ON UNIPOTENT FLOWS IN $\mathcal{H}(1,1)$

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Abstract. We study the action of the horocycle flow on the moduli space of abelian differentials in genus two. In particular, we exhibit a classification of a specific class of probability measures that are invariant and ergodic under the horocycle flow on the stratum $\mathcal{H}(1,1)$.

1. Introduction

We begin by briefly describing translation surfaces in genus two and providing examples of surfaces which are contained in the support of the measures of interest. Then we recall from [C] the description of the submanifolds on which our measures are supported. An alternate description is given by McMullen [Mc1]. For a detailed exposition of translation surfaces and a historical survey of salient results, we refer the reader to Masur-Tabachnikov [Ma-T] or Masur-Hubert-Schmidt-Zorich [M-H-S-Z]. There are numerous sources of reference for Ratner's measure classification theorem for homogeneous spaces; some of these include the following works of Ratner, Margulis-Tomanov, and Morris: [Ra 1], [Ra 2], [Ra 3], [Ra 4], [M-T], and [Mo].

1.1. **Closed submanifolds in $\mathcal{H}(1,1)$.** The moduli space of translation surfaces of genus g is stratified according to the number and order of conical singularities of the surfaces. The Riemann-Roch theorem implies that if an abelian differential on a genus g surface has zeros of orders $m_1, \cdots, m_k$, then $\{m_1, \cdots, m_k\}$ is a partition of $2g-2$. That is, $\sum_{i=1}^{k} m_i = 2g-2$. If we let $\alpha = \{m_i\}_{i=1}^{k}$, then we have

Definition 1. *$\mathcal{H}(\alpha)$ is the stratum of the moduli space of genus g translation surfaces consisting of those surfaces with abelian differentials having k zeros of orders m_i.*

Example. In genus two, there are two strata, $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$. Surfaces in $\mathcal{H}(2)$ have one singularity of order 2 or total angle 6π. An example of a surface in $\mathcal{H}(2)$ is the regular octagon with parallel sides identified. Other examples include the L-tables described by Calta and McMullen. ([C], [Mc1]) On the other hand, surfaces in $\mathcal{H}(1,1)$ have two simple singularities, each of total angle 4π. The regular decagon with parallel sides identified is in this stratum, as are the Z-tables described by Calta and McMullen. ([C], [Mc4]) Our classification theorems pertain to the stratum $\mathcal{H}(1,1)$.

The group $SL(2,\mathbb{R})$ acts on the moduli space of translation surfaces and preserves each stratum $\mathcal{H}(\alpha)$.

Definition 2. *A translation surface S is said to be a* lattice surface *if* $\mathrm{Stab}_{SL(2,\mathbb{R})}(S)$ *is a lattice in $SL(2,\mathbb{R})$.*

Example. A theorem of Veech [V] shows that the regular decagon is a lattice surface. (In fact, the same theorem implies that any regular $2n$-gon with parallel sides identified is a lattice surface.)

We now recall a definition from [C] and describe the submanifolds of interest.

A direction v on a translation surface is said to *completely periodic* if in the direction v, the surface decomposes as a union of cylinders of closed, parallel trajectories bounded by saddle connections. We say that a surface is completely periodic if any direction in which there exists a closed saddle connection is completely periodic. A theorem of Veech [V] implies that any lattice surface is completely periodic, although the converse is false and in fact there are counterexamples in $\mathcal{H}(1,1)$. (See [C] for a concrete examples.)

In genus two, the holonomy field of a completely periodic surface is quadratic [C]. We will use this fact in the description of our submanifolds.

Given a surface $S \in \mathcal{H}(1,1)$ and a cylinder decomposition of S, let w_i, h_i and t_i denote the width, height, and twist parameters of the cylinders. Number the cylinders so that $w_3 = w_2 + w_1$ and define $s_i = h_i + h_3$ and $\tau_i = t_i + t_3$ for $i = 1, 2$. (See [C] for details.) We have the following result:

Theorem 1. *Let $\mathscr{L}_m$ be the set of completely periodic surfaces in $\mathcal{H}(1,1)$ that can be rescaled so that each surface S has a cylinder decomposition with parameters $w_i, s_i, \tau_i \in \mathbb{Q}(\sqrt{d})$ that satisfy:*

$$w_1\bar{s_1} + w_2\bar{s_2} = 0$$
$$w_1\bar{\tau_1} + w_2\bar{\tau_2} = 0$$
$$w_1 s_1 + w_2 s_2 = m.$$

(Here, $^-$ denotes conjugation in $\mathbb{Q}(\sqrt{d})$ and $m \in \mathbb{Q}(\sqrt{d})$. Note that m is the area of S.)

Then $\mathscr{L}_m$ is a closed, $SL(2,\mathbb{R})$-invariant submanifold of $\mathcal{H}(1,1)$. Any primitive lattice surface in $\mathcal{H}(1,1)$ is contained in some $\mathscr{L}_m$.

Note. In the language of McMullen [Mc1], the $\mathscr{L}_m$ are spaces of eigenforms of fixed discriminant.

Since the regular decagon is a lattice surface, it is contained in a submanifold $\mathscr{L}_m$. It should be noted here as well that McMullen showed in [Mc3] that the regular decagon is the only primitive lattice surface in $\mathcal{H}(1,1)$.

In addition to the action of $SL(2,\mathbb{R})$, there is a local pseudo-action of $\mathbb{R}^2$ on each submanifold $\mathscr{L}_m$. Given a surface $S \in \mathscr{L}_m$, we can define a new surface vS for sufficiently small vectors $v \in \mathbb{R}^2$. S can be realized as a union of polygons in $\mathbb{R}^2$ glued along parallel sides; under this identification, there are two equivalence classes of vertices of the polygons, $[p]$ and $[q]$, which give rise to the two singularities of S. Choose an equivalence class $[p]$ and add to each vertex in $[p]$ the same vector v. Then vS is the translation surface realized as the union of polygons with vertices $p+v$ and q, and edges connecting $p+v$ and q for every pair p and q which were connected by an edge in the polygonal decomposition of S. (See [C] for details.) This construction changes the relative homology of S while fixing the absolute homology.

Of particular interest for us in this paper will be translating singularities in $\mathscr{L}_m$ by elements of the group of horizontal vectors $X = \{(x,0) \in \mathbb{R}^2\}$. Any surface $S \in \mathscr{L}_m$ is associated with a maximal open interval $I_S \subseteq X$ such that xS is a well defined surface in $\mathcal{H}(1,1)$ – and hence in $\mathscr{L}_m$ – for any $x \in I_S$. We let $\overline{I_S} \subseteq X$ be the closure of I_S, and we call any vector $x \in \overline{I_S} \setminus I_S$ a *horizontal saddle connection* on S.

For example, it is a straightforward task to construct a Z-table $S \in \mathscr{L}_m$ such that $wS \in \mathscr{L}_m$ for all $w \in X$ with $\|w\| < \ell$ for some fixed $\ell > 0$, but such that $x \in X$ is a horizontal saddle connection between distinct singularities on S with $\|x\| = \ell$. Thus, $xS \notin \mathcal{H}(1,1)$ as x has the effect of identifying the two distinct singularities on S.

With S and x as in the above paragraph, x is a horizontal saddle connection on S. Note that through this construction, we could have that xS is an L-table and thus is contained in $\mathcal{H}(2)$. Alternatively, xS could be a table made of two squares identified on a single vertex and such that the sides of each individual square are identified as to form a torus. That is, xS is two tori joined at a single point.

Remark. Although the group $G = SL(2,\mathbb{R})$ acts on each $\mathscr{L}_m$, there is only a local pseudo-action by $\mathbb{R}^2$, and so we do not obtain an action by the group $G \ltimes \mathbb{R}^2$. However, for any point $S \in \mathscr{L}_m$, there is a neighborhood of the identity $\mathcal{O} \subseteq G \ltimes \mathbb{R}^2$ such that the map $(g,v) \mapsto$

$v(gS)$ defined on $\mathcal{O}$ is a homeomorphism. Thus, there is a local pseudo-action of $G \ltimes \mathbb{R}^2$ on $\mathscr{L}_m$.

1.2. **Statement of the main result.** Let $U \leq SL(2,\mathbb{R})$ be the subgroup of upper-triangular unipotent matrices.

Definition 3. *A measure μ on $\mathscr{L}_m$ is* horizontally invariant *if the space*

$$\mathscr{L}_m^X = \{\, M \in \mathscr{L}_m \mid xM \in \mathcal{H}(1,1) \text{ for all } x \in X \,\}$$

is conull, and if μ is X-invariant.

Theorem 2. *Let μ be an ergodic U-invariant Borel probability measure supported on some $\mathscr{L}_m$. Further, assume that either μ is invariant under the diagonal subgroup of $SL(2,\mathbb{R})$, or that μ is not horizontally invariant, or that μ-almost every $M \in \mathscr{L}_m$ contains no horizontal saddle connections. Then either*

(i) μ is the unique $SL(2,\mathbb{R})$-invariant, ergodic probability measure with Supp$(\mu) = \mathscr{L}_m$,
(ii) μ is arclength on the U-orbit of a U-periodic surface,
(iii) there is an $x \in X$ such that $x_*\mu = \nu_{10}$ where ν_{10} is Haar measure on the closed $SL(2,\mathbb{R})$-orbit of the regular decagon, or
(iv) there is an $x \in X$ such that x is a horizontal saddle connection for every surface in the support of μ. Furthermore, μ is invariant under the natural action of $x^{-1}SL(2,\mathbb{R})x$ on the support of μ.

Remark. If in case (iv) there is some $S \in \mathrm{Supp}(\mu)$ with $xS \in \mathcal{H}(2)$, then $x_*\mu$ is an ergodic, $SL(2,\mathbb{R})$-invariant measure on $\mathcal{H}(2)$. It follows from McMullen's Theorem 1.5 of [Mc2] that $x_*\mu$ is Haar measure on a lattice surface in $\mathcal{H}(2)$.

If there is some $S \in \mathrm{Supp}(\mu)$ such that xS is two tori joined at a point, then $x_*\mu$ is an ergodic, $SL(2,\mathbb{R})$-invariant measure on a product of two moduli spaces of a torus. As this space is homogeneous, Ratner's measure classification theorem applies.

1.3. **Basis of the proof.** The proof utilizes a framework of results that includes an analysis of the structure of moduli space in genus two, the ideas involved in the proof of Ratner's measure classification theorem, and the proof techniques of an analog of Ratner's theorem for spaces of branched covers of lattice surfaces given by Eskin-Marklof-Morris [E-M-M].

Many of the techniques and ideas used in the proof of Theorem 2 are similar to those used in the proof of Ratner's theorems, and to those used in the Eskin-Marklof-Morris measure classification theorem for unipotent flows on spaces of branched covers of lattice surfaces. Examples of these techniques and ideas include the notion of polynomial divergence in unipotent flows, measurement of the directions of transverse divergence under unipotent flows, and entropy inequalities for actions of semisimple elements. In fact, the loose outline for the proof of our theorem is modeled on the proof of Ratner's theorem provided by Margulis-Tomanov [M-T], as in [E-M-M].

Furthermore, in independent work, McMullen and Calta located and explicitly described the submanifolds $\mathscr{L}_m$ ([C], [Mc1]) and McMullen classified the ergodic, $SL(2,\mathbb{R})$-invariant ergodic probability measures on the moduli space in genus two [Mc2]. We make essential use of this classification as well.

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2. Horizontal saddle connections and the support of μ

We fix an ergodic U-invariant probability measure on some $\mathscr{L}_m$, and denote it by μ.

2.1. Well-defined horizontal translations.

Recall that $X \leq \mathbb{R}^2$ is the subgroup of $\mathbb{R}^2$ of horizontal vectors. It will often be convenient for us to identify a vector in X with the real value of its nontrivial coordinate.

In what follows, for any $M \in \mathscr{L}_m$ and any interval $Z \subseteq X$ with $vM \in \mathscr{L}_m$ for all $v \in Z$, we denote $\cup_{v\in Z} vM$ as ZM.

In particular, for any submanifold $\mathscr{L}_m$ and any $s > 0$, we define

$$\mathrm{HC}(s) = \{\, M \in \mathscr{L}_m \mid (-s,s)M \subseteq \mathscr{L}_m \text{ and } [-s,s]M \nsubseteq \mathcal{H}(1,1)\,\}$$

where a real number $t \in (-s,s)$ is identified with $(t,0) \in X$, as described above.

Note that since U and X commute, each $\mathrm{HC}(s)$ is U-invariant, and that

$$\mathscr{L}_m = \mathscr{L}_m^X \coprod \Big(\coprod_{s>0} \mathrm{HC}(s) \Big)$$

Since $\mathscr{L}_m^X$ is also U-invariant, it follows from the ergodicity of μ that either $\mathscr{L}_m^X$ is conull or $\mathrm{Supp}(\mu) \subseteq \mathrm{HC}(s)$ for some $s > 0$.

Lemma 1. *There is an open interval $X_0 \subseteq X$, and a conull set $\mathscr{C}_m \subseteq \mathscr{L}_m$ such that $X_0 M \subseteq \mathscr{L}_m$ for any $M \in \mathscr{C}_m$. Furthermore, we may assume that $X_0 = -X_0$.*

Proof. If $\mathscr{L}_m^X$ is conull, let $X_0 = X$ and $\mathscr{C}_m = \mathscr{L}_m^X$. On the other hand, if $\mathrm{Supp}(\mu) \subseteq \mathrm{HC}(s)$, let $X_0 = (-s, s)$ and $\mathscr{C}_m = \mathrm{HC}(s)$. $\square$

3. Transverse divergence

We denote the set of vertical vectors in the plane as:

$$Y = \{\, (0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R} \,\}.$$

It will be convenient to write $\mathbb{R}^2$ as XY to emphasize the difference between the action of a vertical vector and a horizontal vector. The difference between X and Y is significant because although X is U-invariant, Y is not.

Now $SL(2,\mathbb{R})$ acts on XY so that we may form the semidirect product $SL(2,\mathbb{R}) \ltimes XY$, a 5-dimensional Lie group. We embed $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ into $SL(3,\mathbb{R})$:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{bmatrix}$$

We multiply elements of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ by multiplying their images in $SL(3,\mathbb{R})$.

For any $M \in \mathscr{L}_m$, there is an open neighborhood of the identity $\mathcal{O}_M \subseteq SL(2,\mathbb{R}) \ltimes XY$ such that the map

$$(g, xy) \mapsto xy(gM)$$

defines a homeomorphism of $\mathcal{O}_M$ onto its image in $\mathscr{L}_m$.

For any $s \in \mathbb{R}$, we let $x^s = (s, 0) \in \mathbb{R}^2$ and

$$a^s = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}$$

We let $B \leq SL(2,\mathbb{R})$ be the subgroup of upper-triangular matrices and we define $\mathrm{Stab}_{BX_0}(\mu)$ to be the set of all $h \in BX_0$ such that hM is well-defined for almost every $M \in \mathscr{L}_m$ and such that $h_*\mu = \mu$. Note that $\mathrm{Stab}_{BX_0}(\mu)$ is merely a set, and may not be a group if $X_0 \neq X$.

The following lemma is a key tool used in the proof for Theorem 2. It is essentially Proposition 4.4 from [E-M-M].

First, we establish some notation. Let

$$g_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}$$

and

$$f(g_k,t_k)=\frac{t_ka_k-b_k}{d_k-t_kc_k}.$$

Lemma 2. *Given $\varepsilon>0$, there exists a compact set $\Omega_\varepsilon\subseteq\mathscr{L}_m$ with $\mu(\Omega_\varepsilon)>1-\varepsilon$ and some $0<\delta<1$ for which $x^\delta\in X_0$, such that if*

(i) $\{M_k\}$ is a convergent sequence in Ω_ε with $\lim_{k\to\infty}M_k=M'$,
(ii) $(g_k,x_ky_k)M_k=M'$ for $(g_k,x_ky_k)\in SL(2,\mathbb{R})\ltimes XY$ with $\lim_{k\to\infty}(g_k,x_ky_k)=1$,
(iii) either $|c_k|\le|y_k|$ for all k or $|y_k|\le|c_k|$ for all k,
(iv) $c_ky_k\neq 0$ for all k, and
(v) $t_k=\min\{\frac{\delta}{|c_k|},\frac{\delta}{|y_k|}\}$,

then $\mathrm{Stab}_{BX_0}(\mu)^\circ$ contains an element of AX_0. More precisely, if $|y_k|\le|c_k|$ for all k then $\mathrm{Stab}_{BX_0}(\mu)^\circ$ contains an element of

$$a^{-\log(1\pm\delta)}x^{[\frac{-\delta}{1-\delta},\frac{\delta}{1-\delta}]}$$

and if $|c_k|\le|y_k|$ for all k then $\mathrm{Stab}_{BX_0}(\mu)^\circ$ contains an element of

$$a^{[-\log(1+\delta),-\log(1-\delta)]}\left[x^{[\frac{-\delta}{1-\delta},\frac{-\delta}{1+\delta}]}\cup x^{[\frac{\delta}{1+\delta},\frac{\delta}{1-\delta}]}\right]$$

Proof. As the proof is essentially that of [E-M-M] Proposition 4.4, we will only sketch the argument and refer to [E-M-M] for the details.

Let Ω_ϵ be a compact uniformly generic set for the action of U on $\mathscr{L}_m^X$. There is a compact set $K\subseteq\mathscr{L}_m$ with measure arbitrarily close to 1 such that $K\cap(bx)K=\emptyset$ for any $bx\in BX_0-\mathrm{Stab}_{BX_0}(\mu)$.

Let

$$u^t=\begin{pmatrix}1&t\\0&1\end{pmatrix}$$

Since $M'\in\Omega_\varepsilon$,

$$d(u^tM',K)<\varepsilon \tag{1}$$

for most $t\in[-t_k,t_k]$.

Let $f(g,t)=\frac{ta-b}{d-tc}$. Since the derivative of f is bounded on compact intervals, it follows that

$$d(u^{f(g_k,t)}M_k,K)<\varepsilon \tag{2}$$

for most $t\in[-t_k,t_k]$ as well.

Recall that $\mathcal{O}_{M'}$ is an open neighborhood of the identity in $SL(2,\mathbb{R})XY$ which is homeomorphic to its image in $\mathscr{L}_m$ via the map $(g,xy)\to xygM'$. Now, there is also an open neighborhood, $\mathcal{O}$, of M' in $\mathcal{O}_{M'}M'$

and some $\delta > 0$ such that for any $N \in \mathcal{O}$ and any $v \in XY$ with $||v|| < \delta$ we have $vN \in \mathcal{O}_{M'}M'$. We may assume $M_k \in \mathcal{O}$ for all k.

We claim that after passing to a subsequence, $u^{f(g_k,t_k)}g_kx_ky_ku^{-t_k}$ converges to the element of $\mathrm{Stab}_{BX_0}(\mu)^\circ$ as described in the conclusion of the statement of this lemma. Call this limit $\beta_\delta \in BX_0$.

Now we prove the claim. Our goal is to compute $u^{f(g_k,t_k)}g_kx_ky_ku^{-t_k}$.

Since we assumed that $g_kx_ky_k \to 1$, we know that $a_k, d_k \to 1$ and $c_k, b_k, x_k, y_k \to 0$.

For convenience, we let $f(g_k, t_k) = f_k$.

First note that

$$g_kx_ky_k = \begin{bmatrix} a_k & b_k & 0 \\ c_k & d_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & x_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y_k \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_k & b_k & a_kx_k + b_ky_k \\ c_k & d_k & c_kx_k + d_ky_k \\ 0 & 0 & 1 \end{bmatrix}$$

Now note that

$$g_kx_ky_ku^{-t_k} = \begin{bmatrix} a_k & b_k & a_kx_k + b_ky_k \\ c_k & d_k & c_kx_k + d_ky_k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -t_k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_k & b_k - t_ka_k & a_kx_k + b_ky_k \\ c_k & d_k - t_kc_k & c_kx_k + d_ky_k \\ 0 & 0 & 1 \end{bmatrix}$$

Next, note that

$$u^{f(g_k,t_k)}g_kx_ky_ku^{-t_k} = \begin{bmatrix} 1 & f_k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_k & b_k - t_ka_k & a_kx_k + b_ky_k \\ c_k & d_k - t_kc_k & c_kx_k + d_ky_k \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_k + c_kf_k & (b_k - t_ka_k) + f_k(d_k - t_kc_k) & (a_kx_k + b_ky_k) + f_k(c_kx_k + d_ky_k) \\ c_k & d_k - t_kc_k & c_kx_k + d_ky_k \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_k + c_kf_k & 0 & (a_kx_k + b_ky_k) + f_k(c_kx_k + d_ky_k) \\ c_k & d_k - t_kc_k & c_kx_k + d_ky_k \\ 0 & 0 & 1 \end{bmatrix}$$

We have by definition that $t_k = \min\{\frac{\delta}{|c_k|}, \frac{\delta}{|y_k|}\}$.

Suppose that $|y_k| \leq |c_k|$. Then $t_k = \frac{\delta}{|c_k|}$ so $t_kc_k = \pm\delta$.

Thus, $d_k - t_k c_k = d_k \pm \delta$ and $d_k \pm \delta \to 1 \pm \delta$. So after passing to a subsequence, $a_k + c_k f_k \to 1/(1 \pm \delta)$ since our matrix has determinate equal to 1.

Since $c_k \to 0$, after projecting $\pi : SL(2,\mathbb{R})XY \to SL(2,\mathbb{R})$ we have $\pi(u^{f(g_k,t_k)} g_k x_k y_k u^{-t_k}) \to a^{-\log(1\pm\delta)}$.

As for the "vector-part" of the limit, note that $c_k x_k + d_k y_k \to 0$, so we are only left to determine the limit of $(a_k x_k + b_k y_k) + f_k(c_k x_k + d_k y_k)$

Since $x_k \to 0$ and $y_k \to 0$,

$$\begin{aligned}\lim[(a_k x_k + b_k y_k) + f_k(c_k x_k + d_k y_k)] &= \lim[f_k(c_k x_k + d_k y_k)] \\ &= \lim[\frac{(t_k a_k - b_k)(c_k x_k + d_k y_k)}{d_k - t_k c_k}] \\ &= \frac{\lim[(t_k a_k - b_k)(c_k x_k + d_k y_k)]}{\lim[d_k - t_k c_k]}\end{aligned}$$

Since $d_k \to 1$ and $t_k = \delta/|c_k|$, we have $\lim[d_k - t_k c_k] = 1 \pm \delta$.

Furthermore,

$$\begin{aligned}\lim[(t_k a_k - b_k)(c_k x_k + d_k y_k)] &= \lim[t_k a_k c_k x_k + t_k a_k d_k y_k - b_k c_k x_k - b_k d_k y_k] \\ &= \lim[\pm\delta a_k x_k + a_k d_k \delta \frac{y_k}{|c_k|} - b_k c_k x_k - b_k d_k y_k] \\ &= \lim[0 + \delta \frac{y_k}{|c_k|} - 0 - 0] \\ &= \delta \lim \frac{y_k}{|c_k|}\end{aligned}$$

Since $|y_k| \leq |c_k|$, the above limit is contained in the interval $[-\delta, \delta]$. Thus,

$$\lim[(a_k x_k + b_k y_k) + f_k(c_k x_k + d_k y_k)]$$

is contained in the interval

$$[\frac{-\delta}{1-\delta}, \frac{\delta}{1-\delta}]$$

Thus we have shown

$$\lim u^{f(g_k,t_k)} g_k x_k y_k u^{-t_k} \in a^{-\log(1\pm\delta)} x^{[\frac{-\delta}{1-\delta}, \frac{\delta}{1-\delta}]}$$

Now suppose that $|c_k| \leq |y_k|$. Thus $t_k = \frac{\delta}{|y_k|}$. That is $t_k y_k = \pm\delta$. As in the previous case, we wish to bound the quantity $\lim_{k\to\infty} d_k - t_k c_k$. Because $|c_k| \leq |y_k|$, we have that

$$-\delta \leq t_k c_k \leq \delta$$

Thus, since $d_k \to 1$, we have that $\lim_{k\to\infty}[d_k - t_k c_k] \in [1-\delta, 1+\delta]$. So $\lim \pi(u^{f(g_k,t_k)} g_k x_k y_k u^{-t_k})$ is contained in

$$a^{[-\log(1+\delta), -\log(1-\delta)]}$$

As before, $c_k x_k + d_k y_k \to 0$, and $a_k x_k + b_k y_k \to 0$, so

$$\begin{aligned}
\lim[(a_k x_k + b_k y_k) + f_k(c_k x_k + d_k y_k)] &= \lim[f_k(c_k x_k + d_k y_k)] \\
&= \lim[\frac{(t_k a_k - b_k)(c_k x_k + d_k y_k)}{d_k - t_k c_k}] \\
&= \frac{\lim[(t_k a_k - b_k)(c_k x_k + d_k y_k)]}{\lim[d_k - t_k c_k]}
\end{aligned}$$

The numerator equals

$$\lim[t_k a_k c_k x_k + t_k a_k d_k y_k - b_k c_k x_k - b_k d_k y_k]$$

Because $x_k, y_k, b_k \to 0$, and $c_k, d_k \to 1$, the last two terms in the expansion of the numerator tend to 0. We've already shown that $t_k c_k$ is bounded and since $a_k \to 1$ and $x_k \to 0$, the first term tends to 0. Finally, $t_k y_k = \pm\delta$ and since $a_k, d_k \to 1$, we have that the numerator is $\pm\delta$.

And as we've already shown, $\lim[d_k - t_k c_k] \in [1-\delta, 1+\delta]$. Altogether, we find that the limit of the x-coordinate of our vector lies in

$$\Big[\frac{-\delta}{1-\delta}, \frac{-\delta}{1+\delta}\Big] \cup \Big[\frac{\delta}{1+\delta}, \frac{\delta}{1-\delta}\Big]$$

This proves the claim.

Now, we have that

$$\begin{aligned}
u^{f(g_k,t_k)} M' &= u^{f(g_k,t_k)} g_k x_k y_k M_k \\
&= \big[u^{f(g_k,t_k)} g_k x_k y_k u^{-t_k}\big] u^{t_k} M_k
\end{aligned}$$

Therefore, $\lim_{k\to\infty} u^{f(g_k,t_k)} M' = \lim_{k\to\infty} \beta_\delta u^{t_k} M_k$. From the compactness of K it follows that $\beta_\delta K \cap K \neq \emptyset$ and thus $\beta_\delta \in \mathrm{Stab}_{BX_0}(\mu)$ whenever 1 and 2 are satisfied. Since $\mathrm{Stab}_{BX_0}(\mu)$ is closed and 1 and 2 hold on arbitrarily large subsets, we can let $\delta \to 0$ to see that $\beta_\delta \in \mathrm{Stab}_{BX_0}(\mu)$ and more precisely, that $\beta_\delta \in \mathrm{Stab}_{BX_0}(\mu)^\circ$.

□

4. Stabilizer is U: arclength measures

For any $M \in \mathscr{L}_m$, let $M_X \subseteq \mathscr{L}_m$ be the connected component of M of the space of all surfaces $xM \in \mathscr{L}_m$ for $x \in X$. If $M \in \mathscr{L}_m$, we call $BM_X \subseteq \mathscr{L}_m$ the *U-normalizer space of M.*

Note that each U-normalizer space is U invariant and that $\mathscr{L}_m$ is a disjoint union of U-normalizer spaces.

This next lemma is essentially Proposition 1.6.10 of [Mo].

Lemma 3. *Suppose that* $\mathrm{Stab}_{BX_0}(\mu)^\circ = U$. *Then μ is supported on a U-normalizer space of a single surface.*

Proof. If μ is supported on countably many U-normalizer spaces, then it assigns positive measure to at least one. By ergodicity, it would be supported on a single U-normalizer space. Thus, to prove our claim we can assume that μ is supported on uncountably many U-normalizer spaces, and then arrive at a contradiction.

Assume that μ is supported on uncountably many U-normalizer spaces. With Ω_ε as in Lemma 2,

$$\bigcup_{n\in\mathbb{N}} \Omega_{1/n}$$

is conull, so there must be a fixed n such that μ restricted to $\Omega_{1/n}$ is supported on uncountably many U-normalizer spaces.

Since uncountable sets contain a limit point, there is an $M' \in \Omega_{1/n}$ and a sequence of $M_k \in \Omega_{1/n}$ such that $M_k \to M'$ and $M_k \notin B(M')_X$. Since $M_k \to M'$, there must exist a sequence of elements $(g_k, x_k y_k) \in SL(2,\mathbb{R}) \ltimes XY$ such that $(g_k, x_k y_k)M_k = M'$ with $(g_k, x_k y_k) \to 1$.

Since $(g_k, x_k y_k) \notin BX$, we have for all k that either $c_k \neq 0$ or $y_k \neq 0$. Either way, we can pass to a subsequence and apply Lemma 2 to find a nontrivial element of $\mathrm{Stab}_{BX_0}(\mu)^\circ - U$. This contradicts our assumption that $\mathrm{Stab}_{BX_0}(\mu)^\circ = U$.

□

We will denote the identity component of the group of diagonal matrices in $SL(2,\mathbb{R})$ by A.

Lemma 4. *Suppose that* $\mathrm{Stab}_{BX_0}(\mu)^\circ = U$. *Then there exists some $a \in A$ and some $M' \in \mathscr{L}_m$ such that* $\mathrm{Supp}(\mu) = aUM'$.

Proof. Let BM_X be the U-normalizer space from Lemma 3 and note that the proof of Lemma 3.3 from [E-M-M] shows there is a U-invariant, Borel subset $\Omega_{AX} \subset BM_X$, such that

$$\mu(\Omega_{AX}) = 1$$

and

$$\Omega_{AX} \cap g\Omega_{AX} = \emptyset \text{ for all } g \in AX_0 - \mathrm{Stab}_{AX_0}(\mu)$$

Thus our claim is that $\mu|_{\Omega_{AX}}$ is supported on aUM' for a fixed a and M'. (Notice that aUM' is a U-orbit since a normalizes U, so $\mathrm{Supp}(\mu|_{\Omega_{AX}})$ certainly contains some set of the form aUM'.)

Suppose $a_1UM_1 \subseteq \mathrm{Supp}(\mu|_{\Omega_{AX}})$ for some $a_1 \in A$ and some $M_1 \in M_X$. We will show that $a_1UM_1 = \mathrm{Supp}(\mu|_{\Omega_{AX}})$ and thus will prove our claim. But first it will be helpful to show that $\mathrm{Supp}(\mu|_{\Omega_{AX}}) \subseteq BM_X$ can be thought of as being arbitrarily narrow in the M_X-direction.

The surface M_1 is contained in M_X. Let $I_\varepsilon^{M_1} \subseteq M_X$ be an interval of diameter $\varepsilon > 0$ that contains M_1. The set $BI_\varepsilon^{M_1}$ contains an open set in BM_X that contains a_1UM_1. As the latter set is contained in the support of μ, we have that $\mu(BI_\varepsilon^{M_1}) > 0$. Because A normalizes U, the set $BI_\varepsilon^{M_1}$ is U-invariant. It follows from ergodicity that $BI_\varepsilon^{M_1}$, and thus $\Omega_{AX} \cap BI_\varepsilon^{M_1}$, is conull.

Notice that $\varepsilon > 0$ in the above paragraph was arbitrary. We will assume that ε is sufficiently small depending on a_1. How it depends on a_1 will be explained below. If $a_1UM_1 \neq \mathrm{Supp}(\mu|_{\Omega_{AX}})$, then there is a U-orbit contained in $\mathrm{Supp}(\mu|_{\Omega_{AX}}) \cap BI_\varepsilon^{M_1}$ that is distinct from a_1UM_1. This orbit necessarily has the form a_2UM_2 for some $a_2 \in A$ and some $M_2 \in I_\varepsilon^{M_1}$. Thus, there is a $w \in X$ with norm less than ε such that $wM_1 = M_2$.

Recall that A normalizes X, so $a_1wa_1^{-1} \in X$. Furthermore, by choosing $\varepsilon > 0$ sufficiently small depending on a_1, we may assume that $a_1wa_1^{-1} \in X_0$.

Let

$$g = a_2wa_1^{-1}$$

Note that

$$g = a_2a_1^{-1}(a_1wa_1^{-1}) \in AX_0$$

Since w commutes with U, we also have that

$$g(a_1UM_1) = a_2UM_2$$

The final piece of information we need about g is that $g \neq 1$. This follows from the fact that the two U-orbits, a_1UM_1 and a_2UM_2, are distinct.

Altogether we have the following contradiction:

$$a_2UM_2 \subseteq g\mathrm{Supp}(\mu|_{\Omega_{AX}}) \cap \mathrm{Supp}(\mu|_{\Omega_{AX}}) \subseteq g\Omega_{AX} \cap \Omega_{AX} = \emptyset$$

The final equality in the line above follows from the definition of Ω_{AX} and the fact that $g \notin \mathrm{Stab}_{AX_0}(\mu)^\circ = 1$.

□

Lemma 5. *Suppose that* $\mathrm{Stab}_{BX_0}(\mu)^\circ = U$*. Then there is a* U*-periodic surface* $N \in \mathscr{L}_m$ *such that* $UN \subseteq \mathscr{L}_m$ *is homeomorphic to* S^1 *and* μ *is arc-length measure on* UN*.*

Proof. Let $N = aM'$ where a and M' are as in Lemma 4. Then μ is supported on the U-orbit of N. Because U stabilizes μ, the measure must descend from Haar measure on U. Since the measure is a probability measure, the U-orbit is a closed circle, and μ is arc-length. □

5. Stabilizer is not unipotent: Entropy

By Lemma 5, we may now assume that $\mathrm{Stab}_{BX_0}(\mu)^\circ$ properly contains U. In this section, we assume there is an element of $\mathrm{Stab}_{BX_0}(\mu)^\circ$ that is not unipotent.

There are three cases to be dealt with. First, $\mu(\mathscr{L}_m^X) = 1$ and μ is X-invariant. Second, $\mu(\mathscr{L}_m^X) = 1$ and μ is not X-invariant. Third, $\mu(\mathscr{L}_m^X) = 0$. But before we proceed with individual cases, we will show that almost every surface in $\mathscr{L}_m$ admits relative translations of singularities by arbitrarily long vertical vectors.

5.1. **Fibers of $\mathscr{L}_m$ and μ.** Fix a surface $M \in \mathscr{L}_m$ and let $M_{XY} \subseteq \mathscr{L}_m$ be the set of all surfaces in $\mathscr{L}_m$ that can be realized as $v_1 v_2 \cdots v_m M$ for some $v_1, v_2, ...v_m \in XY$.

Let

$$\Gamma = \{ \, g \in SL(2,\mathbb{R}) \mid gM_{XY} = M_{XY} \, \}$$

Lemma 6. Γ *is discrete.*

Proof. If Γ is not discrete then there is a sequence $\{\gamma_n\} \subseteq \Gamma - 1$ such that $\gamma_n \to 1$. Thus, $\gamma_n M \to M$.

Since $\gamma_n M \in M_{XY}$, we have $\gamma_n M = v_{1,n} v_{2,n} \cdots v_{m,n} M$ for some $v_{1,n}, v_{2,n}, ..., v_{m,n} \in XY$.

In a sufficiently small neighborhood of M, coordinates are given for $\mathscr{L}_d$ by the absolute and relative homology. Since the absolute homology of $v_{1,n} v_{2,n} \cdots v_{m,n} M$ and M agree, the absolute homology of $\gamma_n M$ and M agree which implies that $\gamma_n M = M$. But the set of all $g \in SL(2,\mathbb{R})$ such that $gM = M$ is discrete. Thus, the sequence $\{\gamma_n\}$ is bounded away from 1, a contradiction. □

The space $\mathscr{L}_m$ fibers over $SL(2,\mathbb{R})/\Gamma$ with fibers homeomorphic to M_{XY}. By the previous lemma, $SL(2,\mathbb{R})/\Gamma$ is a manifold so there is a measure μ_π on $SL(2,\mathbb{R})/\Gamma$ and a fiber measure $\mu_{M_{XY}}$ for every fiber M_{XY} such that μ is obtained by integrating the fiber measures over μ_π.

5.2. **Vertical translations are conull.** For any line through the origin of the plane, $\ell \in \mathbb{P}^1(\mathbb{R})$, we let

$$\mathscr{L}_m^\ell = \{ N \in \mathscr{L}_m \mid \ell N \subseteq \mathscr{L}_m \}$$

For any $g\Gamma \in SL(2,\mathbb{R})/\Gamma$, we define

$$\Sigma_{g\Gamma}^\ell = \{ x \in gM_{XY} \mid x \notin \mathscr{L}_m^\ell \}$$

so that $\mathscr{L}_m - \mathscr{L}_m^\ell$ equals

$$\bigcup_{g\Gamma \in G/\Gamma} \Sigma_{g\Gamma}^\ell$$

Lemma 7. *Given $g \in SL(2,\mathbb{R})$, and $\ell, w \in \mathbb{P}^1(\mathbb{R})$, the intersection $\Sigma_{g\Gamma}^\ell \cap \Sigma_{g\Gamma}^w$ is uncountable only if $\ell = w$.*

Proof. Suppose $\{N_\alpha\}_{\alpha \in \mathscr{A}} \subseteq \Sigma_{g\Gamma}^\ell \cap \Sigma_{g\Gamma}^w$ where $\mathscr{A}$ is uncountable.

For any $\alpha \in \mathscr{A}$, let $x_\alpha \in \ell$ and $y_\alpha \in w$ be saddle connection on N_α between distinct singularities. Since $\mathscr{A}$ is uncountable, the set of triples $(N_\alpha, x_\alpha, y_\alpha) \in gM_{XY} \times \ell \times w$ contains an accumulation point (N, x, y). We let $\{(N_i, x_i, y_i)\} \subseteq \{(N_\alpha, x_\alpha, y_\alpha)\}$ be a sequence that converges to (N, x, y)

Let $v_i \in \mathbb{R}^2$ be such that $v_i N_i = N$. Then $x - v_i$ and x_i are saddle connections on N_i, that each converge to x. Thus, we may assume that $x - v_i = x_i$ for all i. Hence, $v_i \in \ell$ for all i. Similarly, $v_i \in w$ for all i, so $\ell = w$. $\square$

The following lemma is essentially Lemma 5.4 from [E-M-M].

Lemma 8. *Let μ be an ergodic U-invariant measure on $\mathscr{L}_m$. Then $\mu(\mathscr{L}_m^Y) = 1$*

Proof. Suppose $\mu(\mathscr{L}_m^Y) < 1$. Then by Fubini's theorem, there is a set $E \subseteq SL(2,\mathbb{R})/\Gamma$ such that $\mu_\pi(E) > 0$ and $\mu_{gM_{XY}}(\Sigma_{g\Gamma}^Y) > 0$ for all $g\Gamma \in E$.

Let λ be Lebesgue measure on $U \cong \mathbb{R}$. By the pointwise ergodic theorem there is a $g_0\Gamma \in SL(2,\mathbb{R})/\Gamma$ and a set $U_0 \subseteq U$ such that $\lambda(U_0) > 0$ and $ug_0\Gamma \in E$ for all $u \in U_0$. Note that $\lambda(U_0) > 0$ implies that U_0 is uncountable.

It can be checked that $u\Sigma_{g_0\Gamma}^{u^{-1}Y} = \Sigma_{ug_0\Gamma}^Y$. This fact and the U-invariance of μ implies

$$\mu_{g_0 M_{XY}}(\Sigma_{g_0\Gamma}^{u^{-1}Y}) = \mu_{ug_0 M_{XY}}(u\Sigma_{g_0\Gamma}^{u^{-1}Y}) = \mu_{ug_0 M_{XY}}(\Sigma_{ug_0\Gamma}^Y) > 0$$

for all $u \in U_0$. In particular, if $\widehat{\mu} = \mu_{g_0 M_{XY}}$ and $\Sigma_{g_0\Gamma}^\ell = \Sigma^\ell$, then there are uncountably many $\ell \in \mathbb{P}^1(\mathbb{R})$ with $\widehat{\mu}(\Sigma^\ell) > 0$.

We form a graph with a vertex for each $\ell \in \mathbb{P}^1(\mathbb{R})$ with $\widehat{\mu}(\Sigma^\ell) > 0$ and an edge for each distinct pair $\ell, w \in \mathbb{P}^1(\mathbb{R})$ with $\widehat{\mu}(\Sigma^\ell \cap \Sigma^w) > 0$.

If a graph with uncountably many vertices has only countably many edges, then there are uncountably many isolated vertices. Thus, there must be uncountably many distinct pairs $\ell, w \in \mathbb{P}^1(\mathbb{R})$ with $\widehat{\mu}(\Sigma^\ell \cap \Sigma^w) > 0$, or else if $\mathcal{I}$ is the set of isolated vertices, then $\widehat{\mu}(\cup_{\ell \in \mathcal{I}} \Sigma^\ell) = \infty$, which would be a contradiction.

If $\ell, w \in \mathbb{P}^1(\mathbb{R})$ are distinct and $\widehat{\mu}(\Sigma^\ell \cap \Sigma^w) > 0$, then let $P(\ell, w) = \{ N \in \Sigma^\ell \cap \Sigma^w \mid \widehat{\mu}(\{N\}) > 0 \}$. Note that $P(\ell, w) \neq \emptyset$ since $\Sigma^\ell \cap \Sigma^w$ is countable.

Form a second graph with a vertex for each $P(\ell, w)$ and edges between $P(\ell, w)$ and $P(\ell', w')$ if $P(\ell, w) \cap P(\ell', w') \neq \emptyset$.

If $N \in g_0 M_{XY}$, then there are only countably many ℓ with $N \in \Sigma^\ell$ since N has a countable set of saddle connections. Thus, only countably many $P(\ell, w)$ contain N. Furthermore, since the set of point masses for $\widehat{\mu}$ is countable, the edge set of our second graph is countable, and therefore there are uncountably many pairs $\ell, w \in \mathbb{P}^1(\mathbb{R})$ with $P(\ell, w)$ pairwise disjoint, so the measure of the union of such sets is infinite. Thus, $\widehat{\mu}(g_0 M_{XY}) = \infty$. This is a contradiction. □

5.3. **First case: Lebesgue.** We will assume in this section that $\mu(\mathscr{L}_m^X) = 1$ and that μ is X-invariant.

Since UX is a codimension 1 unipotent subgroup of the non-unipotent group $\mathrm{Stab}_{BX}(\mu)^\circ$, we have that $BX = \mathrm{Stab}_{BX}(\mu)^\circ$. For $a^s \in A$, we let $h_\mu(a^s)$ be the entropy of the transformation a^s on $\mathscr{L}_m$ with respect to μ. Recall that $h_\mu(a^s) = h_\mu(a^{-s})$.

Since $\mathscr{L}_m$ is foliated by leafs that are locally the orbits of UX, the entropy of a^s is determined by the rate of expansion in the U and X directions, or similarly, by the expansion in the U^t and Y directions. Precisely, the proof of Theorem 9.7 from [M-T] yields

Lemma 9. *Suppose μ is BX-invariant. Then $h_\mu(a^s) = 3|s|$. Also, $h_\mu(a^{-s}) \leq 3|s|$ with equality if and only if μ is U^tY-invariant.*

From Lemma 9 we have

Proposition 3. *If μ is BX-invariant, then it is U^tY-invariant.*

Proof. We have

$$3|s| = h_\mu(a^s) = h_\mu(a^{-s}) \leq 3|s|$$

so the inequality is an equality. □

Now we have

Proposition 4. *If μ is BX-invariant, then μ is the unique ergodic, $SL(2,\mathbb{R})$-invariant measure with support equal to $\mathscr{L}_m$.*

Proof. By Proposition 3, μ is also U^t-invariant, so it is $SL(2,\mathbb{R})$-invariant since the subgroups U, A, and U^t generate $SL(2,\mathbb{R})$.

McMullen classified the $SL(2,\mathbb{R})$-invariant ergodic probability measures on the space of abelian differentials in genus 2; see Theorem 1.5 [Mc2]. It follows from the classification that μ either equals ν_{10} or μ is as desired. But ν_{10} only has a 3-dimensional support, and μ has support equal to $\mathscr{L}_m$ since the support of a measure is closed and μ is invariant under X and Y as well as $SL(2,\mathbb{R})$. □

5.4. **Second case: decagon.** We will assume in this section that $\mu(\mathscr{L}_m^X) = 1$ and that μ is not X-invariant. Because $\mathrm{Stab}_{BX}(\mu)^\circ$ is not unipotent, there is some $x \in X$ such that $x^{-1}Ax \in \mathrm{Stab}_{BX}(\mu)^\circ$. Since x commutes with U, the measure $x_*\mu$ is ergodic and is invariant under B.

In order to apply an argument similar to the proof of Proposition 3, we first have to show that $x_*\mu$ can not detect the X-direction the $\mathscr{L}_m$. This will affect the expansion of the UX-foliation under a^s that is visible to μ, and thus will alter the calculation of $h_{x_*\mu}(a^s)$.

Lemma 10. *There is a conull $\Omega \subseteq \mathscr{L}_m^X$ such that if $x_*\mu$ is not invariant under X, then for all $M \in \Omega$, we have*

$$(UXM) \cap \Omega = (UM) \cap \Omega$$

Proof. Let Ω be as in Lemma 3.3 of [E-M-M], so that Ω is U-invariant and so that if $x \in X$ and $x\Omega \cap \Omega \neq \emptyset$ then $x = 1$. Thus if $u_1xM \in \Omega$ and $u_1xM = u_2M$ then $xM = u_1^{-1}u_2M \in \Omega$, so $x = 1$. □

Analogously, we will need to know that the Y direction does not contribute to the determination of $h_{x_*\mu}(a^{-s})$.

Lemma 11. *There is a conull $\Omega \subseteq \mathscr{L}_m^Y$ such that if $x_*\mu$ is not invariant under X, then for all $M \in \Omega$, we have*

$$(U^tYM) \cap \Omega = (U^tM) \cap \Omega$$

Proof. The proof is essentially the proof of Proposition 5.5 from [E-M-M].

We reproduce it here for convenience. Let $\Omega \subseteq \mathscr{L}_m^Y$ be a generic set such that $x_*\mu(\Omega) = 1$ and $a^sM \in \Omega_\varepsilon$ for most $s > 0$. Let $vy \in U^tY$ and $M, M' \in \Omega$ be such that $vyM = M'$. We wish to show that $y = 0$. Suppose $y \neq 0$, and we will reach a contradiction. Choose a sequence of real numbers $s_k \to \infty$ such that $a^{s_k}M, a^{s_k}M' \in \Omega_\varepsilon$ for all k. Note that $a^{s_k}(vy)a^{-s_k} \to 0$ and that

$$||a^{s_k}va^{-s_k}|| \leq ||a^{s_k}ya^{-s_k}||$$

for $k \gg 0$.

Let $g_k = a^{s_k} v a^{-s_k} \in SL(2,\mathbb{R})$ and $y_k = a^{s_k} y a^{-s_k} \in Y$ and $t_k = \delta/||a^{s_k} y a^{-s_k}||$. By Lemma 2, $\mathrm{Stab}_X(x_*\mu)^\circ \neq 0$. This is a contradiction. □

Using Lemmas 10 and 11, one can apply the proof of Theorem 9.7 from [M-T] to show

Lemma 12. *Suppose $x_*\mu$ is B-invariant and not X-invariant. Then $h_{x_*\mu}(a^s) = 2|s|$. Also, $h_{x_*\mu}(a^s) \leq 2|s|$ with equality if and only if $x_*\mu$ is U^t-invariant.*

Proposition 5. *If $x_*\mu$ is B-invariant, then it is U^t-invariant.*

Proof. By Lemma 12, we have

$$2|s| = h_{x_*\mu}(a^s) = h_{x_*\mu}(a^{-s}) \leq 2|s|$$

so the inequality is an equality. □

Proposition 6. *If $x_*\mu$ is B-invariant, then $x_*\mu = \nu_{10}$.*

Proof. By assumption, $x_*\mu$ is U-invariant and A-invariant. By Proposition 5, $x_*\mu$ is also U^t-invariant. Since the subgroups U, A, and U^t generate $SL(2,\mathbb{R})$, $x_*\mu$ is $SL(2,\mathbb{R})$-invariant. Thus, by Theorem 1.5 of [Mc2], $x_*\mu$ equals ν_{10} or a measure whose support is $\mathscr{L}_m$. By lemma 10, $x_*\mu = \nu_{10}$. □

5.5. **Third case: Lattice surfaces on the boundary.** For our final case, we assume that $\mu(\mathscr{L}_m^X) = 0$. Thus $\mathrm{Supp}(\mu) \subseteq \mathrm{HC}(r)$ for some $r > 0$ from which it follows that there exists some $w \in X$ with $|w| = r$ such that w is a horizontal saddle connection for all $M \in \mathrm{Supp}(\mu)$.

Since $\mathrm{Stab}_{AX_0}(\mu)^\circ$ is not unipotent, $\mathrm{Stab}_{AX_0}(\mu)^\circ$ contains a neighborhood of the identity of a 1-parameter subgroup of AX that is not contained in X. Any such neighborhood is of the form $\{x^{-1}a^s x\}_{|s|<\varepsilon}$ for some $x \in X$.

Note that if w is a horizontal saddle connection on M, then $e^s w - e^s x + x$ is a horizontal saddle connection on $x^{-1}a^s x M$. Since $x^{-1}a^s x$ stabilizes μ, it follows that $x^{-1}a^s x$ stabilizes the support of μ as well. Since saddle connections on a surface are discrete, it follows that $w = e^s w - e^s x + x$ and therefore $x = w$.

It can be easily checked that $x^{-1}U^t x$ acts on the space of surfaces in $\mathscr{L}_m$ that have x as a horizontal saddle connection, so $x^{-1}SL(2,\mathbb{R})x$ acts on μ with a stabilizer that includes $x^{-1}Bx$.

Similar to Lemma 12, $h_\mu(xa^{-s}x^{-1}) = h_\mu(x^{-1}a^s x) = 2|s|$ which implies that μ is $x^{-1}SL(2,\mathbb{R})x$-invariant.

6. Stabilizer is UX

In this final section we will prove the following proposition which completes our proof of Theorem 2.

Proposition 7. *If μ is horizontally invariant, then μ equals the unique ergodic, $SL(2,\mathbb{R})$-invariant measure with support equal to $\mathscr{L}_m$.*

The contents of this section were described to us by Alex Eskin and were motivated by Ratner's proof of Theorem 2 from [Ra 5].

Let ν be the unique ergodic, $SL(2,\mathbb{R})$-invariant measure with support equal to $\mathscr{L}_m$. Let $f : \mathscr{L}_m \to \mathbb{R}_{\geq 0}$ be a continuous, compactly supported function and let $\varepsilon > 0$ be given. We will prove that

$$\Big| \int_{\mathscr{L}_m} f \, d\mu - \int_{\mathscr{L}_m} f d\nu \Big| < \varepsilon$$

and thus prove Proposition 7.

For any $T > 0$ and any $M \in \mathscr{L}_m^X$ we let

$$\mathscr{A}_U(f,T)(M) = \frac{1}{T} \int_0^T f(u^t M)\, dt$$

$$\mathscr{A}_{UX}(f,T)(M) = \frac{1}{T^{\frac{3}{2}}} \int_0^{\sqrt{T}} \int_0^T f(u^t x^s M)\, dt\, ds$$

6.1. **An ergodic theorem.** By the Birkhoff ergodic theorem, there is some $E' \subseteq \mathscr{L}_m$ with $\mu(E') = 1$, and such that if $M \in E'$ then $\lim_{T\to\infty} \mathscr{A}_U(f,T)(M) = \int_{\mathscr{L}_m} f\, d\mu$.

Choose a sequence $\varepsilon_n \to 0$ such that $\Sigma_{i=1}^{\infty} \sqrt{\varepsilon_n}$ converges.

For all n, there is some $E_n \subseteq \mathscr{L}_m$ with $\mu(E_n) > 1 - \varepsilon_n$ and $T_n > 0$ such that if $M \in E_n$ and $T \geq T_n$ then

$$\Big| \mathscr{A}_U(f,T)(M) - \int_{\mathscr{L}_m} f\, d\mu \Big| < \varepsilon_n \tag{3}$$

We let

$$E_n' = \{ M \in \mathscr{L}_m \mid ds(\{ s \in [0, \sqrt{T_n}] \mid x^s M \in E_n \}) \geq (1 - \sqrt{\varepsilon_n}) \sqrt{T_n} \}$$

Lemma 13. *If $M \in E_n'$ and $C = \max\{\sup f\, ,\, 1\}$, then*

$$\Big| \mathscr{A}_{UX}(f,T_n)(M) - \int_{\mathscr{L}_m} f\, d\mu \Big| < \sqrt{\varepsilon_n}(2C+2)$$

Proof. Let $S_n = \{ s \in [0, \sqrt{T_n}] \mid x^s M \in E_n \}$ and $S_n^c = \{ s \in [0, \sqrt{T_n}] \mid x^s M \notin E_n \}$. Now note that both

$$\frac{1}{\sqrt{T_n}} \int_{s \in S_n} \mathscr{A}_U(f,T_n)(x^s M)\, ds$$

and

$$(1-\sqrt{\varepsilon_n})\int_{\mathscr{L}_m} f\,d\mu$$

lie in the interval bounded by $(1-\sqrt{\varepsilon_n})\big(\int_{\mathscr{L}_m} f\,d\mu-\varepsilon_n\big)$ and $\int_{\mathscr{L}_m} f\,d\mu+\varepsilon_n$. Thus,

$$\begin{aligned}
&\Big|\frac{1}{\sqrt{T_n}}\int_{s\in S_n}\mathscr{A}_U(f,T_n)(x^sM)\,ds-(1-\sqrt{\varepsilon_n})\int_{\mathscr{L}_m} f\,d\mu\Big|\leq\\
&\leq\Big|\int_{\mathscr{L}_m} f\,d\mu+\varepsilon_n-(1-\sqrt{\varepsilon_n})\big(\int_{\mathscr{L}_m} f\,d\mu-\varepsilon_n\big)\Big|\\
&\leq\sqrt{\varepsilon_n}\int_{\mathscr{L}_m} f\,d\mu+\varepsilon_n(2-\sqrt{\varepsilon_n})\\
&<\sqrt{\varepsilon_n}(C+2)
\end{aligned}$$

Also note

$$\Big|\frac{1}{T_n^{\frac{3}{2}}}\int_{s\in S_n^c}\int_0^{T_n} f(u^tx^sM)\,dt\,ds-\sqrt{\varepsilon_n}\int_{\mathscr{L}_m} f\,d\mu\Big|\leq\sqrt{\varepsilon_n}C$$

The lemma follows since $\mathscr{A}_{UX}(f,T_n)(M)$ is the sum of

$$\frac{1}{\sqrt{T_n}}\int_{s\in S_n}\mathscr{A}_U(f,T_n)(x^sM)\,ds$$

and

$$\frac{1}{T_n^{\frac{3}{2}}}\int_{s\in S_n^c}\int_0^{T_n} f(u^tx^sM)\,dt\,ds$$

□

Lemma 14. *If $F_n\subseteq\mathscr{L}_m$ is the complement of E_n', then $\mu(F_n)\leq\sqrt{\varepsilon_n}$.*

Proof. Let χ_n denote the characteristic function of the complement of E_n. Since $\mu(E_n)>1-\varepsilon_n$, we have

$$\int_{\mathscr{L}_m}\chi_n(M)\,d\mu(M)\leq\varepsilon_n$$

Since μ is invariant under X, we have for all $0\leq s\leq\sqrt{T_n}$ that

$$\int_{\mathscr{L}_m}\chi_n(x^sM)\,d\mu(M)\leq\varepsilon_n$$

Integrating the inequality above with respect to s gives

$$\int_{\mathscr{L}_m}\left(\int_0^{\sqrt{T_n}}\chi_n(x^sM)\,ds\right)d\mu(M)=\int_0^{\sqrt{T_n}}\left(\int_{\mathscr{L}_m}\chi_n(x^sM)\,d\mu(M)\right)ds$$
$$\leq \varepsilon_n\sqrt{T_n}$$

Notice that $M\notin E_n'$ exactly when

$$\int_0^{\sqrt{T_n}}\chi_n(x^sM)\,ds\geq\sqrt{\varepsilon_n}\sqrt{T_n}$$

Thus, the lemma follows from the third inequality of this proof. □

Proposition 8. *There exists a sequence $T_n\to\infty$ and*

(i) *a subset $E_\mu\subset\mathscr{L}_m$ with $\mu(E_\mu)=1$ such that for $M\in E_\mu$,*

$$\lim_{n\to\infty}\mathscr{A}_{UX}(f,T_n)(M)=\int_{\mathscr{L}_m}f\,d\mu$$

(ii) *a subset $E_\nu\subset\mathscr{L}_m$ with $\nu(E_\nu)=1$ such that for $M\in E_\nu$,*

$$\lim_{n\to\infty}\mathscr{A}_{UX}(f,T_n)(M)=\int_{\mathscr{L}_m}f\,d\nu$$

Proof. By Lemma 14, $\sum\mu(F_n)$ converges. Let E_μ be the set of all $M\in\mathscr{L}_m$ such that M is contained in at most finitely many of the F_n. By the Borel-Cantelli lemma, $\mu(E_\mu)=1$. Also note that for any $M\in E_\mu$, there is some k such that $M\in E_n'$ whenever $n\geq k$. Thus, (i) follows from Lemma 13.

To prove (ii) note that the only properties of μ used were ergodicity with respect to U and X-invariance. These properties are shared by ν as well. To ensure that the sequence T_n is the same for both μ and ν we choose the T_n so that – in addition to being large enough to satisfy all properties above – there also exists a set E_n'' with $\nu(E_n'')>1-\varepsilon_n$, such that

$$\left|\mathscr{A}_U(f,T)-\int_{\mathscr{L}_m}f\,d\nu\right|<\varepsilon_n$$

for all $T\geq T_n$ and all $x\in E_n''$. The rest of the proof of (ii) is identical to that of (i). □

Corollary 9. *Let f and ε be as in the beginning of this section, and let T_n be as in Proposition 8. Then for every $\delta'>0$ there exist subsets $\hat{E}\subset\mathscr{L}_m$ and $\tilde{E}\subset\mathscr{L}_m$ and an integer n_0 such that*

(i) *$\mu(\hat{E})>1-\delta'$ and $\nu(\tilde{E})>1-\delta'$.*

(ii) *For $n>n_0$, and $M\in\hat{E}$, $|\mathscr{A}_{UX}(f,T_n)(M)-\int_{\mathscr{L}_m}f\,d\mu|<\varepsilon/4$.*

(iii) *For $n>n_0$, and $N\in\tilde{E}$, $|\mathscr{A}_{UX}(f,T_n)(N)-\int_{\mathscr{L}_m}f\,d\nu|<\varepsilon/4$.*

6.2. **Recurrence to compact sets.** The below theorem follows directly from Theorem H2 of Minsky-Weiss' [M-W].

Theorem 10. *For any $\delta > 0$ there exists a compact set $K \subset \mathscr{L}_m$ such that if λ is any U-invariant probability measure on $\mathscr{L}_m$ that assigns measure 0 to the set of surfaces that contain a horizontal saddle connection, then $\lambda(K) > 1 - \delta$.*

Recall that $a^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. The proof of the next proposition is credited to Elon Lindenstrauss and Maryam Mirzakhani.

Proposition 11. *Let $\delta > 0$ and let $t_n \to \infty$ be any sequence. Suppose K is as in the previous theorem and that λ is a U-invariant probability measure on $\mathscr{L}_m$ that assigns measure 0 to the set of surfaces that contain a horizontal saddle connection. Let F denote the set of $M \in K$ such that there exists a subsequence τ_n of t_n, with $a^{-\tau_n} M \in K$ for all n. Then $\lambda(F) > 1 - \delta$.*

Proof. Because A normalizes U, we can apply Theorem 10 to each pushforward measure $(a^{-t_n})_* \lambda$. Thus, for all $n \in \mathbb{N}$,

$$\lambda(a^{t_n} K) > 1 - \delta$$

Therefore,

$$\lambda\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} a^{t_k} K\right) \geq 1 - \delta$$

The lemma follows. □

6.3. **An adaptation of an argument of Ratner's.** For any $M \in \mathscr{L}_m$, there is some $D > 0$ such that

$$a^{[-D,D]} y^{[-D,D]} v^{[-D,D]} M \subseteq \mathscr{L}_m$$

We let $W^D(M) = a^{[-D,D]} y^{[-D,D]} v^{[-D,D]} M$.

Choose $\delta > 0$ with $\delta \ll 1$, and let K and F be as in Theorem 10 and Proposition 11. Since K is compact, we may choose $D > 0$ uniformly so that the above inclusion is satisfied for all $M \in K$. Furthermore, with ε and the compactly supported f as in the beginning of this section, we may assume that $D \ll 1$ is such that for any $M \in \mathscr{L}_m$, if there is some $N = a^{t_1} y^{t_2} v^{t_3} M$ where $|t_i| \leq D$, then

$$|f(M) - f(N)| < \varepsilon/8 \tag{4}$$

There are constants $R \ll 1$ and $D' \ll D$ uniform for all $M \in K$ such that $W^{D'}(u^t x^s M)$ is a well-defined subset of $\mathscr{L}_m$ for $0 \leq t, s \leq \sqrt{R}$, and such that if $N \in W^{D'}(M)$ and $0 \leq t, s \leq \sqrt{R}$, then $u^{\gamma(N,t,s)} x^{\beta(N,t,s)} N \in$

$W^D(u^t x^s M)$ for some $\gamma(N,t,s)\,,\,\beta(N,t,s) > 0$. Note that the map $\phi : [0,\sqrt{R}]^2 \to (\mathbb{R}_{\geq 0})^2$ given by $\phi(t,s) = \big(\gamma(N,t,s)\,,\,\beta(N,t,s)\big)$ is a diffeomorphism of $[0,\sqrt{R}]^2$ onto its image, and for any $\eta > 0$ we can choose $D' \ll 1$ such that

$$\Big|\frac{\text{volume}_{UX}(J)}{\text{volume}_{UX}(\phi(J))} - 1\Big| < \eta \tag{5}$$

for any Borel set $J \subseteq [0,1]^2$.

For $M \in K$, we let

$$\begin{aligned} V^{D'}(M) &= \{\, u^{\gamma(N,t,s)} x^{\beta(N,t,s)} N \mid N \in W^{D'}(M) \text{ and } 0 \leq s,t \leq \sqrt{R}\varepsilon/C \,\} \\ &\subseteq \bigcup_{0 \leq s,t \leq \sqrt{R}} W^D(u^t x^s M) \end{aligned}$$

By the compactness of K there is some $\delta_0 > 0$ such that $\nu\big(V^{D'}(M)\big) > \delta_0$ for all $M \in K$.

Choose $\delta' < \min\{\delta, \delta_0\}$, let $\hat{E}$ and $\tilde{E}$ be as in Corollary 9, and let $M \in F \cap \hat{E}$.

Let T_n be as in Proposition 8 and let $t_n = \log(T_n/R)/2$. We let τ_n be the subsequence of t_n associated to M as guaranteed by Proposition 11. We let $W^{D'}_{\tau_n}(M) = a^{\tau_n} W^{D'}(a^{-\tau_n} M)$ and $V^{D'}_{\tau_n}(M) = a^{\tau_n} V^{D'}(a^{-\tau_n} M)$. Notice that the A-invariance of ν implies

$$\nu\big(V^{D'}_{\tau_n}(M)\big) > \delta_0 \tag{6}$$

for all τ_n.

If $N \in W^{D'}_{\tau_n}(M)$ — so that $a^{-\tau_n} N \in W^{D'}(a^{-\tau_n} M)$ — and $0 \leq t' \leq T_n$ and $0 \leq s' \leq \sqrt{T_n}$, then we define $\Gamma(N,t',s')\,,\,B(N,t',s') > 0$ as

$$\Gamma(N,t',s') = \gamma(N, t'R/T_n, s'\sqrt{R}/\sqrt{T_n}) T_n/R$$

and

$$B(N,t',s') = \beta(N, t'R/T_n, s'\sqrt{R}/\sqrt{T_n}) \sqrt{T_n}/\sqrt{R}$$

From 5 it follows that the map $\phi_n : [0,T_n] \times [0,\sqrt{T_n}] \to (\mathbb{R}_{\geq 0})^2$ given by $\phi_n(t',s') = \big(\Gamma(N,t',s')\,,\,B(N,t',s')\big)$ satisfies

$$\Big|\frac{\text{volume}_{UX}(J)}{\text{volume}_{UX}(\phi_n(J))} - 1\Big| < \eta \tag{7}$$

for any Borel set $J \subseteq [0,T_n] \times [0,\sqrt{T_n}]$.

By 6 and our choice of δ', there is an $L \in V^{D'}_{\tau_n}(M) \cap \tilde{E}$. Notice that $L = a^{\tau_n} u^{\gamma(a^{-\tau_n}N,t,s)} x^{\beta(a^{-\tau_n}N,t,s)} a^{-\tau_n} N$ for some $N \in W^{D'}_{\tau_n}(M)$.

We have

$$
\begin{aligned}
&u^{\Gamma(N,t',s')}x^{B(N,t',s')}N = \\
&= u^{\gamma(N,t'R/T_n,s'\sqrt{R}/\sqrt{T_n})T_n/R}x^{\beta(N,t'R/T_n,s'\sqrt{R}/\sqrt{T_n})\sqrt{T_n}/\sqrt{R}}a^{\tau_n}a^{-\tau_n}N \\
&= a^{\tau_n}u^{\gamma(N,t'R/T_n,s'\sqrt{R}/\sqrt{T_n})}x^{\beta(N,t'R/T_n,s'\sqrt{R}/\sqrt{T_n})}a^{-\tau_n}N \\
&\in a^{\tau_n}W^D(u^{t'R/T_n}x^{s'\sqrt{R}/\sqrt{T_n}}a^{-\tau_n}M) \\
&= a^{\tau_n}W^D(a^{-\tau_n}u^{t'}x^{s'}M) \\
&= W^D_{\tau_n}(u^{t'}x^{s'}M) \\
&= a^{[-D,D]}y^{[-D\sqrt{R}/\sqrt{T_n},D\sqrt{R}/\sqrt{T_n}]}v^{[-DR/T_n,DR/T_n]}u^{t'}x^{s'}M \\
&\subseteq W^D(u^{t'}x^{s'}M)
\end{aligned}
$$

It now follows from 4 that

$$\left|\mathscr{A}_{UX}(f,T_n)(M) - \frac{1}{T_n^{\frac{3}{2}}}\int_0^{\sqrt{T_n}}\int_0^{T_n} f(u^{\Gamma(N,t',s')}x^{B(N,t',s')}N)dt'ds'\right| < \varepsilon/8$$

and then from 7 that for $n \gg 0$

$$\left|\mathscr{A}_{UX}(f,T_n)(N) - \frac{1}{T_n^{\frac{3}{2}}}\int_0^{\sqrt{T_n}}\int_0^{T_n} f(u^{\Gamma(N,t',s')}x^{B(N,t',s')}N)dt'ds'\right| < \varepsilon/8$$

The two above inequalities give

$$\left|\mathscr{A}_{UX}(f,T_n)(M) - \mathscr{A}_{UX}(f,T_n)(N)\right| < \varepsilon/4 \tag{8}$$

Also note that for $n \gg 1$ and with $R \ll 1$

$$|\mathscr{A}_{UX}(f,T_n)(N) - \mathscr{A}_{UX}(f,T_n)(L)| < \epsilon/4 \tag{9}$$

Recall that $M \in \hat{E}$ and that $L \in \tilde{E}$. In view of 8, 9, and of Corollary 9 (ii) and (iii), we have

$$\Big|\int_{\mathscr{L}_m} f\,d\mu - \int_{\mathscr{L}_m} f d\nu\Big| < \varepsilon$$

This completes the proof of Proposition 7.

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