathcal{O}_S)$-action on V. Applying Satz 3 of [Be 1], there is a compact set $C \subseteq \mathbf{R_u}(\mathbf{P}^+)(K_S)$ such that $\mathbf{R_u}(\mathbf{P}^+)(K_S) = \mathbf{R_u}(\mathbf{P}^+)(\mathcal{O}_S)C$. Since $\mathbf{A}(\mathcal{O}_S)$ normalizes $\mathbf{R_u}(\mathbf{P}^+)(K_S)$, we have:

$$\begin{aligned} \mathbf{R_u}(\mathbf{P}^+)(K_S)V &\subseteq \mathbf{R_u}(\mathbf{P}^+)(K_S)\mathbf{A}(\mathcal{O}_S)D \\ &\subseteq \mathbf{A}(\mathcal{O}_S)\mathbf{R_u}(\mathbf{P}^+)(K_S)D \\ &\subseteq \mathbf{A}(\mathcal{O}_S)\mathbf{R_u}(\mathbf{P}^+)(\mathcal{O}_S)CD \\ &\subseteq \mathbf{H}(\mathcal{O}_S)CD \end{aligned}$$

Since the region $CD \subseteq X$ is bounded, we can choose a number $r_0 > 0$ such that

$$\mathbf{R_u}(\mathbf{P}^+)(K_S)V \subseteq \mathbf{H}(\mathcal{O}_S)CD \subseteq X(r_0)$$

It is inside the space $\mathbf{R_u}(\mathbf{P}^+)(K_S)V$ where we shall produce cycles that remain nontrivial in the homology of $X(r)$ for $r \geq r_0$.

3.4. A direction away from $X(r_0)$

Recall our choice of $\alpha_0 \in \Delta_K$ from the beginning of Section 3.2. This root is nontrivial when restricted to $\mathbf{T_1}$, so for any $v \in S$, the set

$$\{ \, a \in \mathbf{T_1}(K_v) \mid |\alpha_0(a)|_v > 1 \, \}$$

is nonempty and open in the Hausdorff topology induced by the metric $|\cdot|_v$ on K_v that arises from v.

Since $\mathbf{T_1}$ is K-isomorphic to an affine line with a point removed, it satisfies the weak approximation property with respect to S. That is, the diagonal embedding

$$\mathbf{T_1}(K) \longrightarrow \prod_{v \in S} \mathbf{T_1}(K_v)$$

has a dense image. Therefore, there exists some $a_+ \in \mathbf{T_1}(K)$ such that

$$|\alpha_0(a_+)|_v > 1$$

for all $v \in S$.

It will be important for us later to have a direction in Σ, that leads away from every $X(r)$. The direction we will use is given by the sequence $(a_+^n e)_{n \in \mathbb{N}}$. Note that the above condition on a_+ assures us that the sequence $(a_+^n)_{n \in \mathbb{N}}$ is not contained in any compact subset of $\mathbf{H}(K_S)$. Therefore, $(a_+^n e)_{n \in \mathbb{N}}$ does specify a direction in Σ.

Let X^∞ be the visual boundary of X. It can be identified in a natural way with the spherical Tits building for $\mathbf{H}(K_S)$. Note that X^∞ is the spherical join of the spherical buildings for the groups $\mathbf{H}(K_v)$ with $v \in S$. We let $\Sigma^\infty \subseteq X^\infty$ be the apartment corresponding to $\Sigma \subseteq X$, and we let $a_+^\infty \in \Sigma^\infty$ be the accumulation point of $(a_+^n e)_{n \in \mathbb{N}}$.

We let Π_+^∞ be the unique simplex in X^∞ that is maximal among all simplices stabilized by the action of $\mathbf{P}^+(K_S)$. Note that Π_+^∞ is the spherical join over S of the simplices associated with the groups $\mathbf{P}^+(K_v)$ in the spherical buildings for $\mathbf{H}(K_v)$.

The conditions on our choice of a_+ were imposed to insure that the orbit of e under its iterates would accumulate inside Π_+^∞. Specifically, we have:

Lemma 3.4. *The point $a_+^\infty \in \Sigma^\infty$ is contained in Π_+^∞.*

Proof. Using the definition of spherical joins, we can reduce to the case when S contains a single valuation v. What follows is routine; see e.g. 2.4 [Pr].

We let $\mathbf{Q}$ be the K_v-parabolic subgroup of $\mathbf{H}$ with

$$\mathbf{Q}(K_v) = \{\, g \in \mathbf{H}(K_v) \mid g a_+^\infty = a_+^\infty \,\}$$

The proof of this lemma amounts to showing that

$$\mathbf{P}^+(K_v) \leq \mathbf{Q}(K_v)$$

Note that $g a_+^\infty = a_+^\infty$ if and only if

$$d(g a_+^n e\,,\, a_+^n e) = d(a_+^{-n} g a_+^n e\,,\, e)$$

is a bounded sequence. Since distance from e is a proper function, and because the action of $\mathbf{H}(K_v)$ on X is proper, we can alternatively characterize $\mathbf{Q}(K_v)$ as the group

$$\{\, g \in \mathbf{H}(K_v) \mid (a_+^{-n} g a_+^n)_{n\in\mathbb{N}} \subseteq \mathbf{H}(K_v) \text{ is precompact} \,\}$$

Let $\mathfrak{u}$ and $\mathfrak{h}$ be the Lie algebras of $\mathbf{R_u}(\mathbf{P}^+)$ and $\mathbf{H}$ respectively. We denote the set of positive roots given by our ordering of Φ_K in Section 3.2 as $\Phi_K^+ \subseteq \Phi_K$, and we write the set of roots that are linear combinations of elements in $\Delta_K - \alpha_0$ as $[\Delta_K - \alpha_0]$.

If $\mathbf{T}$ is the maximal K-split torus in $\mathbf{H}$ that was chosen to produce the roots Φ_K, then our choice of $\mathbf{P}^+$ from Section 3.2 implies that

$$\mathfrak{u} = \bigoplus_{\alpha \in \Phi_K^+ - [\Delta_K - \alpha_0]} \{\, v \in \mathfrak{h} \mid \operatorname{Ad} t(v) = \alpha(t) v \text{ for all } t \in \mathbf{T} \,\}$$

Note that $\Phi_K^+ - [\Delta_K - \alpha_0]$ is exactly the subset of Φ_K consisting of sums of the form $\sum_{\alpha_i \in \Delta_K} n_i \alpha_i$ with $n_i \geq 0$ for all i and $n_0 \geq 1$. By our definition of $\mathbf{T_1} \leq \mathbf{T}$ as being contained in the kernel of every root in $\Delta_K - \alpha_0$, and as $a_+ \in \mathbf{T_1}$, we can express $\mathfrak{u}$ as a finite direct sum

$$\mathfrak{u} = \bigoplus_{n \geq 1} \{\, v \in \mathfrak{u} \mid \operatorname{Ad} a_+(v) = n\alpha_0(a_+) v \,\}$$

Since $|\alpha_0(a_+)|_v > 1$, we see that for any $u \in \mathbf{R_u}(\mathbf{P}^+)(K_v)$,

$$a_+^{-n} u a_+^n \to 1$$

as $n \to \infty$. Hence, if $u \in \mathbf{R_u}(\mathbf{P}^+)(K_v)$ and $z \in \mathbf{Z_H}(\mathbf{T_1})(K_v)$, then

$$a_+^{-n} u z a_+^n = a_+^{-n} u a_+^n z \to z$$

In particular, the above sequence is precompact. As a consequence, uz, and thus all of

$$\mathbf{P}^+(K_v) = \mathbf{R_u}(\mathbf{P})(K_v) \rtimes \mathbf{Z_H}(\mathbf{T_1})(K_v)$$

is contained in $\mathbf{Q}(K_v)$. $\square$

With a little more effort, it can be shown that $a_+^\infty \in \Pi_+^\infty - \partial\Pi_+^\infty$, but we will not need this fact.

Now we know the direction of a_+^∞. Our last point of business in this section is to see that this direction leads away from the orbit $\mathbf{H}(\mathcal{O}_S)e$. This argument is standard.

Lemma 3.5. *For any $r > 0$, there exists an $n \in \mathbb{N}$ such that $a_+^n e \notin X(r)$.*

Proof. Choose any nontrivial $\gamma \in \mathbf{R_u}(\mathbf{P}^+)(\mathcal{O}_S)$. As in the proof of the preceding lemma, $a_+^{-n}\gamma a_+^n \to 1$.

¿From Theorem I.1.12 of [Ra 2], the sequence $(a_+^n)_{n\in\mathbb{N}} \subseteq \mathbf{H}(K_S)$ induces a sequence in the quotient space $\mathbf{H}(\mathcal{O}_S)\backslash\mathbf{H}(K_S)$ that is not contained in any compact set. The lemma follows. □

3.5. A blueprint at infinity

In this section we will construct a cycle inside X^∞ in the direction given by the sequence $(a_+^{-n}e)_{n\in\mathbb{N}}$. This is the direction in Σ that is opposite to a_+^∞. In Section 3.7, translates of this cycle will be "coned off" from points of the form $a_+^n e$. Then, these cones will be intersected with $\mathbf{R_u}(\mathbf{P}^+)(K_S)V$ to produce cycles in $X(r_0)$.

We let Π_-^∞ be the simplex opposite to Π_+^∞ in the spherical apartment Σ^∞. This simplex is the unique maximal simplex in X^∞ that is fixed under the action of $\mathbf{P}^-(K_S)$, where $\mathbf{P}^-$ is the maximal proper K-parabolic subgroup of $\mathbf{H}$ that contains $\mathbf{Z_H}(\mathbf{T_1})$ and is opposite to $\mathbf{P}^+$.

We let Δ_-^∞ be the simplicial star of Π_-^∞ in the apartment Σ^∞. That is, Δ_-^∞ is the union of all simplices in Σ^∞ that contain Π_-^∞.

The description of a chain in the boundary. Let σ be a codimension 1 simplex in Σ^∞ that is contained in the boundary of Δ_-^∞. The geodesic continuation of σ in Σ_∞ is a great sphere, that is, the boundary of a closed simplicial hemisphere $R_\alpha \subseteq \Sigma^\infty$ (called a *root space*). Among the two possible hemispheres, R_α and $R_{-\alpha}$, in Σ^∞ that contain σ in their boundary (called *opposite root spaces*), we fix notation so that $R_{-\alpha}$ contains Π_-^∞.

Lemma 3.6. *There exists a group element $u_{-\alpha} \in \mathbf{R_u}(\mathbf{P}^-)(K_S)$ fixing $R_{-\alpha}$ pointwise and satisfying the condition*

$$\Sigma^\infty \cap u_{-\alpha}\Sigma^\infty = R_{-\alpha}$$

Proof. We may assume that S consists of a single valuation v. The general case follows from the definition of the spherical join.

Let $\mathbf{Q}$ be the minimal K_v-parabolic subgroup of $\mathbf{H}$ corresponding to the chamber containing σ and Π_-^∞. Let Φ_{K_v} be the set of roots of

$\mathbf{H}$ with respect to the maximal K_v-split torus in $\mathbf{A}$, let $\Phi_{K_v}^{\mathrm{nd}} \subseteq \Phi_{K_v}$ be the set of nondivisible roots, and let $\Delta_{K_v} \subseteq \Phi_{K_v}^{\mathrm{nd}}$ be the set of simple roots associated with our choice of $\mathbf{Q}$.

As explained in 5.6 of [Ti], there is a root $-\alpha \in \Phi_{K_v}^{\mathrm{nd}}$ such that any nontrivial element $u_{-\alpha}$ of the root group $\mathbf{U}_{(-\alpha)}(K_v) \leq \mathbf{H}(K_v)$ fixes $R_{-\alpha}$ pointwise and satisfies $\Sigma^\infty \cap u_{-\alpha}\Sigma^\infty = R_{-\alpha}$. (A similar statment holds by replacing $-\alpha$ throughout with α, where the root $\alpha \in \Phi_{K_v}^{\mathrm{nd}}$ is the negative of α.)

Note that all we have left to show is $\mathbf{U}_{(-\alpha)} \leq \mathbf{R_u}(\mathbf{P}^-)$.

Recall the standard correspondence that assigns to any subset $I \subseteq \Delta_{K_v}$ a K_v-parabolic subgroup of $\mathbf{H}$ containing $\mathbf{Q}$, denoted $\mathbf{Q}_I$; see e.g. 21.12 [Bo]. Since σ is of codimension 1 in Σ^∞, the K_v-parabolic subgroup of $\mathbf{H}$ corresponding to σ is of the form $\mathbf{Q}_{\{\beta\}}$ for a single simple root $\beta \in \Delta_{K_v}$. We also have that $\mathbf{U}_{(-\alpha)}(K_v) \leq \mathbf{Q}_{\{\beta\}}(K_v)$ and $\mathbf{U}_{(\alpha)}(K_v) \leq \mathbf{Q}_{\{\beta\}}(K_v)$ since $\sigma \subseteq R_{-\alpha} \cap R_\alpha$ is fixed by $\mathbf{U}_{(-\alpha)}(K_v)$ and $\mathbf{U}_{(\alpha)}(K_v)$. It follows from 21.12 of [Bo] that either $-\alpha = \beta$ or $\alpha = \beta$.

Since $\mathbf{U}_{(-\alpha)}(K_v)$ fixes $R_{-\alpha}$ pointwise, the chamber corresponding to $\mathbf{Q}$ is also fixed under the action of $\mathbf{U}_{(-\alpha)}(K_v)$. Hence, $\mathbf{U}_{(-\alpha)} \leq \mathbf{Q}$ implying that $-\alpha$ is positive under the ordering on Φ_{K_v} consistent with Δ_{K_v}. Now it must be that $-\alpha = \beta$.

Since $\Pi_-^\infty \nsubseteq \sigma$, we have $\mathbf{Q}_{\{-\alpha\}} = \mathbf{Q}_{\{\beta\}} \nleq \mathbf{P}^-$. Therefore, if we assume $J \subseteq \Delta_{K_v}$ is such that $\mathbf{Q}_J = \mathbf{P}^-$, then $-\alpha \notin J$. It follows that $\mathbf{U}_{(-\alpha)} \leq \mathbf{R_u}(\mathbf{P}^-)$ as desired. $\square$

Any K_v-parabolic subgroup of $\mathbf{H}$ that is contained in $\mathbf{P}^-$ must contain $\mathbf{R_u}(\mathbf{P}^-)$. Thus, $u_{-\alpha} \in \mathbf{R_u}(\mathbf{P}^-)(K_S)$ fixes Δ_-^∞ pointwise. Therefore, $\Delta_-^\infty \subseteq R_{-\alpha}$ which, in turn, implies that $\Delta_-^\infty \cap R_\alpha$ is the union of some codimension 1 simplices in the boundary of Δ_-^∞ (including σ). We name this union F_α and call it a *geodesically continued face of* Δ_-^∞. We take a minimal (hence finite) family of root spaces $\{R_\alpha\}_{\alpha\in A}$ which exhaust the boundary of Δ_-^∞ as the union of the corresponding geodesically continued faces of Δ_-^∞.

Applying Lemma 3.6 to the opposite parabolic and opposite root space, we have that for each for each $\alpha \in A$, there is a group element $u_\alpha \in \mathbf{R_u}(\mathbf{P}^+)(K_S)$ that fixes R_α pointwise and satisfies $\Sigma^\infty \cap u_\alpha\Sigma^\infty = R_\alpha$. Hence, $\Delta_-^\infty \cap u_\alpha\Delta_-^\infty = F_\alpha$.

We define the group $U \leq \mathbf{R_u}(\mathbf{P}^+)(K_S)$ to be generated by the finite set of u_α as above. As it will be useful in Section 3.7, we also choose our u_α to fix the point e. This can always be arranged by replacing the u_α with conjugates by elements of $\mathbf{A}(K_S)$.

It is well known that every element of $\mathbf{R_u}(\mathbf{P}^+)(K_S)$ has order a power of p (see e.g. 4.1 [Bo]), so U is a p-group. Since the group $\mathbf{R_u}(\mathbf{P}^+)(K_S)$ is nilpotent, U is a finitely-generated nilpotent torsion group and thus is finite (see e.g. Theorem 9.17 [Mac]). We conclude that U is a finite p-group.

By abuse of notation, we shall denote the formal sum of chambers in Δ^∞_- simply by Δ^∞_-. Now we form the $(k(\mathbf{H},S)-1)$-chain $\sum_{u\in U} u\Delta^\infty_-$.

Properties of the chain in the boundary. In the remainder of this section, we will show that $\sum_{u\in U} u\Delta^\infty_-$ is a cycle describing a simplicial decomposition of $U\Delta^\infty_- = \bigcup_{u\in U} u\Delta^\infty_-$.

Lemma 3.7. *If $u \in U$ is nontrivial and $\mathfrak{C}^\infty \subseteq \Delta^\infty_-$ is a chamber, then $u\mathfrak{C}^\infty \nsubseteq \Delta^\infty_-$.*

Proof. Suppose $u\mathfrak{C}^\infty \subseteq \Delta^\infty_-$. Then we have $\Pi^\infty_- \subseteq \mathfrak{C}^\infty \cap u\mathfrak{C}^\infty$ by the definition of Δ^∞_-. As the action of $\mathbf{H}(K_S)$ on X_∞ is type preserving, $u\Pi^\infty_- = \Pi^\infty_-$. This implies that $u \in \mathbf{P}^-(K_S) \cap \mathbf{R_u}(\mathbf{P}^+)(K_S) = 1$. □

Lemma 3.8. *The chain $\sum_{u\in U} u\Delta^\infty_-$ is a cycle over $\mathbb{F}_p$.*

Proof. Suppose that $u \in U$ is nontrivial and that $\Delta^\infty_- \cap u\Delta^\infty_-$ contains an interior point x of a maximal simplex of a geodesically continued face of Δ^∞_-, say F_α. We begin by verifying that u fixes F_α pointwise, and that $F_\alpha = \Delta^\infty_- \cap u\Delta^\infty_-$.

Indeed, u fixes pointwise a simplex of F_α that contains x, since u acts by type preserving simplicial automorphisms on X^∞. The antipodal point of x in Σ^∞ is contained in the boundary of a chamber of Σ^∞ containing Π^∞_+; we call this chamber $\mathfrak{C}^\infty$. As in the comment immediately following proof of Lemma 3.6, we see that $\mathfrak{C}^\infty$ is fixed by $u \in \mathbf{R_u}(\mathbf{P}^+)(K_S)$.

The hemisphere R_α is the convex hull spanned by the simplex of F_α that contains x and the chamber $\mathfrak{C}^\infty$. Therefore, u fixes every point in $R_\alpha \supseteq F_\alpha$.

For the remaining claim that $F_\alpha = \Delta^\infty_- \cap u\Delta^\infty_-$: If there was a point $y \in \Delta^\infty_- \cap u\Delta^\infty_-$ outside of R_α, then u would have to fix y since R_α is fixed pointwise by u and the action is by isometries. Hence, u fixes pointwise the convex hull of R_α and y. But that is all of Σ^∞, and any $u \in \mathbf{R_u}(\mathbf{P}^+)(K_S)$ fixing Σ^∞ pointwise is the identity. So we have verified our claims.

We are now prepared to show that the homological boundary of $\sum_{u\in U} u\Delta^\infty_-$ is 0 modulo p. Applying the boundary homomorphism yields:

$$\partial\Big(\sum_{u\in U} u\Delta^\infty_-\Big) = \sum_{u\in U} \partial(u\Delta^\infty_-) = \sum_{u\in U}\sum_{\alpha\in A} uF_\alpha$$

where, again stretching notation slightly, F_α denotes the formal sum of all simplices in the geodesically continued face with the orientation induced from the orientation on Δ^∞_-.

The claims we verified above show that, for $u, v \in U$ and all $\alpha \in A$, either $uF_\alpha \cap vF_\alpha$ is contained in the topological boundary of uF_α or alternatively, uF_α and vF_α are equal as chains. Thus, we choose a complete set $\{f_1, f_2, \ldots, f_n\}$ of representatives for the chains in $\{uF_\alpha\}_{u\in U, \alpha\in A}$ so that

$$\sum_{u\in U}\sum_{\alpha\in A} uF_\alpha = \sum_{i=1}^{n} |U_i| f_i$$

where $U_i \leq U$ is the stabilizer of f_i. Since U is a finite p-group, $|U_i|$ is a power of p. Moreover, since each F_α is stabilized by a nontrivial $u_\alpha \in U$, each group U_i is nontrivial. Therefore,

$$\partial\Big(\sum_{u\in U} u\varDelta_{-}^{\infty}\Big) \equiv 0 \pmod{p}$$

□

Observation. By the preceding lemmas, $U\varDelta_{-}^{\infty}$ represents a class in the homology group $\widetilde{\mathrm{H}}_{k(\mathbf{H},S)-1}(U\varDelta_{-}^{\infty}\,;\,\mathbb{F}_p)$.

3.6. A line of communication from infinity to $X(r_0)$

In the next section, we will build cycles in $X(r_0)$ by transferring the topological data from $U\varDelta_{-}^{\infty}$ into $X(r_0)$ by method of "casting shadows" of $U\varDelta_{-}^{\infty}$ on $\mathbf{R_u}(\mathbf{P}^+)(K_S)V$. For the shadow to contain the same topological data as $U\varDelta_{-}^{\infty}$, it is important, for example, to have the shadow of $\varDelta_{-}^{\infty}$ in V be compact. The purpose of this section is to establish that fact, although we state this problem below using different language.

Recall that $\varSigma^\infty$ can be regarded as the space of all geodesic rays in $\varSigma$ based at e. We let $V^\infty \subseteq \varSigma^\infty$ be the set of all geodesic rays contained in V emanating from e. Note that V^∞ is an equatorial sphere in $\varSigma^\infty$.

We call a point in $\varSigma^\infty$ *rational* if it is represented by a geodesic ray based at e that passes through another (and hence infinitely many) points of $\mathbf{A}(K_S)e$. Let $\varSigma^\infty_{\mathbb{Q}}$ denote the set of rational points in $\varSigma^\infty$. Since $\mathbf{A}(K_S)$ acts on $\varSigma$ as a lattice of translations of full rank $k(\mathbf{H}, S) = \dim(\varSigma)$, the set $\varSigma^\infty_{\mathbb{Q}}$ is dense in $\varSigma^\infty$.

Similarly, we let $V^\infty_{\mathbb{Q}}$ denote the set of those points in V^∞ that can be joined to e by a geodesic ray passing through infinitely many points of $\mathbf{A}(\mathcal{O}_S)e$. From our choice of V before the Example in Section 3.2, it is also clear that $V^\infty_{\mathbb{Q}}$ is dense in V^∞.

Lemma 3.9. *We have $V^\infty_{\mathbb{Q}} = V^\infty \cap \varSigma^\infty_{\mathbb{Q}}$.*

Proof. The action of $\mathbf{A}(\mathcal{O}_S)$ factors through the inclusion $\mathbf{A}(\mathcal{O}_S) \hookrightarrow \mathbf{A}(K_S)$. Since $\mathbf{A}(\mathcal{O}_S)$ acts on V as a lattice of maximum rank $k(\mathbf{H}, S) - 1 = \dim(V)$, the affine lattices $\mathbf{A}(\mathcal{O}_S)e$ and $V \cap \mathbf{A}(K_S)e$ are commensurable. Hence, they define identical rational structures at infinity. $\square$

The goal of this section is:

Lemma 3.10. *We have $\Delta_-^\infty \cap V^\infty = \emptyset$.*

Proof. We proceed by contradiction. So assume $\Delta_-^\infty \cap V^\infty \neq \emptyset$. Our first step will be to show that $\Delta_-^\infty \cap V_{\mathbb{Q}}^\infty \neq \emptyset$. There are two cases. First, V^∞ contains an interior point of Δ_-^∞. Then the intersection $\Delta_-^\infty \cap V^\infty$ is open in V^∞ and contains a rational point since these are dense in V^∞. That is $\Delta_-^\infty \cap V_{\mathbb{Q}}^\infty \neq \emptyset$. Second, if V^∞ does not contain an interior point of Δ_-^∞ then since V^∞ has codimension 1 in Σ^∞ and because Δ_-^∞ is spherically convex in Σ^∞, we deduce that V^∞ contains a boundary simplex of Δ_-^∞. Since the affine lattice $\mathbf{A}(K_S)e \subseteq \Sigma$ is commensurable to the affine lattice of vertices in the Euclidean Coxeter complex underlying the apartment Σ, rational points are dense in every simplex in Σ^∞. Therefore, $V^\infty \cap \left(\Sigma_{\mathbb{Q}}^\infty \cap \Delta_-^\infty\right) \neq \emptyset$. Using Lemma 3.9, we again find a point in $\Delta_-^\infty \cap V_{\mathbb{Q}}^\infty$.

Now choose $b \in \mathbf{A}(\mathcal{O}_S)$ such that $b^n e$ converges to a point $b^\infty \in \Delta_-^\infty \cap V_{\mathbb{Q}}^\infty$ as $n \to \infty$, and let $\mathbf{D} \leq \mathbf{A}$ be the Zariski closure of the cyclic group $\langle b \rangle$. After replacing b with a suitable power of b, we may assume that $\mathbf{D}$ is connected.

Recall that for each $v \in S$, the group $\mathbf{R_u}(\mathbf{P}^-)$ is contained in any minimal K_v-parabolic subgroup of $\mathbf{H}$ that is contained in $\mathbf{P}^-$. Therefore, $\mathbf{R_u}(\mathbf{P}^-)(K_S)$ fixes Δ_-^∞ pointwise and, consequently, fixes the point $b^\infty \in \Delta_-^\infty$. As in the proof of Lemma 3.4,

$$\begin{aligned}
\mathbf{R_u}(\mathbf{P}^-)(\mathcal{O}_S) &\leq \mathbf{R_u}(\mathbf{P}^-)(K_S) \\
&\leq \{\, g \in \mathbf{H}(K_S) \mid g b^\infty = b^\infty \,\} \\
&= \{\, g \in \mathbf{H}(K_S) \mid (b^{-n} g b^n)_{n \in \mathbb{N}} \text{ is precompact} \,\}
\end{aligned}$$

Therefore, for any $\gamma \in \mathbf{R_u}(\mathbf{P}^-)(\mathcal{O}_S)$, the sequence $(b^{-n}\gamma b^n)_{n \in \mathbb{N}} \subseteq \mathbf{H}(\mathcal{O}_S)$ is both discrete and precompact. Hence, it is finite. We conclude that

$$b^{-n}\gamma b^n = b^{-m}\gamma b^m$$

for distinct n and m. Now, γ centralizes b^{n-m}.

Let $\mathbf{D}_\gamma$ be the subgroup of $\mathbf{A}$ that is the Zariski closure of the group generated by b^{n-m}. Then, γ centralizes $\mathbf{D}_\gamma$.

Note that $\mathbf{D}_\gamma \leq \mathbf{D}$, and let

$$\varphi : \mathbf{D} \to \mathbf{D}/\mathbf{D}_\gamma$$

be the quotient map. Since $\varphi(\langle b\rangle)$ is finite, it is equal to its own Zariski closure. Thus,

$$\mathbf{D}/\mathbf{D}_\gamma = \varphi(\mathbf{D}) \subseteq \varphi(\langle b\rangle)$$

Using the finiteness of $\varphi(\langle b\rangle)$ again, we have that the dimensions of $\mathbf{D}_\gamma$ and $\mathbf{D}$ are equal. Therefore, $\mathbf{D}_\gamma = \mathbf{D}$ by the connectivity of $\mathbf{D}$. Thus, we have shown that $\mathbf{R_u}(\mathbf{P}^-)(\mathcal{O}_S)$ centralizes $\mathbf{D}$.

Since $b^n e \to b^\infty$, iterates of b define an unbounded sequence in $\mathbf{D}(K_v)$ for at least one $v \in S$. It follows that $\mathbf{D}$ contains a nontrivial K_v-split torus $\mathbf{D_d}$. Indeed, if $\mathbf{D}$ were K_v-anisotropic, then $\mathbf{D}(K_v)$ would be compact.

We denote the centralizer of $\mathbf{D_d}$ in $\mathbf{H}$ by $\mathbf{L}$. Therefore, $\mathbf{L}$ is a reductive group that contains $\mathbf{A}$ as a maximal torus (20.4 [Bo]).

We have shown that

$$\mathbf{R_u}(\mathbf{P}^-)(\mathcal{O}_S) \leq \mathbf{L}(K_v)$$

As $\mathbf{R_u}(\mathbf{P}^-)$ is K-isomorphic as a variety to affine space (see 21.20 [Bo]), $\mathbf{R_u}(\mathbf{P}^-)(\mathcal{O}_S)$ is Zariski dense in $\mathbf{R_u}(\mathbf{P}^-)$ (use 3.1.1.ii [Mar]). Thus,

$$\mathbf{R_u}(\mathbf{P}^-) \leq \mathbf{L}$$

It follows that if α is any root of $\mathbf{H}$ relative to $\mathbf{A}$ that corresponds to a root group $\mathbf{U}_{(\alpha)} \leq \mathbf{H}$ contained in $\mathbf{R_u}(\mathbf{P}^-)$, then α is also a root of $\mathbf{L}$ relative to $\mathbf{A}$. Thus, $-\alpha$ is a root of $\mathbf{L}$ relative to $\mathbf{A}$ as well, so $\mathbf{U}_{(-\alpha)} \leq \mathbf{L}$. Hence,

$$\mathbf{R_u}(\mathbf{P}^+) \leq \mathbf{L}$$

Since $\mathbf{R_u}(\mathbf{P}^+)$ and $\mathbf{R_u}(\mathbf{P}^-)$ generate $\mathbf{H}$ (14.21 (iii) [Bo]), we have $\mathbf{H} \leq \mathbf{L}$. That is $\mathbf{H} = \mathbf{L}$, so the center of $\mathbf{H}$ contains the infinite group generated by b. This is our contradiction. □

3.7. Cycle assembly in $X(r_0)$

Let $C_\Delta \subseteq \Sigma$ be the cone of all geodesic rays contained in Σ, based at e, and limiting to points in Δ_-^∞.

Recall our choice of $a_+ \in \mathbf{H}(K_S)$ as a translation of Σ such that

$$a_+^n e \to a_+^\infty \in \Pi_+^\infty$$

Recall also that $\Pi_+^\infty \subseteq \Sigma^\infty$ is the collection of antipodal points for points in $\Pi_-^\infty \subseteq \Sigma^\infty$ and that Δ_-^∞ is the union of chambers in Σ^∞ containing Π_-^∞.

Therefore, Lemma 3.10 implies for $n \geq 1$ that any geodesic ray emanating from $a_+^n e$ and limiting to Δ_-^∞ is separated by V. Hence, there is a well-defined geodesic projection toward $a_+^n e$ that gives rise to a homeomorphism

$$\Delta_-^\infty \longrightarrow V \cap a_+^n C_\Delta$$

Recall that we chose U to fix e. Thus, $a_+^n U a_+^{-n}$ fixes $a_+^n e$. It follows that for all $u \in U$, there are well-defined geodesic projections toward $a_+^n e$ that give rise to homeomorphisms

$$\pi_n^u \colon a_+^n u \Delta_-^\infty = a_+^n u a_+^{-n} \Delta_-^\infty \longrightarrow a_+^n u a_+^{-n}(V \cap a_+^n C_\Delta)$$

We claim that these homeomorphisms piece together to give a surjection

$$\pi_n : a_+^n U \Delta_-^\infty \longrightarrow a_+^n U a_+^{-n}(V \cap a_+^n C_\Delta)$$

Recall that we showed in the first step of the proof for Lemma 3.8 that any $u \in U$ fixes $\Delta_-^\infty \cap u\Delta_-^\infty$ pointwise. Thus, $a_+^n u a_+^{-n}$ fixes $a_+^n \Delta_-^\infty \cap a_+^n u \Delta_-^\infty$ pointwise. Since $a_+^n u a_+^{-n}$ fixes $a_+^n e$ as well, it fixes all geodesic rays from $a_+^n \Delta_-^\infty \cap a_+^n u \Delta_-^\infty$ to $a_+^n e$. Hence, the geodesic projections π_n^1 and π_n^u agree on $a_+^n \Delta_-^\infty \cap a_+^n u \Delta_-^\infty$. Since we can write any intersection $a_+^n u \Delta_-^\infty \cap a_+^n u' \Delta_-^\infty$ as $a_+^n u a_+^{-n}(a_+^n \Delta_-^\infty \cap a_+^n u^{-1} u' \Delta_-^\infty)$ it follows that the geodesic projections π_n^u and $\pi_n^{u'}$ agree on $a_+^n u \Delta_-^\infty \cap a_+^n u' \Delta_-^\infty$ for any $u, u' \in U$, and thus π_n is well-defined.

Note that π_n is continuous since all maps π_n^u are continuous. Also note that the image of π_n is contained in $\mathbf{R_u}(\mathbf{P}^+)(K_S)V \subseteq X(r_0)$. The collection of $\sum_{u \in U} \pi_n\big(a_+^n u \Delta_-^\infty\big)$ are the cycles we have been searching for throughout this paper.

Lemma 3.11. *There is a point $s \in \Sigma$, a chamber $\mathfrak{s} \subseteq \Sigma$, and a sector $\mathfrak{S} \subseteq C_\Delta$ such that:*

(i) $s \in \mathfrak{s} \subseteq \mathfrak{S}$; and
(ii) For each nontrivial $u \in U$,

$$\mathfrak{S} \cap \varrho_{\Sigma,\mathfrak{s}}(uC_\Delta) = \emptyset$$

where $\varrho_{\Sigma,\mathfrak{s}} : X \to \Sigma$ is the building retraction for the pair $(\Sigma, \mathfrak{s})$.

Proof. Let $\mathfrak{S}' \subseteq C_\Delta$ and $\mathfrak{T} \subseteq X$ be sectors that do not contain a common subsector. Consider an apartment $\Sigma_* \subseteq X$ that contains disjoint subsectors $\mathfrak{S}_0 \subseteq \mathfrak{S}'$ and $\mathfrak{T}_0 \subseteq \mathfrak{T}$. For any chamber $\mathfrak{c} \subseteq \mathfrak{S}_0$, the retraction $\varrho_{\Sigma,\mathfrak{c}}$ restricts to an isometry from Σ_* to Σ that fixes $\mathfrak{S}_0$ pointwise. Thus, we have

$$\mathfrak{S}_0 \cap \varrho_{\Sigma,\mathfrak{c}}(\mathfrak{T}_0) = \emptyset$$

Choose $D \geq 0$ such that $\mathfrak{T}$ is contained within the closed metric D-neighborhood of $\mathfrak{T}_0$. Now choose $\mathfrak{S} \subseteq \mathfrak{S}_0$ such that the closed metric D-neighborhood of $\mathfrak{S}$ in Σ is completely contained within $\mathfrak{S}_0$. Then for any two chambers $\mathfrak{s} \subseteq \mathfrak{S}$ and $\mathfrak{t} \subseteq \mathfrak{T}$, the distance from $\varrho_{\Sigma,\mathfrak{s}}(\mathfrak{t})$ to $\varrho_{\Sigma,\mathfrak{s}}(\mathfrak{T}_0)$ is at most D since $\varrho_{\Sigma,\mathfrak{s}}$ does not increase distances. As the distance from $\varrho_{\Sigma,\mathfrak{s}}(\mathfrak{T}_{\mathfrak{o}})$ to $\mathfrak{S}$ is at least D, we find

$$\mathfrak{S} \cap \varrho_{\Sigma,\mathfrak{s}}(\mathfrak{T}) = \emptyset$$

By Lemma 3.7, uC_Δ can be covered by finitely many $\mathfrak{T}$ as above for any nontrivial $u \in U$. Thus, we can assume, after perhaps passing to a subsector of $\mathfrak{S}$, that $\mathfrak{S} \cap \varrho_{\Sigma,\mathfrak{s}}(\mathfrak{T}) = \emptyset$ for all such u and $\mathfrak{T}$. Hence, the lemma is satisfied for any choice of $s \in \mathfrak{s}$. $\square$

We fix s, $\mathfrak{s}$, and $\mathfrak{S}$ as above, and for every $n \in \mathbb{N}$ we let

$$\phi_n : \Sigma - \{a_+^n s\} \to \Sigma^\infty$$

be the visual projection to the boundary from the point $a_+^n s$.

Lemma 3.12. *For every $r \geq r_0$, the inclusion $X(r_0) \hookrightarrow X(r)$ induces a nontrivial homomorphism*

$$\widetilde{\mathrm{H}}_{k(\mathbf{H},S)-1}(X(r_0)\,;\mathbb{F}_p) \longrightarrow \widetilde{\mathrm{H}}_{k(\mathbf{H},S)-1}(X(r)\,;\mathbb{F}_p)$$

Proof. Choose $n \in \mathbb{N}$ such that V separates $a_+^n\mathfrak{S}$ into a compact component (containing s) and a noncompact component, and such that

$$a_+^n e \notin X\big(r + d(e,s)\big)$$

The latter condition can be arranged by Lemma 3.5, and it implies that

$$\varrho^{-1}_{\Sigma,\,a_+^n\mathfrak{s}}(a_+^n s) = \{a_+^n s\} \nsubseteq X(r)$$

where $\varrho_{\Sigma,\,a_+^n\mathfrak{s}}$ is the retraction corresponding to the pair $(\Sigma,\,a_+^n\mathfrak{s})$ Therefore, the following composition is well defined:

$$a_+^n U \Delta_-^\infty \to X(r_0) \hookrightarrow X(r) \to \Sigma - \{a_+^n s\} \to \Sigma^\infty$$

where the map on the left is π_n, the map second from the right is $\varrho_{\Sigma,\,a_+^n\mathfrak{s}}$, and the map on the far right is ϕ_n.

Since $\varrho_{\Sigma,\,a_+^n\mathfrak{s}}$ is simply $\varrho_{\Sigma,\mathfrak{s}}$ conjugated by a_+^n, Lemma 3.11 implies that there is an open neighborhood of $\phi_n(V \cap a_+^n\mathfrak{S}) \subseteq \Delta_-^\infty$ that has 1-point pre-images of points under the above composition. Hence, using excision—as in determining degrees of maps between spheres (see e.g. Proposition 2.30 of [Ha])—one sees that the image of the homology class represented by $a_+^n U \Delta_-^\infty$ is nontrivial under the homomorphism induced by the above composition

$$\widetilde{\mathrm{H}}_{k(\mathbf{H},S)-1}(a_+^n U \Delta_-^\infty\,;\mathbb{F}_p) \longrightarrow \widetilde{\mathrm{H}}_{k(\mathbf{H},S)-1}(\Sigma^\infty\,;\mathbb{F}_p)$$

Our result follows as the above homomorphism factors through

$$\widetilde{\mathrm{H}}_{k(\mathbf{H},S)-1}(X(r_0)\,;\mathbb{F}_p) \longrightarrow \widetilde{\mathrm{H}}_{k(\mathbf{H},S)-1}(X(r)\,;\mathbb{F}_p)$$

showing that the class represented by $\sum_{u\in U} \pi_n\big(a_+^n u \Delta_-^\infty\big)$ is nontrivial in the homology of $X(r)$. $\square$

Our proof of Proposition 3.1 is complete.

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