Quasi-isometric rigidity of higher rank S-arithmetic lattices

KEVIN WORTMAN

We show that S-arithmetic lattices in semisimple Lie groups with no rank one factors are quasi-isometrically rigid.

20F65, 20G30, 22E40

1 Introduction

Cocompact lattices in semisimple Lie groups over local fields with no rank one factors are quasi-isometrically rigid. This was shown by Kleiner-Leeb [K-L] in general, and Eskin-Farb [E-F 1] later gave a different proof in the case of real Lie groups.

Eskin then applied the "quasiflats with holes" theorem for symmetric spaces of Eskin-Farb [E-F 1] to prove that any quasi-isometry of a non-cocompact irreducible lattice in a real semisimple Lie group with no rank one factors is a finite distance from a commensurator [Es]. As a consequence, any such lattice is quasi-isometrically rigid. Basic examples of such lattices include $\mathbf{SL_n}(\mathbb{Z})$ for $n \geq 3$. Druţu has given another proof of Eskin's theorem [Dr] using asymptotic cones and the results of [K-L].

Eskin's theorem has a place in a larger body of work of Schwartz, Farb-Schwartz, and Eskin. In particular, it has been shown that any quasi-isometry of an irreducible non-cocompact lattice in a semisimple real Lie group, which is not locally isomorphic to $\mathbf{SL_2}(\mathbb{R})$, is a finite distance from a commensurator ([Sch 1], [Fa-Sch], [Sch 2], and [Es]); see [Fa] for a full account.

While the theorem of Kleiner-Leeb applied to cocompact S-arithmetic lattices in semisimple Lie groups with no rank one factors, the question of quasi-isometric rigidity for non-cocompact S-arithmetic lattices remained unexplored for a few years. The first account of quasi-isometric rigidity for non-cocompact S-arithmetic lattices (and the only account aside from this paper) was given by Taback [Ta]. Taback's theorem states that any quasi-isometry of $\mathbf{SL_2}(\mathbb{Z}[1/p])$ is a finite distance in the sup-norm from a commensurator. Thus, Taback's theorem provided evidence that quasi-isometries of S-arithmetic lattices could be characterized in the same way as their arithmetic counterparts.

Following the work of Eskin, we apply the quasiflats with holes theorem of [W1] for products of symmetric spaces and Euclidean (affine) buildings to show that non-cocompact S-arithmetic lattices in semisimple Lie groups with no rank one factors are quasi-isometrically rigid. Examples of such lattices include $\mathbf{SL_n}(\mathbb{Z}[1/p])$ and $\mathbf{SL_n}(\mathbb{F}_q[t])$ for $n \geq 3$, where $\mathbb{F}_q[t]$ is a polynomial ring with indeterminate t and coefficients in the finite field $\mathbb{F}_q$. (See Section 5 for more examples.)

As a special case of our results, we show that any finitely generated group quasi-isometric to $\mathbf{SL_n}(\mathbb{Z}[1/p])$, is in fact isomorphic to $\mathbf{SL_n}(\mathbb{Z}[1/p])$ "up to finite groups" as long as $n \geq 3$.

Our proof also shows that cocompact lattices in semisimple Lie groups with no rank one factors are quasi-isometrically rigid, thus providing a unified proof of the theorems of Kleiner-Leeb, Eskin-Farb, and Eskin. In particular, we give a proof of the theorem of Kleiner-Leeb – a proof which does not use the theory of asymptotic cones.

Summary of definitions to come. In order to state our results, we briefly provide some definitions. We will expand on these definitions in Section 2.

For any topological group H, we let $\mathrm{Aut}(H)$ be the group of topological group automorphisms of H.

For any valuation v of a global field K, let K_v be the completion of K with respect to v. If S is a set of valuations of K, then we let $\mathcal{O}_S \leq K$ be the ring of S-integers.

We call an algebraic K-group $\mathbf{G}$ *placewise not rank one* with respect to S if $K_v-\mathrm{rank}(\mathbf{G}) \neq 1$ for all $v \in S$. We denote the adjoint representation by $\mathbf{Ad}$, and we let G be the direct product of the groups $\mathbf{Ad}(\mathbf{G})(K_v)$ over all $v \in S$ for which $\mathbf{G}$ is K_v-isotropic.

Last, we let $\mathcal{QI}(\mathbf{G}(\mathcal{O}_S))$ be the quasi-isometry group of $\mathbf{G}(\mathcal{O}_S)$, and we let $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ be the commensurator group of $\mathbf{G}(\mathcal{O}_S)$. We warn the reader here that our definition of $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ is slightly atypical (see Section 2).

Quasi-isometries of S-arithmetic groups. Our main result is

Theorem 1.1 *Let K be a global field and S a finite nonempty set of inequivalent valuations containing all of the archimedean ones. Suppose $\mathbf{G}$ is a connected simple K-group that is placewise not rank one with respect to S.*

(i) *If $\mathbf{G}$ is K-isotropic and K is a number field, then there is an isomorphism:*

$$\mathcal{QI}\big(\mathbf{G}(\mathcal{O}_S)\big) \cong \mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)).$$

(ii) *If* $\mathbf{G}$ *is* K*-isotropic and* K *is a function field, then there exist an inclusions:*

$$\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)) \hookrightarrow \mathcal{QI}\big(\mathbf{G}(\mathcal{O}_S)\big) \hookrightarrow \mathrm{Aut}(G).$$

Furthermore, the image of $\mathcal{QI}(\mathbf{G}(\mathcal{O}_S))$ *in* $\mathrm{Aut}(G)$ *has measure zero.*

(iii) *If* $\mathbf{G}$ *is* K*-anisotropic, then there is an isomorphism:*

$$\mathcal{QI}\big(\mathbf{G}(\mathcal{O}_S)\big) \cong \mathrm{Aut}(G).$$

As an example of Theorem 1.1(i), we have

$$\mathcal{QI}(\mathbf{SL_3}(\mathbb{Z}[1/p])) \cong \mathbf{PGL_3}(\mathbb{Q}) \rtimes \mathbb{Z}/2\mathbb{Z},$$

where the topology on the right side of the isomorphism is induced by the topology of $\mathbb{Q}$ as the diagonal subspace of $\mathbb{R} \times \mathbb{Q}_p$. This example is described in more detail in Section 5, where we also present five other examples.

We note that the theorem above leaves room for improvement, as the K-isotropic case for function fields is not completely determined. However, results in this case are still slightly stronger than they are for the fully resolved K-anisotropic case.

Quasi-isometric rigidity. From Theorem 1.1 we can deduce

Corollary 1.2 *Suppose* K*,* S*, and* $\mathbf{G}$ *are as in Theorem 1.1, and suppose that* $\mathbf{G}$ *is of adjoint type. Let* Λ *be a finitely generated group, and assume there is a quasi-isometry*

$$\phi : \Lambda \to \mathbf{G}(\mathcal{O}_S).$$

(i) If $\mathbf{G}$ *is* K*-isotropic and* K *is a number field, then there exists a finite index subgroup* Λ_S *of* Λ *and a homomorphism* $\varphi : \Lambda_S \to \mathbf{G}(\mathcal{O}_S)$ *with a finite kernel and finite co-image such that*

$$\sup_{\lambda \in \Lambda_S} d\Big(\varphi(\lambda), \phi(\lambda)\Big) < \infty.$$

(ii) If $\mathbf{G}$ *is* K*-isotropic and* K *is a function field, then there exists a finite group* F *and an exact sequence*

$$1 \to F \to \Lambda \to \Gamma \to 1,$$

such that Γ *is a non-cocompact lattice in* $\mathrm{Aut}(G)$*.*

(iii) If $\mathbf{G}$ *is* K*-anisotropic, then there exists a finite group* F *and an exact sequence*

$$1 \to F \to \Lambda \to \Gamma \to 1,$$

such that Γ *is a cocompact lattice in* $\mathrm{Aut}(G)$*.*

Bibliographic note. We will present a proof of Theorem 1.1 that covers all of the cases above, some of which are well known.

Part (iii) of Theorem 1.1 and Corollary 1.2 was shown by Kleiner-Leeb [K-L]. Part (iii) was also shown when K is a number field and S equals the set of archimedean valuations by Eskin-Farb [E-F 1]. (Note that the theorems in [K-L] and [E-F 1] are stated in equivalent terms of isometries of Euclidean buildings and/or symmetric spaces.)

Part (i) of Theorem 1.1 and Corollary 1.2 was shown by Eskin [Es] with the additional assumption that S equals the set of archimedean valuations. Druţu has also given a proof of (i) assuming S is the set of archimedean valuations [Dr]. The proof in [Dr] uses results from [K-L].

Corollary 1.2 follows directly from Theorem 1.1 and, for part (i), Margulis' super-rigidity theorem. The proof of this corollary using Theorem 1.1 is routine. See, for example, Section 9 of [Es].

Similarities and differences between our proof and Eskin's. The proof of Eskin's theorem involves studying the large-scale geometry of symmetric spaces on which higher rank real semisimple Lie groups act. Our proof of Theorem 1.1 applies the "quasiflats with holes" theorem from [W1] (which itself is an extension of the quasiflats with holes theorem of Eskin-Farb [E-F 1]) to extend Eskin's proof by allowing for the presence Euclidean buildings. (Recall that Euclidean buildings are the natural spaces acted on by semisimple Lie groups over nonarchimedean local fields.) We rely on many of Eskin's arguments in using large-scale geometry to construct a boundary function defined almost everywhere.

Where our proof differs substantially from Eskin's, is in the way we complete the boundary function. We are forced to confront this problem with different methods, since the proof in [Es] relies on the fact that the Furstenberg boundary of a real semisimple Lie group is a Euclidean manifold. This is not the case in general, as the Furstenberg boundary of a semisimple Lie group over a nonarchimedean local field is a Cantor set. Being unable to rely as heavily on topological arguments, we turn to algebraic methods to find a completion. (See Section 4 for an expanded outline of our proof.)

Strong rigidity. Our main result can be viewed as a strengthening of strong rigidity.

Recall that the strong rigidity theorems – first proved by Mostow and later expanded on greatly by Prasad, Margulis, and Venkataramana – state that any isomorphism between irreducible lattices in semisimple Lie groups, which are not locally isomorphic to

$\mathbf{SL_2}(\mathbb{R})$, extends to an isomorphism of the ambient semisimple group. Thus, the ambient semisimple group is completely determined by the isomorphism class of a lattice ([Mo], [Pr 1], [Pr 2], [Mar], and [Ve]).

Our result states that the quasi-isometry class alone of an S-arithmetic lattice meeting the conditions of Theorem 1.1 is enough to determine the ambient semisimple group.

We note that the proofs of strong rigidity in cases (i) and (ii) of our main theorem (given by Margulis and Venkataramana respectively) are rooted in ergodic theory. Our unified proof of cases (i), (ii), and (iii) is based on the large-scale geometry of symmetric spaces and Euclidean buildings. As such, we return to Mostow's original ideas and present a proof that is of a more geometric nature than the ergodic theoretical proofs of strong rigidity.

Number fields versus function fields. Although our results are not complete in the function field case, we point out that this is only due to the absence of a characterization of commensurators which does not exist in the function field case (see Proposition 7.2).

Throughout the portion of the proof dealing with large-scale geometry, the function field case allows for significant simplifications. The simplifications stem from the fact that two Weyl chambers in a Euclidean building are Hausdorff equivalent if and only if their intersection contains a Weyl chamber. Of course this is false for symmetric spaces.

Acknowledgements. I thank my Ph.D. thesis advisor, Benson Farb, for giving me the opportunity to work on this problem and for believing I could solve it.

Thanks to Alex Eskin for helpful insights and for helping me discover some mistakes I made along the way.

I am also happy to thank Nimish Shah for showing me how to prove Proposition 7.2 below, Steven Spallone for explaining numerous mathematical concepts to me over the past five years, as well as Bruce Kleiner, Dan Margalit, and Karen Vogtmann for suggestions about the exposition of this paper.

I would like to acknowledge the University of Chicago for supporting me as a graduate student while I developed the ideas in this paper, and Cornell University for the pleasant working environment given to me while I completed its writing.

Last and most important, I am grateful for Barbara Csima, Benson Farb, and Dan Margalit; their support, encouragement, and patience made me into a mathematician.

2 Definitions

We will take some time now to be precise with our definitions.

Quasi-isometries. For constants $\kappa \geq 1$ and $C \geq 0$, a (κ, C) *quasi-isometric embedding* of a metric space X into a metric space Y is a function $\phi : X \to Y$ such that for any $x_1, x_2 \in X$:

$$\frac{1}{\kappa} d\big(x_1, x_2\big) - C \leq d\big(\phi(x_1), \phi(x_2)\big) \leq \kappa d\big(x_1, x_2\big) + C.$$

We call ϕ a (κ, C) *quasi-isometry* if ϕ is a (κ, C) quasi-isometric embedding and there is a number $D \geq 0$ such that every point in Y is within distance D of some point in the image of X.

Quasi-isometry groups. For a metric space X, we define the relation $\sim$ on the set of functions $X \to X$ by $\phi \sim \psi$ if

$$\sup_{x \in X} d\big(\phi(x), \psi(x)\big) < \infty.$$

We form the set of all self-quasi-isometries of X, and denote the quotient space modulo $\sim$ by $\mathcal{QI}(X)$. We call $\mathcal{QI}(X)$ the *quasi-isometry group* of X as it has a natural group structure arising from function composition. Note that if X and Y are quasi-isometric metric spaces, then there is a natural isomorphism $\mathcal{QI}(X) \cong \mathcal{QI}(Y)$.

Word metrics. A finitely generated group Γ is naturally equipped with a proper left-invariant *word metric*. This is the metric obtained by setting the distance between $\gamma \in \Gamma$ and $1 \in \Gamma$ to be the infimum of the length of all words written in a fixed finite generating set that represent γ.

The word metric depends on the choice of finite generating set, but only up to quasi-isometry. Hence, the group $\mathcal{QI}(\Gamma)$ is independent of the choice of a finite generating set for Γ.

***S*-integers.** Recall that finite algebraic extensions of either $\mathbb{Q}$ or the field $\mathbb{F}_p(t)$ of rational functions with indeterminate t and coefficients in a finite field $\mathbb{F}_p$, are called *global fields*. If K is a global field then we denote the set of all inequivalent valuations on K by V_K, and we denote the set of all inequivalent archimedean valuations of K by V_K^∞.

For any valuation $v \in V_K$, let K_v be the topological completion of K with respect to v. The field K_v is a locally compact nondiscrete field. Any field satisfying these topological properties is called a *local field*.

For a finite nonempty set of valuations $S \subseteq V_K$ containing V_K^∞, we define the ring of S-*integers* in K to be

$$\mathcal{O}_S = \{\, x \in K \mid 1 \geq |x|_v \text{ for all } v \in V_K - S \,\}.$$

Rank. If a simple algebraic group $\mathbf{G}$ is defined over a field L, we say it is an L-*group*. An L-group $\mathbf{G}$ is called L-*isotropic* if $L - \text{rank}(\mathbf{G}) > 0$, and called L-*anisotropic* otherwise. (Recall that $L - \text{rank}(\mathbf{G})$ is the maximum dimension of an algebraic subgroup of $\mathbf{G}$ which is diagonalizable over L.)

For a global field K and a simple K-group $\mathbf{G}$, let $V_K^{\mathbf{G},a} \subseteq V_K$ be the set of valuations v for which $\mathbf{G}$ is K_v-anisotropic. Recall that $v \in V_K^{\mathbf{G},a}$ is equivalent to the condition that $\mathbf{G}(K_v)$ is compact.

We define $\mathbf{G}$ to be *placewise not rank one* with respect to a chosen finite set of valuations S, if $K_v - \text{rank}(\mathbf{G}) \neq 1$ for all $v \in S$.

S-**arithmetic groups.** A group is called S-*arithmetic* if it is isomorphic to $\mathbf{G}(\mathcal{O}_S)$ for some K-group $\mathbf{G}$ and for some finite nonempty set $S \subseteq V_K$ containing V_K^∞.

Throughout the remainder, $\mathbf{G}$ is connected, simple, and placewise not rank one with respect to S. Under these conditions it is well known that $\mathbf{G}(\mathcal{O}_S)$ is a finitely generated group, so it admits a proper word metric.

Lattices. A locally compact group H supports a Haar measure μ. A discrete subgroup $\Gamma < H$ is called a *lattice* if H/Γ has finite volume with respect to μ. This is necessarily the case if H/Γ is compact. Such lattices are called *cocompact*; they are called *non-cocompact* otherwise.

We write $\mathbf{Ad}(\mathbf{G})$ for the image of $\mathbf{G}$ under the adjoint representation of $\mathbf{G}$. The adjoint representation has a finite kernel which equals the center of $\mathbf{G}$.

Define

$$G = \prod_{v \in S - V_K^{\mathbf{G},a}} \mathbf{Ad}(\mathbf{G})(K_v).$$

The diagonal homomorphism of $\mathbf{G}(\mathcal{O}_S)$ into G has a finite kernel. We write the image of the diagonal homomorphism as $\mathbf{G}(\mathcal{O}_S)^\Delta$. The reduction theory of Borel, Behr, and

Harder established that $\mathbf{G}(\mathcal{O}_S)^\Delta$ is a lattice in G and that $\mathbf{G}(\mathcal{O}_S)^\Delta$ is cocompact if and only if $\mathbf{G}$ is K-anisotropic.

We point out here that $\mathbf{G}(\mathcal{O}_S)^\Delta$ is clearly irreducible as a lattice in G. Recall that a lattice $\Gamma < G$ is *reducible* if Γ contains a finite index subgroup of the form $\Gamma_1 \times \Gamma_2$ where

$$\Gamma_i = \Gamma \cap \prod_{T_i} \mathbf{Ad}(\mathbf{G})(K_v),$$

and T_1 and T_2 nontrivially partition $S - V_K^{\mathbf{G},a}$. Otherwise, Γ is *irreducible*.

Let $\mathrm{Aut}(G)$ be the group of all topological group automorphisms of G. Since G has a trivial center, it embeds into $\mathrm{Aut}(G)$ via inner automorphisms. Furthermore, G is a closed cocompact subgroup of $\mathrm{Aut}(G)$, so $\mathbf{G}(\mathcal{O}_S)^\Delta$ is also a lattice in $\mathrm{Aut}(G)$. Furthermore, $\mathbf{G}(\mathcal{O}_S)^\Delta$ is cocompact in $\mathrm{Aut}(G)$ if and only if it is cocompact in G.

Commensurators. An automorphism $\psi \in \mathrm{Aut}(G)$ *commensurates* $\mathbf{G}(\mathcal{O}_S)^\Delta$ if $\psi(\mathbf{G}(\mathcal{O}_S)^\Delta) \cap \mathbf{G}(\mathcal{O}_S)^\Delta$ is a finite index subgroup of both $\psi(\mathbf{G}(\mathcal{O}_S)^\Delta)$ and $\mathbf{G}(\mathcal{O}_S)^\Delta$.

Define $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)) < \mathrm{Aut}(G)$ as the group of automorphisms that commensurate $\mathbf{G}(\mathcal{O}_S)^\Delta$. Notice that $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ is different from the standard definition of the commensurator group of $\mathbf{G}(\mathcal{O}_S)$ in two ways: we project $\mathbf{G}(\mathcal{O}_S)$ into G, and we do not restrict ourselves to inner automorphisms.

Let $\mathrm{Aut}(K)$ be the group of field automorphisms of K. There is an action of $\mathrm{Aut}(K)$ on the set of affine K-varieties. Indeed, if $\mathbf{W}$ is an affine K-variety, then we let ${}^\sigma\mathbf{W}$ be the variety obtained by applying σ to the coefficients of the polynomials that define $\mathbf{W}$. We define $\mathrm{Aut}(K)_{\mathbf{G}}$ to be the group of automorphisms $\sigma \in \mathrm{Aut}(K)$ such that ${}^\sigma\mathbf{G}$ is K-group isomorphic to $\mathbf{G}$.

Since valuations are obtained by embedding K into various local fields, there is an obvious action of $\mathrm{Aut}(K)$ on the set of valuations V_K. We let $\mathrm{Aut}(K)_{\mathbf{G},S}$ be the subgroup of $\mathrm{Aut}(K)_{\mathbf{G}}$ consisting of those $\sigma \in \mathrm{Aut}(K)_{\mathbf{G}}$ such that $\sigma(S - V_K^{\mathbf{G},a}) = S - V_K^{\mathbf{G},a}$.

The group $\mathrm{Aut}(K)$ is finite when K is a global field, so both $\mathrm{Aut}(K)_{\mathbf{G}}$ and $\mathrm{Aut}(K)_{\mathbf{G},S}$ are finite also.

We will see in Section 7 that $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ is an extension

$$1 \to \mathbf{Aut}(\mathbf{Ad}(\mathbf{G}))(K) \to \mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)) \to \mathrm{Aut}(K)_{\mathbf{G},S} \to 1,$$

where $\mathbf{Aut}(\mathbf{Ad}(\mathbf{G}))$ is the K-group of algebraic group automorphisms of $\mathbf{Ad}(\mathbf{G})$.

If $\mathbf{G}$ is defined over a subfield of K that is fixed pointwise by $\mathrm{Aut}(K)_{\mathbf{G},S}$, then the above extension splits. Furthermore, if $\mathbf{G}$ is K-split, then there is a split extension

$$1 \to \mathbf{Ad}(\mathbf{G})(K) \to \mathbf{Aut}(\mathbf{Ad}(\mathbf{G}))(K) \to \mathbf{Out}(\mathbf{Ad}(\mathbf{G}))(K) \to 1,$$

where $\mathbf{Out}(\mathbf{Ad}(\mathbf{G}))$ is the K-group of outer automorphisms of $\mathbf{Ad}(\mathbf{G})$ (or alternatively the K-group of automorphisms of the Dynkin diagram of $\mathbf{Ad}(\mathbf{G})$).

Combining the two remarks above, we have that if $\mathbf{G}$ is K-split and defined over a subfield of K that is fixed pointwise by $\mathrm{Aut}(K)_{\mathbf{G},S}$, then

$$\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)) \cong \Big(\mathbf{Ad}(\mathbf{G})(K) \rtimes \mathbf{Out}(\mathbf{Ad}(\mathbf{G}))(K) \Big) \rtimes \mathrm{Aut}(K)_{\mathbf{G},S}.$$

Regardless of whether the extensions defining $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ split, $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ contains $\mathbf{Ad}(\mathbf{G})(K)$ as a finite index subgroup since the outer automorphism group of a simple algebraic group is finite. Therefore, we can define a topology on $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ by assigning the topology on $\mathbf{Ad}(\mathbf{G})(K)$ to be the subspace topology resulting from the diagonal embedding

$$\mathbf{Ad}(\mathbf{G})(K) \longrightarrow \prod_{v \in S - V_K^{\mathbf{G},a}} \mathbf{Ad}(\mathbf{G})(K_v).$$

Examples. A reader not familiar with S-arithmetic groups is encouraged at this point to skip ahead to Section 5 where a series of examples is presented.

3 Notes

Now that our definitions are in place, we revisit Theorem 1.1.

Remarks on Theorem 1.1(i). In the K-isotropic case for number fields in Theorem 1.1, the group of K-rational points of $\mathbf{Ad}(\mathbf{G})$ is a finite index subgroup of $\mathcal{QI}(\mathbf{G}(\mathcal{O}_S))$. Hence the group operation on $\mathcal{QI}(\mathbf{G}(\mathcal{O}_S))$ recovers K and a finite quotient of $\mathbf{G}$. These are two of the three ingredients used to create $\mathbf{G}(\mathcal{O}_S)$. The third ingredient, S, cannot in general be recovered from the quasi-isometry group, but it can be identified up to an element of the finite group $\mathrm{Aut}(K)_{\mathbf{G}}$.

Let's briefly make the paragraph above more precise.

Theorem 1.1 states that $\mathcal{QI}(\mathbf{G}(\mathcal{O}_S))$ is determined up to a topological group isomorphism as $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$. By a theorem of Borel-Tits ([Bo-T] Cor. 6.7), $\mathbf{Ad}(\mathbf{G})(K)^+$

is the minimal finite index subgroup of $\mathcal{QI}(\mathbf{G}(\mathcal{O}_S))$ where $\mathbf{Ad}(\mathbf{G})(K)^+$ is the subgroup of $\mathbf{Ad}(\mathbf{G})(K)$ generated by the K-points of the unipotent radicals of the K-parabolic subgroups of $\mathbf{Ad}(\mathbf{G})$. Therefore, any topological group isomorphism of $\mathcal{QI}(\mathbf{G}(\mathcal{O}_S))$ induces a topological group isomorphism

$$f : \mathbf{Ad}(\mathbf{G})(K)^+ \longrightarrow \mathbf{Ad}(\mathbf{G})(K)^+,$$

where we assume the domain of f has the topology derived from S.

Another well known theorem of Borel-Tits ([Bo-T] Theorem (A)) states that $f = \beta \circ \sigma^0$ where $\sigma \in \mathrm{Aut}(K)_{\mathbf{G}}$, and

$$\sigma^0 : \mathbf{Ad}(\mathbf{G})(K) \longrightarrow {}^{\sigma}\mathbf{Ad}(\mathbf{G})(K)$$

is the homomorphism defined by applying σ to the matrix entries of $\mathbf{Ad}(\mathbf{G})(K)$, and

$$\beta : {}^{\sigma}\mathbf{Ad}(\mathbf{G}) \longrightarrow \mathbf{Ad}(\mathbf{G})$$

is a K-isomorphism of algebraic groups.

Since f is a homeomorphism, σ is a homeomorphism as well. Therefore, the topology on the image of $\sigma : K \to K$ is given by the set σS, since S determines the topology of the domain of σ.

Note that if $\sigma \in \mathrm{Aut}(K)_{\mathbf{G}}$ and $\beta : {}^{\sigma}\mathbf{Ad}(\mathbf{G}) \to \mathbf{Ad}(\mathbf{G})$ is a K-isomorphism of topological groups, then σ^0 restricts to an isomorphism $\mathbf{Ad}(\mathbf{G})(\mathcal{O}_S) \cong {}^{\sigma}\mathbf{Ad}(\mathbf{G})(\mathcal{O}_{\sigma S})$ and $\beta({}^{\sigma}\mathbf{Ad}(\mathbf{G})(\mathcal{O}_{\sigma S}))$ is commensurable with $\mathbf{Ad}(\mathbf{G})(\mathcal{O}_{\sigma S})$ (see e.g. [Mar] I.3.1.1.iv). Hence, recovering S up to an element of $\mathrm{Aut}(K)_{\mathbf{G}}$ provides us with enough information to reconstruct $\mathbf{G}(\mathcal{O}_S)$ up to finite groups. In light of this, we could not hope for quasi-isometries to pinpoint S any more than up to an element of $\mathrm{Aut}(K)_{\mathbf{G}}$.

For clarity, we observe that

$$\mathbf{SL_n}\big(\mathbb{Z}[i, 1/(2+i)]\big) \cong \mathbf{SL_n}\big(\mathbb{Z}[-i, 1/(2-i)]\big)$$

is an example of how the set of valuations can fail to be identified completely by quasi-isometries since, in this example, the set cannot even be distinguished by isomorphisms of groups.

Remarks on Theorem 1.1(iii). In the K-anisotropic case, the simple group $\mathbf{Ad}(\mathbf{G})$ is encoded in the quasi-isometry group, but the global field K is not.

For example, examine the quadratic form

$$\Phi = \sum_{i=1}^{5} x_i^2.$$

Let **SO** be the special orthogonal group of Φ, so that **SO** is $\mathbb{Q}$-anisotropic and $\mathbb{Q}(\sqrt{11})$-anisotropic.

There are exactly two elements of $V^{\infty}_{\mathbb{Q}(\sqrt{11})}$ — which we name v_1^∞ and v_2^∞ — and $\mathbb{Q}(\sqrt{11})_{v_i^\infty} \cong \mathbb{R}$ for $i = 1, 2$. If we choose the valuation $v_{(4+\sqrt{11})} \in V_{\mathbb{Q}(\sqrt{11})}$ defined by the prime ideal $(4+\sqrt{11}) \subseteq \mathbb{Z}[\sqrt{11}]$, then $\mathbb{Q}(\sqrt{11})_{v_{(4+\sqrt{11})}}$ is isomorphic to the field of 5-adic numbers, $\mathbb{Q}_5$.

Let $S = \{v_1^\infty, v_2^\infty, v_{(4+\sqrt{11})}\}$. By the theorem of Kleiner-Leeb,

$$\mathcal{QI}\Big(\mathbf{SO}\left(\mathcal{O}_S\right)\Big) \cong \mathbf{SO}(\mathbb{Q}_5).$$

(That **SO** is placewise not rank one with respect to S follows form the fact that $i \in \mathbb{Q}_5$.)

Next, we take our global field to be $\mathbb{Q}$. We let $S' = \{v^\infty, v_{(5)}\}$, where v^∞ is the archimedian valuation on $\mathbb{Q}$ and $v_{(5)}$ is the 5-adic valuation. Then Kleiner-Leeb's theorem also gives us

$$\mathcal{QI}\Big(\mathbf{SO}\left(\mathcal{O}_{S'}\right)\Big) \cong \mathbf{SO}(\mathbb{Q}_5).$$

Hence, quasi-isometries could not distinguish between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{11})$ in these two examples.

Remarks on Theorem 1.1(ii). My current level of knowledge for the general S-arithmetic group when K is a function field and **G** is K-isotropic is at an intermediate level. In this setting we have stronger results than in the K-anisotropic case, but less is known than in the number field case.

There is some evidence that we should be able to remove the assumption that K is a number field from part (i) of Theorem 1.1. The number field case itself provides evidence that part (i) should hold for the function field case, and it has been shown that Theorem 1.1(i) holds for $\mathbf{SL_n}(\mathbb{F}_q[t])$ when $n \geq 3$ [W2].

The distinction between number fields and function fields in the K-isotropic case exists because our proof for number fields takes advantage of Ratner's theorem for unipotent flows [Ra]. Ratner's theorem is a powerful tool, and it appears to be unknown in positive characteristic.

Note that, in contrast with lattices in semisimple Lie groups over p-adic number fields, lattices in semisimple Lie groups over function fields can be non-cocompact. In fact, Harder showed that if K is a global function field and **G** is a simple K-group, then **G** can be K-anisotropic only if **G** is of type $\mathbb{A}_n$ [Har]. Therefore, resolving the K-isotropic case for function fields has heightened importance.

Remarks on Corollary 1.2. In the remarks on Theorem 1.1(i) it was pointed out that in the K-isotropic case for a number field K, the quasi-isometry group of $\mathbf{G}(\mathcal{O}_S)$ carries the information needed to reconstruct $\mathbf{G}(\mathcal{O}_S)$. Hence, an arbitrary finitely generated group Λ that is quasi-isometric to $\mathbf{G}(\mathcal{O}_S)$ will also carry the information needed to reconstruct $\mathbf{G}(\mathcal{O}_S)$ as Λ and $\mathbf{G}(\mathcal{O}_S)$ will have the same quasi-isometry groups. This is the content of part (i) of Corollary 1.2.

Note that (i) states that the only way to deform $\mathbf{G}(\mathcal{O}_S)$ in the space of all finitely generated groups without moving it outside of its initial quasi-isometry class is through algebraic methods.

If we knew that Theorem 1.1(i) held in the function field case, then Corollary 1.2(i) would apply to the function field case as well. In particular, case (i) of the above corollary holds when $\mathbf{G}(\mathcal{O}_S)$ is replaced by $\mathbf{SL_n}(\mathbb{F}_q[t])$ for $n \geq 3$.

Rigidity for groups with poor finiteness properties. Any finitely generated group that was previously known to be quasi-isometrically rigid contains a finite index subgroup that is simultaneously complex linear, torsion-free, of type F_∞, and of finite cohomological dimension. Thus, the final comment in the preceding paragraph displays the first quasi-isometric rigidity result for a finitely generated group with poor finiteness properties.

Indeed, it is well known that $\mathbf{SL_n}(\mathbb{F}_q[t])$ is not virtually torsion free. Hence, $\mathbf{SL_n}(\mathbb{F}_q[t])$ is not complex linear, and any finite index subgroup has infinite cohomological dimension. In addition, $\mathbf{SL_3}(\mathbb{F}_q[t])$ is known *not* to be finitely presentable (a result of Behr [Be]), and independent work of Abels and Abramenko shows that the class of groups of the form $\mathbf{SL_n}(\mathbb{F}_q[t])$ where $n \geq 3$ contains groups of type F_k, but not of type F_{k+1} for all $k \geq 1$ (see [Abl] and [Abr]). Recall that a group π is of type F_k if there exists an Eilenberg-Mac Lane $K(\pi, 1)$ complex with finite k-skeleton, and π is of type F_∞ if it is of type F_k for all k.

4 Outline

Our proof of Theorem 1.1 borrows heavily from [Es].

We proceed by realizing any element of $\mathcal{QI}(\mathbf{G}(\mathcal{O}_S))$ as a quasi-isometric embedding

$$\phi : N(\Gamma) \longrightarrow X,$$

where X is a product of a symmetric space and a Euclidean building, and $N(\Gamma) \subseteq X$ is a set (defined in Section 8) that both contains, and is contained in, a metric neighborhood of a $\mathbf{G}(\mathcal{O}_S)$ orbit. The existence of such a quasi-isometric embedding follows from a theorem of Lubotzky-Mozes-Raghunathan [L-M-R].

Our goal is to show that ϕ is within a finite distance of an element of $\mathrm{Isom}(X) \cong \mathrm{Aut}(G)$.

Constructing a boundary function defined a.e. In logical order, our proof begins with Section 8. Following Eskin, we apply basic ergodic theory to show that the generic flat $F \subseteq X$ has most of its volume contained in $N(\Gamma)$. We denote this generic collection of flats by $\mathfrak{U}$, and we note that in general, $\mathfrak{U}$ is a proper subset of the set of all flats in X.

For any flat $F \in \mathfrak{U}$, the quasi-isometric embedding ϕ restricts to a quasi-isometric embedding

$$\Omega_F' \longrightarrow X,$$

where $\Omega_F' \subseteq F \cap N(\Gamma)$ is a suitably large subset of F. By precomposing with a closest point projection, we have maps

$$\phi_F : F \longrightarrow X.$$

We analyze the image of these maps using the quasiflats with holes theorem of [W1], and we use the asymptotic behavior of the images to construct a function

$$\partial\phi : U_\partial \to \mathcal{B}(G),$$

where $\mathcal{B}(G)$ is the spherical Tits building for G and $U_\partial \subseteq \mathcal{B}(G)$ is a subcomplex that has full measure in the Furstenberg boundary.

For this task, we mostly defer to the proof in [Es] which covers the case when X is a symmetric space. Indeed, Eskin's proof uses the geometry of symmetric spaces mostly to establish a few foundational lemmas. These lemmas are used to analyze the behavior at infinity of the quasiflats with holes. We supply the analogous foundational lemmas for the general space X, and then Eskin's proof applies to the more general setting.

Continuity of the boundary function on neighborhoods of faces. Section 9 is the final section of this paper. The first three lemmas of the section are meant as replacements for foundational lemmas in [Es], so that we can apply a proof from [Es] to derive a fourth lemma: the restriction of $\partial\phi$ to the simplicial neighborhood of a face of a maximal simplex in U_∂ is continuous.

Completing the boundary function. Our goal is to extend the domain of $\partial\phi$ to all of $\mathcal{B}(G)$. Then we can use Tits' theorem to show that $\partial\phi$ corresponds to an element of $\mathrm{Aut}(G)$. This step is the content of Section 6. Despite the fact that this section is the third part of our proof if it were presented in logical order, it is placed in the early portion of this paper as it is less technical than material from Sections 8 and 9, and as it contains material unlike that found in [Es].

Eskin's approach to finding an extension of $\partial\phi$, for the case when K is a number field and $S = V_K^\infty$, was to find a topological completion of $\partial\phi$. A restriction of $\partial\phi$ to a co-null subset of the Furstenberg boundary is shown to be bi-Hölder. Then $\partial\phi$ can be completed to a domain of $\mathcal{B}(G)$.

Eskin's argument relied on the fact that the Furstenberg boundary of a real semisimple Lie group is an analytic manifold and a topological manifold. In contrast, the Furstenberg boundary of a semisimple Lie group over a nonarchimedean local field is a Cantor set. Therefore, our approach is forced to deviate from Eskin's at this point.

We complete $\partial\phi$ algebraically, using the Borel-Tits classification of abstract homomorphisms between simple groups. We restrict $\partial\phi$ to a collection of countably many chambers in U_∂ (a spherical building for $\mathbf{G}$ over global fields) and argue that the restriction is induced by an injective homomorphism of rational points of algebraic groups. The homomorphism is specified by pairs: isomorphisms of algebraic groups and inclusions of global fields into local fields. We show the field inclusions are continuous using the continuity of the boundary function on simplicial neighborhoods of faces of maximal simplices. Then we extend the restriction to an automorphism of G by completing the field inclusions. Finally, we show that the extension of the restriction is also an extension of $\partial\phi$.

See also [Dr] in the case when K is a number field and $S = V_K^\infty$ for a more combinatorial approach to this problem.

To conclude Section 6, a result of [Es] is applied to show that the automorphism of G which corresponds to $\partial\phi$, stabilizes $\mathbf{G}(\mathcal{O}_S)$ up to Hausdorff equivalence. We denote the group of all such automorphisms by $\mathrm{Aut}_{\mathrm{Hd}}(G; \mathbf{G}(\mathcal{O}_S))$. Therefore,

$$\mathcal{QI}(\mathbf{G}(\mathcal{O}_S)) \cong \mathrm{Aut}_{\mathrm{Hd}}(G; \mathbf{G}(\mathcal{O}_S)).$$

Automorphisms coarsely preserving lattices. If $\mathbf{G}$ is K-anisotropic, then G and $\mathbf{G}(\mathcal{O}_S)$ are Hausdorff equivalent so $\mathrm{Aut}_{\mathrm{Hd}}(G; \mathbf{G}(\mathcal{O}_S)) = \mathrm{Aut}(G)$. In Section 7 we show that $\mathrm{Aut}_{\mathrm{Hd}}(G; \mathbf{G}(\mathcal{O}_S))$ is a null subset of $\mathrm{Aut}(G)$ otherwise. We also show that $\mathrm{Aut}_{\mathrm{Hd}}(G; \mathbf{G}(\mathcal{O}_S)) = \mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ when G is K-isotropic and K is a number field.

5 Examples

This section will be especially useful for geometric group theorists who are not specialists in S-arithmetic lattices.

In this section we present six examples illustrating various aspects of Theorem 1.1. To focus on previously unknown results, the examples below will all be for the case that $\mathbf{G}$ is K-isotropic and $S \neq V_K^\infty$.

Example (A) The basic global field is $\mathbb{Q}$. It supports a countably infinite family of inequivalent valuations (which we think of as metrics for the global field): an "infinite" valuation and an l-adic valuation for every prime number l. It is well known that these are the only valuations supported on $\mathbb{Q}$.

The infinite valuation $v^\infty : \mathbb{Q} \to \mathbb{R}$ is obtained by embedding $\mathbb{Q}$ into $\mathbb{C}$ and then restricting the standard metric on $\mathbb{C}$. Any valuation on a global field that is obtained through an embedding into $\mathbb{C}$ is called *archimedean*. By completing $\mathbb{Q}$ metrically with respect to v^∞ we obtain the real numbers. In the notation of Section 1, this is written as $\mathbb{Q}_{v^\infty} = \mathbb{R}$.

The only archimedean valuation on $\mathbb{Q}$ is v^∞, but there are still the *nonarchimedean* l-adic valuations $v_{(l)}$ for prime numbers l. First, we define for any integer k, the natural number $\deg_l(k)$ as the exponent of l occurring in the prime factorization of k. Then, we define $v_{(l)} : \mathbb{Q} \to \mathbb{R}$ by

$$\left|\frac{n}{m}\right|_{v_{(l)}} = \exp\Big(\deg_l(m) - \deg_l(n)\Big).$$

Hence, the defining feature of the l-adic valuation is that it treats the size of powers of l backwards from what our intuition is used to from the archimedean valuation. That is $|l^n|_{v_{(l)}} \to 0$ as $n \to \infty$, and $|1/l^n|_{v_{(l)}} \to \infty$ as $n \to \infty$.

The l-adic valuation on $\mathbb{Q}$ is not complete. If we complete $\mathbb{Q}$ with respect to $v_{(l)}$, we obtain the *l-adic numbers* $\mathbb{Q}_{v_{(l)}}$ which is written simply as $\mathbb{Q}_l$. The l-adic numbers are locally compact and totally disconnected.

If we fix a prime number p and let $S = \{v^\infty, v_{(p)}\}$, then

$$\mathcal{O}_S = \{\, x \in \mathbb{Q} \mid 1 \geq |x|_{v_{(l)}} \text{ for all primes } p \neq l \,\} = \mathbb{Z}[1/p].$$

Because $L - \mathrm{rank}(\mathbf{SL_3}) = 2$ for all fields L, Theorem 1.1 applies to $\mathbf{SL_3}(\mathbb{Z}[1/p])$. Since $\mathbb{Q}$ admits no nontrivial automorphisms, the image of $\mathbf{SL_3}$ under the adjoint representation is $\mathbf{PGL_3}$, and transpose-inverse is the only outer automorphism of $\mathbf{PGL_3}$, we have

$$\mathcal{QI}(\mathbf{SL_3}(\mathbb{Z}[1/p])) \cong \mathbf{PGL_3}(\mathbb{Q}) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Notice that as abstract groups,

$$\mathcal{QI}(\mathbf{SL_3}(\mathbb{Z}[1/p])) \cong \mathcal{QI}(\mathbf{SL_3}(\mathbb{Z}[1/l]))$$

for any primes p and l. However this isomorphism is not topological. Indeed, $\mathcal{QI}(\mathbf{PGL_3}(\mathbb{Z}[1/p]))$ is the quotient of a space of functions so it has a quotient topology descending from the compact-open topology. This topology is equivalent to the subspace topology on $\mathbf{PGL_3}(\mathbb{Q})$ inherited from the diagonal embedding

$$\mathbf{PGL_3}(\mathbb{Q}) \to \mathbf{PGL_3}(\mathbb{R}) \times \mathbf{PGL_3}(\mathbb{Q}_p).$$

With this natural topological structure, the sequence of quasi-isometry classes given by

$$\begin{pmatrix} 1 & 0 & p^{-n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $n \in \mathbb{N}$ is discrete in $\mathcal{QI}(\mathbf{PGL_3}(\mathbb{Z}[1/p]))$, but not in $\mathcal{QI}(\mathbf{PGL_3}(\mathbb{Z}[1/l]))$. In particular, $\mathbf{SL_3}(\mathbb{Z}[1/p])$ and $\mathbf{SL_3}(\mathbb{Z}[1/l])$ are not quasi-isometric if $p \neq l$.

Example (B) Expanding on the previous example, we let P be any finite set of prime numbers. Then for the finite set of valuations $S = \{v^\infty\} \cup \{v_{(p)}\}_{p\in P}$, the ring $\mathcal{O}_S$ is:

$$\{\, x \in \mathbb{Q} \mid 1 \geq |x|_{v_{(l)}} \text{ for all primes } l \notin P \,\} = \mathbb{Z}[1/m_P],$$

where $m_P = \prod_{p\in P} p$.

Expanding on the previous example in another direction, recall that for any field L, the rank of $\mathbf{SL_n}$ over L is $n-1$. Hence, as long as $n \geq 3$ we have

$$\mathcal{QI}(\mathbf{SL_n}(\mathbb{Z}[1/m_P])) \cong \mathbf{PGL_n}(\mathbb{Q}) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Again we note that $\mathcal{QI}(\mathbf{SL_n}(\mathbb{Z}[1/m_P]))$ has a natural topology equivalent to the topology obtained via the diagonal embedding

$$\mathbf{PGL_n}(\mathbb{Q}) \to \mathbf{PGL_n}(\mathbb{R}) \times \prod_{p\in P} \mathbf{PGL_n}(\mathbb{Q}_p).$$

Hence $\mathcal{QI}(\mathbf{SL_n}(\mathbb{Z}[1/m_P]))$ becomes “more discrete" as the finite set P grows.

Also notice that the semisimple Lie group

$$\mathbf{PGL_n}(\mathbb{R}) \times \prod_{p \in P} \mathbf{PGL_n}(\mathbb{Q}_p)$$

is an index two subgroup of the topological closure of $\mathcal{QI}(\mathbf{SL_n}(\mathbb{Z}[1/m_P]))$. Hence, the quasi-isometry class of $\mathbf{SL_n}(\mathbb{Z}[1/m_P])$ identifies the ambient semisimple Lie group that contains $\mathbf{SL_n}(\mathbb{Z}[1/m_P])$ as a lattice.

Example (C) Examine the quadratic form

$$\Phi = x_1^2 + 2x_2^2 - \sqrt{2}x_3^2 + \sum_{i=4}^{5}(x_i^2 - x_{i+2}^2).$$

As Φ is defined over $\mathbb{Q}(\sqrt{2})$, the special orthogonal group $\mathbf{SO}_\Phi$ is a $\mathbb{Q}(\sqrt{2})$-group.

There are exactly two archimedean valuations supported on $\mathbb{Q}(\sqrt{2})$. They are obtained from the embeddings $a + \sqrt{2}b \mapsto a + \sqrt{2}b \in \mathbb{C}$ and $a + \sqrt{2}b \mapsto a - \sqrt{2}b \in \mathbb{C}$. Call these valuations v_1^∞ and v_2^∞ respectively, and note that $\mathbb{Q}(\sqrt{2})_{v_1^\infty}$ and $\mathbb{Q}(\sqrt{2})_{v_2^\infty}$ are each isomorphic to $\mathbb{R}$ as topological fields, but each in a different way.

We want to add a nonarchimedean valuation to our example. Since 3 does not split as a product of two primes in $\mathbb{Z}[\sqrt{2}]$, there is a unique extension of the 3-adic valuation to $\mathbb{Q}(\sqrt{2})$ (written as $v_{(3)}$), and $\mathbb{Q}(\sqrt{2})_{v_{(3)}} \cong \mathbb{Q}_3(\sqrt{2})$.

Let $S = \{v_1^\infty, v_2^\infty, v_{(3)}\}$. Then $\mathcal{O}_S = \mathbb{Z}[\sqrt{2}, 1/3]$. We can apply Theorem 1.1 since the rank of $\mathbf{SO}_\Phi$ over both $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2})_{v_2^\infty}$ is 2, and the rank of $\mathbf{SO}_\Phi$ over both $\mathbb{Q}(\sqrt{2})_{v_1^\infty}$ and $\mathbb{Q}(\sqrt{2})_{v_{(3)}}$ is 3. (That $\mathbb{Q}(\sqrt{2})_{v_{(3)}} - \mathrm{rank}(\mathbf{SO}_\Phi) = 3$ follows from the fact that $\sqrt{-2} \in \mathbb{Q}_3$.)

There is a nontrivial element of $\mathrm{Aut}(\mathbb{Q}(\sqrt{2}))$. Namely σ where $\sigma(a+b\sqrt{2}) = a-b\sqrt{2}$. However, while $\sigma S = S$, there is no $\mathbb{Q}(\sqrt{2})$-isomorphism between ${}^\sigma\mathbf{SO}_\Phi$ and $\mathbf{SO}_\Phi$. Indeed, ${}^\sigma\mathbf{SO}_\Phi$ and $\mathbf{SO}_\Phi$ are not even isomorphic over $\mathbb{R}$ as ${}^\sigma\Phi$ has signature $(5,2)$ and Φ has signature $(4,3)$. Hence, $\mathrm{Aut}(\mathbb{Q}(\sqrt{2}))_{\mathbf{G},S}$ is trivial (as is $\mathbf{Out}(\mathbf{SO}_\Phi)$) so Theorem 1.1 yields

$$\mathcal{QI}\Big(\mathbf{SO}_\Phi(\mathbb{Z}[\sqrt{2}, 1/3])\Big) \cong \mathbf{SO}_\Phi(\mathbb{Q}(\sqrt{2})).$$

Example (D) The symplectic group $\mathbf{SP_6}$ has rank 3 over any field. For the global field $\mathbb{Q}(i)$, we take the lone archimedean valuation v^∞ (given by restricting the standard metric on $\mathbb{C}$) along with the $(2+i)$-adic and the $(2-i)$-adic valuations to comprise the set S. (Note that $2+i$ and $2-i$ are prime in $\mathbb{Z}[i]$.)

Obviously $\mathbb{Q}(i)_{v^\infty} \cong \mathbb{C}$, and because $(2+i)(2-i)=5$, both $\mathbb{Q}(i)_{v_{(2+i)}}$ and $\mathbb{Q}(i)_{v_{(2-i)}}$ are isomorphic to $\mathbb{Q}_5$. Now

$$\mathcal{QI}\Big(\ \mathbf{SP_6}\ (\mathbb{Z}\big[i, \frac{1}{2+i}, \frac{1}{2-i}\big])\Big) \cong \mathbf{PSP_6}(\mathbb{Q}(i)) \rtimes \mathbb{Z}/2\mathbb{Z},$$

where $\mathbf{PSP_6}$ is the adjoint group of $\mathbf{SP_6}$. The nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ represents the automorphism σ of $\mathbb{Q}(i)$ defined by $\sigma(i) = -i$. Complex conjugation clearly stabilizes S, and ${}^{\sigma}\mathbf{PSP_6} = \mathbf{PSP_6}$ since $\mathbf{PSP_6}$ is defined over $\mathbb{Q}$.

Example (E) Let $\mathbb{F}_q$ be the finite field with q elements, and let $\mathbb{F}_q(t)$ be the field of rational functions with indeterminate t and coefficients in $\mathbb{F}_q$. This is the primary example of a global function field. All other global function fields are finite algebraic extensions of $\mathbb{F}_q(t)$ in analogy with the role $\mathbb{Q}$ plays for number fields.

The characteristic of $\mathbb{F}_q(t)$ is nonzero so there are no embeddings of this field into $\mathbb{C}$ and, hence, no archimedean valuations.

Examine the valuation of $\mathbb{F}_q(t)$ at infinity, v_∞, defined on quotients of polynomials by

$$\Big|\frac{p(x)}{q(x)}\Big|_{v_\infty} = \exp\Big(\deg(p(t)) - \deg(q(t))\Big).$$

Note that v_∞ measures the degree of the pole of a rational function at $\infty \in \mathbb{P}^1(\overline{\mathbb{F}}_q)$, where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$.

We could define an analogous valuation, v_p, for every point $p \in \mathbb{P}^1(\overline{\mathbb{F}}_q)$. The ring of functions $f \in \mathbb{F}_q(t)$ for which $|f|_{v_p} \leq 1$ for all $p \in \mathbb{P}^1(\overline{\mathbb{F}}_q) - \{\infty\}$ are precisely those rational functions which have no poles in $\mathbb{P}^1(\overline{\mathbb{F}}_q) - \{\infty\}$. Equivalently, the ring above is simply the ring of polynomials with indeterminate t. In the notation used in Section 1, we have $\mathcal{O}_S = \mathbb{F}_q[t]$ for $S = \{v_\infty\}$.

Completing $\mathbb{F}_q(t)$ with respect to v_∞ produces the locally compact field of formal Laurent series $\mathbb{F}_q((t^{-1}))$ with indeterminate t^{-1}. Hence, we have by Theorem 1.1 that

$$\mathcal{QI}\Big(\mathbf{SL_n}(\mathbb{F}_q[t])\Big) < \Big(\ \mathbf{PGL_n}\ \big(\mathbb{F}_q((t^{-1}))\big) \rtimes \mathbb{Z}/2\mathbb{Z}\Big) \rtimes \mathrm{Aut}\ \big(\mathbb{F}_q((t^{-1}))\big)$$

for all $n \geq 3$. We remark that $\mathrm{Aut}\ \big(\mathbb{F}_q((t^{-1}))\big)$ is profinite and in particular is compact.

It will be shown in [W2] however, that for this particular example the quasi-isometry group is determined exactly as it is in the number field case. That is,

$$\mathcal{QI}\big(\mathbf{SL_n}(\mathbb{F}_q[t])\big) \cong \Big(\ \mathbf{PGL_n}\ \big(\mathbb{F}_q(t)\big) \rtimes \mathbb{Z}/2\mathbb{Z}\Big) \rtimes B,$$

where B is a finite solvable subgroup of $\mathbf{PGL_2}(\mathbb{F}_q)$. Precisely, B is the group of $\mathbb{F}_q$-points of $\mathbf{PGL_2} \cong \mathbf{Aut}\,(\mathbb{P}^1)$ that stabilize our distinguished point $\infty \in \mathbb{P}^1(\overline{\mathbb{F}}_q)$.

Example (F) We give a final example involving function fields for which I do not at this time know of a proof that the quasi-isometry group is exactly the subgroup of $\mathrm{Aut}(G)$ consisting commensurators.

Examine the smooth elliptic curve C over $\mathbb{F}_5$ given by the equation $y^2 = t^3 - t$. The field of $\mathbb{F}_5$-rational functions on C is $\mathbb{F}_5(t, \sqrt{t^3 - t})$, and it is a separable extension of $\mathbb{F}_5(t)$.

Note that $(t = 2, y = 1)$ and $(t = 1, y = 0)$ define points on C which we name p and q respectively. We define valuations of $\mathbb{F}_5(t, \sqrt{t^3 - t})$ with respect to the points p and q as we did in the previous example, and we let $S = \{v_p, v_q\}$. Then $\mathcal{O}_S$ is the ring of regular functions on $C - \{p, q\}$.

Since $[\mathbb{F}_5(t, \sqrt{t^3 - t}) : \mathbb{F}_5(t)] = 2$, and since the point of C given by $(t = 2, y = 4)$ and the point p each lie above $2 \in \mathbb{P}^1(\mathbb{F}_5)$, we know by the so-called fundamental identity of valuation theory that $\mathbb{F}_5(t, \sqrt{t^3 - t})_{v_p} \cong \mathbb{F}_5((t - 2))$.

As the point $q \in C$ is the only point on C with $t = 1$ (i.e. q is a point of ramification) $\sqrt{t^3 - t} \notin \mathbb{F}_5(t)_{w_1}$ where w_1 denotes the valuation of $\mathbb{F}_5(t)$ at the point $1 \in \mathbb{P}^1(\overline{\mathbb{F}_5})$. Hence, $\mathbb{F}_5(t, \sqrt{t^3 - t})_{v_q} \cong \mathbb{F}_5((t - 1))(\sqrt{t^3 - t})$.

Now we are set to apply Theorem 1.1 which states in this case that

$$\mathcal{QI}\Big(\mathbf{SP_6}(\mathcal{O}_S)\Big)$$

is contained as a measure zero subgroup of the direct product of

$$\mathbf{PSP_6}\left(\mathbb{F}_5((t-2))\right) \rtimes \mathrm{Aut}\left(\mathbb{F}_5((t-2))\right)$$

with

$$\mathbf{PSP_6}\left(\mathbb{F}_5((t-1))(\sqrt{t^3-t})\right) \rtimes \mathrm{Aut}\left(\mathbb{F}_5((t-1))(\sqrt{t^3-t})\right).$$

This is a stronger result than the one that is known to hold in the K-anisotropic case, but it is an incomplete result. There is evidence to suggest that there should be an isomorphism

$$\mathcal{QI}\Big(\mathbf{SP_6}(\mathcal{O}_S)\Big) \cong \mathbf{PSP_6}\left(\mathbb{F}_5(t, \sqrt{t^3-t})\right).$$

Note that it can be shown that $\mathrm{Aut}(\mathbb{F}_5(t, \sqrt{t^3 - t}))_{\mathbf{G},S}$ is trivial since there are no nontrivial automorphisms of C which fix the point p and the point q.

Corollary 1.2(i) would hold for $\mathbf{SP_6}(\mathcal{O}_S)$ if the above isomorphism existed.

6 Completing the boundary function

Let $\mathbf{G}(\mathcal{O}_S)$ be as in Theorem 1.1. Since $\mathbf{G}(\mathcal{O}_S)$ and $\mathbf{Ad}(\mathbf{G})(\mathcal{O}_S)$ are commensurable up to finite kernels (see e.g. [Mar] I.3.1.1.iv),

$$\mathcal{QI}(\mathbf{G}(\mathcal{O}_S)) \cong \mathcal{QI}(\mathbf{Ad}(\mathbf{G})(\mathcal{O}_S)).$$

Thus we may, and will, assume throughout the remainder that $\mathbf{G}$ is of adjoint type.

Let

$$G = \prod_{v \in S - V_K^{\mathbf{G},a}} \mathbf{G}(K_v).$$

Let X be the natural product of nonpositively curved symmetric spaces and Euclidean buildings on which G acts by isometries and such that $\mathrm{Isom}(X)/G$ is compact. In this case $\mathrm{Isom}(X) \cong \mathrm{Aut}(G)$.

Throughout we let n equal the rank of X. (Recall the rank of X is the maximal dimension of a flat in X.)

Two boundaries. For any point $e \in X$, there is a natural topology on the space of directions from e which forms a simplicial complex $\mathcal{B}(G)$, called the *spherical Tits building for* G. The spherical building is $(n-1)$-dimensional, and it is the same as the spherical building for G that is produced using the standard BN pair construction. Hence, group automorphisms of G induce simplicial automorphisms of $\mathcal{B}(G)$.

A subset $L \subseteq X$ is called a *wall* if it is a codimension 1 affine subspace of a flat that is contained in at least two distinct flats. A *Weyl chamber* in X is the closure of a connected component of a flat $F \subseteq X$ less all the walls containing a fixed point $x \in F$. Most of the time we will not care about the point x which was used to create a Weyl chamber. In those cases when the distinction is important, we say any such Weyl chamber is *based* at x. (This is different terminology than was used in [W1]. See the word of caution following the discussion of the Furstenberg metric.)

The *Furstenberg boundary of* X is the compact space of maximal simplices in $\mathcal{B}(G)$. We denote it by $\widehat{X}$. It can be defined equivalently as the space of Weyl chambers in X modulo the relation that two Weyl chambers are equivalent if they are a finite Hausdorff distance from each other.

If $X = X_\infty \times X_\mathfrak{p}$, where X_∞ is a symmetric space and $X_\mathfrak{p}$ and a Euclidean building, then $\widehat{X} = \widehat{X}_\infty \times \widehat{X}_\mathfrak{p}$.

Furstenberg metric. There are metrics on $\widehat{X}_\infty$ and $\widehat{X}_\mathfrak{p}$ that are invariant under a fixed isotropy subgroup of $\mathrm{Isom}(X_\infty)$ and $\mathrm{Isom}(X_\mathfrak{p})$ respectively. The metric on $\widehat{X}_\infty$ is well-known.

To define the metric on $\widehat{X}_\mathfrak{p}$, we begin by choosing a point $x \in X_\mathfrak{p}$ and a representative Weyl chamber $\mathfrak{S} \subseteq X_\mathfrak{p}$ for every equivalence class in $\widehat{X}_\mathfrak{p}$ such that $\mathfrak{S}$ is based at x. Thus, we regard $\widehat{X}_\mathfrak{p}$ as the space of all Weyl chambers based at x.

For any Weyl chamber based at x, say $\mathfrak{S}$, let $\gamma_\mathfrak{S} : [0, \infty) \to \mathfrak{S}$ be the geodesic ray such that $\gamma_\mathfrak{S}(0) = x$ and such that $\gamma_\mathfrak{S}(\infty)$ is the center of mass of the boundary at infinity of $\mathfrak{S}$ with its usual spherical metric.

We endow $\widehat{X}_\mathfrak{p}$ with the metric $\widehat{d}_\mathfrak{p}$ where

$$\widehat{d}_\mathfrak{p}(\mathfrak{Y}, \mathfrak{Z}) = \begin{cases} \pi, & \text{if } \gamma_\mathfrak{Y} \cap \gamma_\mathfrak{Z} = \{x\}; \\ \exp\big(- |\gamma_\mathfrak{Y} \cap \gamma_\mathfrak{Z}|\big), & \text{otherwise.} \end{cases}$$

In the above, $|\gamma_\mathfrak{Y} \cap \gamma_\mathfrak{Z}|$ is the length of the geodesic segment $\gamma_\mathfrak{Y} \cap \gamma_\mathfrak{Z}$.

Note that $\widehat{d}_\mathfrak{p}$ is invariant under the action of the stabilizer of x and is a complete ultrametric on $\widehat{X}_\mathfrak{p}$. That $\widehat{d}_\mathfrak{p}$ is an ultrametric means that it is a metric and

$$\widehat{d}_\mathfrak{p}(\mathfrak{Y}, \mathfrak{Z}) \leq \max\{\widehat{d}_\mathfrak{p}(\mathfrak{Y}, \mathfrak{X}), \widehat{d}_\mathfrak{p}(\mathfrak{X}, \mathfrak{Z})\} \qquad \text{for any } \mathfrak{Y}, \mathfrak{Z}, \mathfrak{X} \in \widehat{X}_\mathfrak{p}.$$

We endow $\widehat{X}$ with the metric $\widehat{d} = \max\{\widehat{d}_\infty, \widehat{d}_\mathfrak{p}\}$.

Caution. In [W1], Weyl chambers in buildings are called sectors, and the metric $\widehat{d}_\mathfrak{p}$ is given a different form. In [W1], we made arguments by projecting onto the factors of X, and most of the paper analyzed the geometry of Euclidean buildings. Thus, our proof was geared towards terminology and tools more common for buildings. In this paper, we favor terminology and metrics for buildings which are more compatible with their better established symmetric space counterparts.

A boundary function defined a.e. In Section 8, we will define a group Γ that acts on X and is isomorphic to $\mathbf{G}(\mathcal{O}_S)$ up to finite groups (Γ is a lattice in the simply connected cover of G). We will also define a Γ-invariant set $N(\Gamma) \subseteq X$ such that $\Gamma \backslash N(\Gamma)$ is compact. A theorem of Lubotzky-Mozes-Raghunathan [L-M-R] states that Γ is quasi-isometric to any metric neighborhood of an orbit of Γ in X. Hence, if we are given a quasi-isometry of $\mathbf{G}(\mathcal{O}_S)$, we may replace it with an equivalent quasi-isometric embedding

$$\phi : N(\Gamma) \longrightarrow N(\Gamma) \subseteq X.$$

Every direction in X (i.e. every geodesic ray) is contained in a flat. In Section 8 we will show that enough flats in X have enough of their volume contained in $N(\Gamma)$ to enable us to construct a boundary function

$$\partial\phi : U_\partial \to \mathcal{B}(G),$$

where U_∂ is a subcomplex of $\mathcal{B}(G)$ that has full measure in $\widehat{X}$. The function $\partial\phi$ is a simplicial isomorphism of U_∂ onto its image.

We state below a lemma on a topological property of $\partial\phi$ that is proved in Section 9. First, we define $\mathcal{N}(f)$ as the simplicial neighborhood in $\mathcal{B}(G)$ of a fixed $(n-2)$-dimensional simplex $f \subset \mathcal{B}(G)$. That is, $\mathcal{N}(f)$ is the set of all chambers in $\mathcal{B}(G)$ containing f. We define $\mathcal{N}_U(f)$ to be the simplicial neighborhood of f in U_∂, or $\mathcal{N}(f) \cap U_\partial$.

Lemma 9.4 *If $f \subset U_\partial$ is a simplex of dimension $n-2$, then $\partial\phi|_{\mathcal{N}_U(f)}$ is continuous in the Furstenberg metric.*

Our goal is to show that $\partial\phi$ is the restriction of an automorphism of $\mathcal{B}(G)$ which is continuous on $\widehat{X}$. Then by Tits' Theorem, $\partial\phi$ is induced by an element of $\mathrm{Aut}(G) \cong \mathrm{Isom}(X)$. Knowing this would enable us to apply an argument of Eskin's to show further that $\partial\phi$ corresponds to an isometry of X which is a finite distance in the sup norm from ϕ.

Embeddings of spherical buildings. An *embedding of spherical buildings* $\mathcal{B}_1$ into $\mathcal{B}_2$ is a function $f : \mathcal{B}_1 \to \mathcal{B}_2$ that restricts to a simplicial isomorphism between $\mathcal{B}_1$ and $f(\mathcal{B}_1)$.

We wish to describe a particularly nice class of embeddings that play a key role in our proof. These are embeddings which arise from extremely well behaved homomorphisms of rational points of simple groups. We begin by describing the latter.

Let k be an arbitrary field and $\mathbf{H}$ a simple k-group. If k' is an extension of k, then there are injective group homomorphisms of $\mathbf{H}(k)$ into $\mathbf{H}(k')$ of the form $\beta \circ \psi^0$, where $\psi : k \to k'$ is an injective homomorphism of fields and $\beta : {}^{\psi}\mathbf{H} \to \mathbf{H}$ is a k'-isomorphism of algebraic groups. Any such homomorphism will be called *standard*.

Now let $\mathcal{B}(\mathbf{H}(k))$ and $\mathcal{B}(\mathbf{H}(k'))$ be the spherical buildings for $\mathbf{H}(k)$ and $\mathbf{H}(k')$ respectively. A standard homomorphism induces an embedding $f : \mathcal{B}(\mathbf{H}(k)) \to \mathcal{B}(\mathbf{H}(k'))$. We call any such embedding *standard* as well.

Implicit in theorems of Tits and Borel-Tits, is

Proposition 6.1 *Let* $\mathbf{H}$ *be a simple connected* k*-group of adjoint type and assume* k *is infinite. If* k' *is an extension of* k *with* $k - \mathrm{rank}(\mathbf{H}) = k' - \mathrm{rank}(\mathbf{H}) \geq 2$*, then any embedding* $\rho : \mathcal{B}(\mathbf{H}(k)) \to \mathcal{B}(\mathbf{H}(k'))$ *is standard.*

Proof Let $\mathbf{H}(k)^+$ be the subgroup of $\mathbf{H}(k)$ generated by the k-points of the unipotent radicals of k-parabolic subgroups of $\mathbf{H}$. In Chapter 5 of [Ti 2], Tits shows how to construct an injective group homomorphism $\rho_* : \mathbf{H}(k)^+ \to \mathbf{H}(k')$ which is induced by ρ. We have used the equal rank condition here.

We would like to be able to apply the well known theorem of Borel-Tits that classifies certain abstract homomorphisms between rational points of simple groups as being standard ([Bo-T] Theorem (A)).

By construction, ρ_* has a nontrivial image. Hence, our assumptions on $\mathbf{H}$ and k satisfy all of the hypotheses on ρ_* needed to apply the theorem of Borel-Tits except, possibly, for the condition that the image of ρ_* is Zariski dense in $\mathbf{H}$. If we let $\mathbf{M}$ be the the Zariski closure of the image of ρ_*, then our goal is to show that $\mathbf{M} = \mathbf{H}$.

By Corollary 6.7 of [Bo-T], we know that $\mathbf{H}(k)^+$ has no proper finite index subgroup. Hence, $\mathbf{M}$ must be connected. Also note that $\mathbf{M}$ modulo its radical, $\mathbf{R}(\mathbf{M})$, has positive dimension since $\mathbf{H}(k)^+$ is not solvable. In particular there exists a connected simple factor $\mathbf{L}$ of positive dimension of $\mathbf{M}/\mathbf{R}(\mathbf{M})$.

We postcompose ρ_* with the natural sequence of homomorphisms,

$$\mathbf{M} \to \mathbf{M}/\mathbf{R}(\mathbf{M}) \to \mathbf{L} \to \mathbf{Ad}(\mathbf{L}),$$

to obtain a homomorphism $\mathbf{H}(k)^+ \to \mathbf{Ad(L)}(k')$ with a nontrivial, Zariski dense image. Now we can apply Theorem (A) of [Bo-T] to conclude that there exists a field homomorphism $\psi : k \to k'$ and an isogeny ${}^{\psi}\mathbf{H} \to \mathbf{Ad}(\mathbf{L})$. Therefore,

$$\dim(\mathbf{H}) = \dim({}^{\psi}\mathbf{H}) = \dim(\mathbf{Ad}(\mathbf{L})) \leq \dim(\mathbf{M}/\mathbf{R}(\mathbf{M})) \leq \dim(\mathbf{M}).$$

Because $\mathbf{H}$ is connected and $\mathbf{M} \leq \mathbf{H}$, we conclude that $\mathbf{M} = \mathbf{H}$ as desired. We are then able to apply Theorem (A) of [Bo-T] to our original homomorphism ρ_* and arrive at our desired conclusion. □

A global sub-building. We would like to be able to apply Proposition 6.1 to an algebraically defined sub-building of $\mathcal{B}(G)$. We will need to begin by finding an extension of K, for each $v \in S - V_K^{\mathbf{G},a}$, that is contained in K_v and that satisfies the hypothesis of Proposition 6.1. This is the purpose of the following

Lemma 6.2 *For each $v \in S$, there is a finite algebraic extension L^v of K such that L^v is contained in K_v and $L^v - \mathrm{rank}(\mathbf{G}) = K_v - \mathrm{rank}(\mathbf{G})$.*

Proof Given a maximal K_v-torus $\mathbf{T} < \mathbf{G}$, there is a group element $g \in \mathbf{G}(K_v)$ such that ${}^g\mathbf{T}$ is defined over K, where ${}^g\mathbf{T}$ denotes the conjugate of $\mathbf{T}$ by g. See Section 7.1 Corollary 3 in [Pl-Ra] for a proof of this fact. It is assumed that K is a number field throughout most of [Pl-Ra], but the proof of this fact does not make an essential use of the number field assumption, aside from the proof of the K-rationality of the maximal toric variety of $\mathbf{G}$. For a proof of this last fact over arbitrary fields K, see [Bo-Sp].

Assume that $\mathbf{T}$ and g are as above and that $K_v - \mathrm{rank}(\mathbf{T}) = K_v - \mathrm{rank}(\mathbf{G})$. It is well known that there is a finite separable extension F^v of K over which ${}^g\mathbf{T}$ splits (see e.g. [Bo 2] 8.11). Hence, if $\mathbf{X}({}^g\mathbf{T})_L$ is the group of characters of ${}^g\mathbf{T}$ defined over an extension L of K, we have

$$\mathbf{X}({}^g\mathbf{T})_{K_v} = \mathbf{X}({}^g\mathbf{T})_{F_v} \cap \mathbf{X}({}^g\mathbf{T})_{K_v} = \mathbf{X}({}^g\mathbf{T})_{F_v \cap K_v}.$$

(Recall that a torus splits over a field L if and only if all of its characters are defined over L.)

Therefore, we let $L^v = F^v \cap K_v$ so that

$$K_v - \mathrm{rank}(\mathbf{T}) = K_v - \mathrm{rank}({}^g\mathbf{T}) = L^v - \mathrm{rank}({}^g\mathbf{T}).$$

Hence,

$$K_v - \mathrm{rank}(\mathbf{G}) \leq L^v - \mathrm{rank}(\mathbf{G}).$$

Since $L^v < K_v$, the inequality is an equality. □

We define the group

$$G_R = \prod_{v \in S - V_K^{\mathbf{G},a}} \mathbf{G}(L^v).$$

Let $\mathcal{B}(G_R)$ be the spherical building for G_R. By our choice of L^v, the building $\mathcal{B}(G_R)$ has countably many chambers, the dimensions of $\mathcal{B}(G_R)$ and $\mathcal{B}(G)$ are equal, and $\mathcal{B}(G)$ naturally contains $\mathcal{B}(G_R)$ as a subcomplex.

By conjugating $\mathbf{G}(\mathcal{O}_S)$, we can assume that $\mathcal{B}(G_R) \subseteq U_\partial$. Indeed, since $\mathcal{B}(G_R)$ has countably many chambers, we can appeal to Lemma 8.9 below.

Extending the global embedding. Define $\partial\phi_R$ as the restriction of $\partial\phi$ to $\mathcal{B}(G_R)$. The induced group homomorphism

$$\partial\phi_{R*} : \prod_{v\in S-V_K^{\mathbf{G},a}} \mathbf{G}(L^v)^+ \longrightarrow G$$

has a nontrivial image in each factor of G by construction. Also, Tits proved that each $\mathbf{G}(L^v)^+$ is an abstract simple group ([Ti 1] Main Theorem). It follows that $\partial\phi_{R*}$, and hence $\partial\phi_R$, preserves factors up to permutation.

Therefore we can apply Proposition 6.1 to conclude that $\partial\phi_R$ is induced by a family of standard homomorphisms. Precisely, there is a permutation τ of $S - V_K^{\mathbf{G},a}$, and for each $v \in S - V_K^{\mathbf{G},a}$ there exists an injective field homomorphism

$$\psi_v : L^v \to K_{\tau(v)}$$

and a $K_{\tau(v)}$-isomorphism of algebraic groups

$$\beta_v : {}^{\psi_v}\mathbf{G} \to \mathbf{G}$$

such that $\partial\phi_{R*}$ is the product of the homomorphisms

$$\beta_v \circ \psi_v^0 : \mathbf{G}(L^v)^+ \to \mathbf{G}(K_{\tau(v)}).$$

Now extending $\partial\phi_R$ amounts to extending each ψ_v. This is the technique of the proposition below. Before we continue though, we require an extra piece of notation.

Let $f \subset \mathcal{B}(G_R)$ be an $(n-2)$-dimensional simplex. We denote the simplicial neighborhood of f in $\mathcal{B}(G_R)$, or $\mathcal{N}(f) \cap \mathcal{B}(G_R)$, by $\mathcal{N}_R(f)$.

We continue with

Proposition 6.3 *The map $\partial\phi_R : \mathcal{B}(G_R) \to \mathcal{B}(G)$ uniquely extends to an embedding $\overline{\partial\phi_R} : \mathcal{B}(G) \to \mathcal{B}(G)$ which is uniformly continuous on the Furstenberg boundary.*

Proof Choose an apartment $\Sigma \subseteq \mathcal{B}(G_R) \subseteq \mathcal{B}(G)$ and a chamber $c \subseteq \Sigma$. For any $(n-2)$-dimensional simplex $f \subseteq c$, there exists a root space $R_f \subseteq \Sigma$ (as defined in [Ti 2] 1.12) such that $f \subseteq \partial R_f$.

By Proposition 3.27 in [Ti 2], any chamber in $\mathcal{N}_R(f)$ is contained in an apartment of $\mathcal{B}(G_R)$ which contains R_f. Therefore, by Proposition 5.6(i) of [Ti 2], there exists a valuation $w(f) \in S - V_K^{\mathbf{G},a}$ and an $L^{w(f)}$-defined root subgroup $\mathbf{U_{w(f)}} < \mathbf{G}$, such that $\mathbf{U_{w(f)}}(L^{w(f)})$ acts faithfully and transitively on $\mathcal{N}_R(f) - \{ c \}$.

The valuation $w(f)$ depends on a choice of f. However, for any valuation $v \in V_K^{\mathbf{G},a}$, we can choose a face $f_v \subseteq c$ such that $w(f_v) = v$. We assume we have chosen such a face f_v for all $v \in S - V_K^{\mathbf{G},a}$.

If $b_v \in \mathcal{N}_R(f_v) - \{ c_v \}$, then for any $u \in \mathbf{U_v}(L^v)$ we have $ub_v \in \mathcal{B}(G_R)$. Therefore,

$$\partial\phi(ub_v) = \partial\phi_R(ub_v) = \beta \circ \psi_v^0(u)\partial\phi_R(b_v)$$

Since $\mathcal{N}_R(f_v) \subseteq \mathcal{N}_U(f)$, it follows from Lemma 9.4 that $\beta \circ \psi_v^0$, and hence ψ_v, is continuous for all $v \in S - V_K^{\mathbf{G},a}$.

Using translation under addition, we see that ψ_v is also uniformly continuous. Therefore, we can complete ψ_v to $\overline{\psi_v} : \overline{L^v} \to K_{\tau(v)}$. Each $\overline{\psi_v}$ is injective since any field homomorphism is injective.

Now let $\overline{\partial\phi_R} : \mathcal{B}(G) \to \mathcal{B}(G)$ be the embedding induced by the homomorphisms $\beta_v \circ \overline{\psi_v}^0 : \mathbf{G}(\overline{L^v}) \to \mathbf{G}(K_{\tau(v)})$. The map $\overline{\partial\phi_R}$ is clearly continuous on the Furstenberg boundary, and since the Furstenberg boundary is compact, $\overline{\partial\phi_R}$ is uniformly continuous. □

If K is a number field then $\overline{\partial\phi_R}$ is an automorphism. In general though, it is not necessarily the case that a self-embedding of a spherical building is an automorphism. Take for example the spherical building for the standard flag complex of $\mathbb{P}^k(\mathbb{F}_q((t)))$ which is both isomorphic to, and properly contains, the flag complex for $\mathbb{P}^k(\mathbb{F}_q((t^2)))$.

The surjectivity of $\overline{\partial\phi_R}$ will be shown in Lemma 6.8 and must wait until we can show that $\overline{\partial\phi_R}$ extends $\partial\phi$. Then we can use the fact that $\partial\phi$ has a dense image.

Extending the a.e. defined boundary function. Our goal is to show that $\partial\phi$ is extended by $\overline{\partial\phi_R}$.

Earlier we chose each global field L^v to be large in an algebraic sense with respect to each K_v. We can also assume that each L^v is topologically large with respect to each K_v by choosing $L^v < K_v$ to be a dense subfield. Indeed, if L^v is not dense we could replace L^v with a finite extension that is dense in K_v. This will ensure that $\mathcal{B}(G_R)$ carries some of the topological information of $\mathcal{B}(G)$. In particular we have

Lemma 6.4 *For any $(n-2)$-dimensional simplex $f \subset \mathcal{B}(G_R)$, the set $\mathcal{N}_R(f)$ is dense in $\mathcal{N}(f) \subseteq \mathcal{B}(G)$ under the subspace topology of the Furstenberg topology.*

Proof Let $\Sigma_f \subseteq \mathcal{B}(G_R)$ be an apartment containing f, and suppose $c_f \subseteq \Sigma_f$ is a chamber containing f.

As in the proof of the previous lemma, there is a valuation $v \in S - V_K^{\mathbf{G},a}$ and an L^v-defined root subgroup $\mathbf{U} < \mathbf{G}$, such that $\mathbf{U}(L^v) < G_R$ acts faithfully and transitively on the set $\mathcal{N}_R(f) - \{ c_f \}$. It also follows from Proposition 5.6(i) of [Ti 2], that $\mathbf{U}(K_v) < G$ acts faithfully and transitively on the set $\mathcal{N}(f) - \{ c_f \}$. Therefore, $\mathbf{U}(L^v)$ is homeomorphic to $\mathcal{N}_R(f) - \{ c_f \}$, and $\mathbf{U}(K_v)$ is homeomorphic to $\mathcal{N}(f) - \{ c_f \}$.

Since L^v is dense in K_v, and because $\mathbf{U}$ is isomorphic as an L^v-variety to affine space, we have that $\mathbf{U}(L^v)$ is dense in $\mathbf{U}(K_v)$. Therefore, we have the following series of dense inclusions

$$\begin{aligned} \mathcal{N}_R(f) - \{ c_f \} &\subseteq \mathcal{N}(f) - \{ c_f \} \\ &\subseteq \mathcal{N}(f) \end{aligned}$$

□

Let F_R be the set of $(n-2)$-dimensional simplices in $\mathcal{B}(G_R)$ and define

$$D_R = \bigcup_{f \in F_R} \mathcal{N}_U(f)$$

We use the topological properties of $\mathcal{B}(G_R)$, and of $\overline{\partial\phi_R}$, to deduce topological properties of $\partial\phi|_{D_R}$ in the following

Lemma 6.5 *The function $\partial\phi|_{D_R} : D_R \to \mathcal{B}(G)$ is Furstenberg continuous.*

Proof Let $\varepsilon > 0$ and a chamber $c_1 \subset D_R$ be given.

By Proposition 6.3, there is a $\delta_R > 0$ such that

$$\widehat{d}\Big(\partial\phi(w_1)\,,\,\partial\phi(w_2)\Big) < \varepsilon/3$$

for all chambers $w_1, w_2 \subset \mathcal{B}(G_R)$ with $\widehat{d}(w_1, w_2) < \delta_R$.

Suppose $c_2 \subset D_R$ is a chamber with $\widehat{d}(c_1, c_2) < \delta_R/3$. By Lemma 6.4 and Lemma 9.4, there are chambers $c_i' \subset \mathcal{B}(G_R)$ that intersect c_i in an $(n-2)$-dimensional simplex, and such that $\widehat{d}(c_i, c_i') < \delta_R/3$ and $\widehat{d}(\partial\phi(c_i), \partial\phi(c_i')) < \varepsilon/3$. Hence

$$\widehat{d}(\partial\phi(c_1), \partial\phi(c_2)) \leq \widehat{d}(\partial\phi(c_1'), \partial\phi(c_2')) + \Sigma_{i=1}^2 \widehat{d}(\partial\phi(c_i), \partial\phi(c_i')) < \varepsilon. \qquad \square$$

Since $\partial\phi|_{D_R}$ and $\overline{\partial\phi_R}$ are continuous we have

Lemma 6.6 *For any simplex $q \subset D_R$, we have $\partial\phi(q) = \overline{\partial\phi_R}(q)$.*

Proof Both $\partial\phi|_{D_R}$ and $\overline{\partial\phi_R}|_{D_R}$ are continuous so they are uniquely determined by $\partial\phi_R$. Indeed, according to Lemma 6.4, $\mathcal{B}(G_R)$ is Furstenberg dense in D_R. $\square$

In Section 8, a maximal K_v-split torus $\mathbf{A_v} < \mathbf{G}$ is chosen for each $v \in S - V_K^{\mathbf{G},a}$. The tori are used to supply an ergodic theory argument that allows for the creation of the boundary function $\partial\phi : U_\partial \to \mathcal{B}(G)$.

Let $\Sigma_A \subset \mathcal{B}(G)$ be the apartment stabilized by the group

$$\prod_{v \in S - V_K^{\mathbf{G},a}} \mathbf{A_v}(K_v) < G.$$

By conjugating $\mathbf{G}(\mathcal{O}_S)$, we may assume that Σ_A is an apartment in $\Delta(G_R)$. Let W be the Weyl group with respect to Σ_A, and denote a fixed chamber in Σ_A by a^+. Let a^- be the chamber in Σ_A opposite of a^+. For each $w \in W$ we let $P_w < G$ be the stabilizer of wa^+.

In Section 8, we will see that there exists a co-null subset $\mathcal{U} \subseteq G$ such that $U_\partial = \mathcal{U}a^+$. By Fubini's theorem, we can conjugate $\mathbf{G}(\mathcal{O}_S)$ such that $P_w \cap \mathcal{U}$ is co-null in P_w for all $w \in W$.

Define

$$U_\partial^w = \{\, gwa^- \in \widehat{X} \mid g \in P_w \cap \mathcal{U} \,\}$$

and

$$\mathcal{U}^w = \{\, g \in \mathcal{U} \mid gwa^- \in \mathcal{U}_\partial^w \,\}.$$

Note that wa^- is opposite of wa^+, so we have that $P_w wa^-$ is a full measure subset of $\widehat{X}$. Since $P_w \cap \mathcal{U}$ is co-null in P_w, it follows that U_∂^w is a full measure subset of $\widehat{X}$. Hence, $\mathcal{U}^w \subseteq G$ is co-null for all $w \in W$. Consequently, $\cap_{w \in W}\mathcal{U}^w \subseteq G$ is co-null.

We replace $\mathcal{U}$ with $\cap_{w \in W}\mathcal{U}^w$. As a result, if $c \subset U_\partial$ is a chamber, then there is an apartment Σ_c which is completely contained in U_∂, and such that the chamber opposite from c in Σ_c is contained in Σ_A. For any chamber $c \subset U_\partial$, we let

$$\delta_A(c) = \min_{\Sigma_c}\{\, d_{\Sigma_c}(c, \Sigma_A) \,\},$$

where the min is taken over all Σ_c as above with respect to the Tits metric d_{Σ_c} on Σ_c.

We can now improve upon Lemma 6.6.

Lemma 6.7 *For any simplex $q \subset U_\partial$, we have $\partial\phi(q) = \overline{\partial\phi_R}(q)$.*

Proof For a chamber $c \subset U_\partial$, we prove that $\partial\phi(c) = \partial\phi_R(c)$ by induction on $\delta_A(c)$.

If $\delta_A(c) \leq 1$, then the result follows from the previous lemma. Now suppose the result is true for any chamber $f \subset U_\partial$ with $\delta_A(f) \leq k-1$, and let $c \subset U_\partial$ be a chamber with $\delta_A(c) = k$.

Let $\Sigma_c \subset U_\partial$ be an apartment containing c, and such that the chamber in Σ_c opposite of c is contained in Σ_A. Choose a chamber $f \subset \Sigma_c$ such that $d_{\Sigma_c}(c,f) = 1$ and $\delta_A(f) < k$. If f^{op} is the chamber in Σ_c opposite of f, then $\delta_A(f^{op}) \leq 1$. By our induction hypothesis, $\partial\phi(f) = \partial\phi_R(f)$ and $\partial\phi(f^{op}) = \partial\phi_R(f^{op})$.

It will be shown in Lemma 8.8 that $\partial\phi$ preserves apartments. Therefore, $\partial\phi(\Sigma_c)$ is an apartment. In fact, $\partial\phi(\Sigma_c)$ is the unique apartment containing $\partial\phi_R(f)$ and $\partial\phi_R(f^{op})$. Note that $\partial\phi_R(\Sigma_c)$ is also the unique apartment containing $\partial\phi_R(f)$ and $\partial\phi_R(f^{op})$.

We conclude our proof by observing that both $\partial\phi(c)$ and $\partial\phi_R(c)$ must be the unique chamber in $\partial\phi(\Sigma_c) = \partial\phi_R(\Sigma_c)$ that contains $\partial\phi(c \cap f) = \partial\phi_R(c \cap f)$, but not $\partial\phi(f) = \partial\phi_R(f)$. □

The extension is an automorphism. Now that we have shown that $\overline{\partial\phi_R}$ extends $\partial\phi$, we have to prove that $\overline{\partial\phi_R}$ is surjective, and hence an automorphism of $\mathcal{B}(G)$. Then it follows that $\overline{\partial\phi_R}$ corresponds to an automorphism of G, or alternatively, an isometry of X.

Lemma 6.8 *The map $\overline{\partial\phi_R}$ is an automorphism of $\mathcal{B}(G)$.*

Proof Let ϕ^* be a coarse inverse for ϕ, and define $U_{\partial *}$ and $\overline{\partial\phi^*_R}$ analogously to U_∂ and $\overline{\partial\phi_R}$.

Let $\Sigma \in U^*_\partial$, and let $F \subseteq X$ be the flat corresponding to Σ. Note that $\phi \circ \phi^*$ preserves the portion of F that lies near an orbit of $\mathbf{G}(\mathcal{O}_S)$ in X (see Section 8). Since F is the only flat in X that is a finite Hausdorff distance from itself, it follows that

$$\overline{\partial\phi_R} \circ \overline{\partial\phi^*_R}(\Sigma) = \Sigma.$$

Hence,

$$U^*_\partial \subseteq \overline{\partial\phi_R}\big(\mathcal{B}(G)\big).$$

Note that the map $\overline{\partial\phi_R}$ either has a closed null image or is surjective since $K_{\tau(v)}$ is a $\overline{\psi_v}(\overline{L^v})$-vector space. The lemma follows since U^*_∂ is co-null in $\widehat{X}$. □

Automorphisms that correspond to quasi-isometries. Let Hd denote the Hausdorff distance between closed subsets of G. We define the group

$$\mathrm{Aut}_{\mathrm{Hd}}(G;\mathbf{G}(\mathcal{O}_S)) = \{\, \varphi \in \mathrm{Aut}(G) \mid \mathrm{Hd}(\varphi(\mathbf{G}(\mathcal{O}_S)) \, , \, \mathbf{G}(\mathcal{O}_S)) < \infty \,\}.$$

Using Lemma 6.7 and Lemma 8.3(vii), Eskin's proof that the automorphism $\overline{\partial\phi_R} \in \mathrm{Aut}(G) \cong \mathrm{Isom}(X)$ corresponds to an isometry of X that is a finite distance from ϕ ([Es] Step 7) can be applied to show

Proposition 6.9 *There is an isomorphism*

$$\mathcal{QI}(\mathbf{G}(\mathcal{O}_S)) \cong \mathrm{Aut}_{\mathrm{Hd}}(G;\mathbf{G}(\mathcal{O}_S))$$

The proof proceeds by identifying points in X as intersections of flats in X. Flats are parameterized by apartments in $\mathcal{B}(G)$, so $\overline{\partial\phi_R}$ completely determines where points in X are mapped to under the corresponding isometry of X. Any point in a $\mathbf{G}(\mathcal{O}_S)$-orbit is a bounded distance from the intersection of flats whose boundaries are in U_∂. Therefore, ϕ maps points in a $\mathbf{G}(\mathcal{O}_S)$-orbit to within a bounded distance of their images under the isometry corresponding to $\overline{\partial\phi_R}$.

Eskin's proof makes no mention of the topological nature of this isomorphism, but it clearly follows. The fact that the isomorphism is topological is more interesting in the S-arithmetic setting since merely the abstract group type of the quasi-isometry group of an arithmetic lattice in a real semisimple Lie group determines the lattice up to commensurability.

7 Automorphisms coarsely preserving lattices

We want to determine the group $\mathrm{Aut}_{\mathrm{Hd}}(G;\mathbf{G}(\mathcal{O}_S))$ and complete our proof of Theorem 1.1.

The case of anisotropic groups. Notice that if $\mathbf{G}$ is K-anisotropic, then $\mathrm{Aut}_{\mathrm{Hd}}(G;\mathbf{G}(\mathcal{O}_S))$ is isomorphic to $\mathrm{Aut}(G)$. Indeed, $\mathbf{G}(\mathcal{O}_S)$ is a cocompact lattice in G so $\mathrm{Hd}(G\, , \, \mathbf{G}(\mathcal{O}_S)) < \infty$. Thus, our proof of Theorem 1.1(iii) is complete (assuming the results from Sections 8 and 9).

The function field case for isotropic groups. The proof of Theorem 1.1(ii) concludes with Lemma 7.1 below. We include the proof here to group it with similar results, but its proof uses notation and concepts defined in Section 8. The reader may want to return to the proof of this small fact after having read what will follow.

Lemma 7.1 *If* $\mathbf{G}$ *is* K*-isotropic, then the group* $\mathrm{Aut}_{\mathrm{Hd}}(G;\mathbf{G}(\mathcal{O}_S))$ *is a measure zero subgroup of* $\mathrm{Aut}(G)$*.*

Proof For a given element of $\mathrm{Aut}_{\mathrm{Hd}}(G;\mathbf{G}(\mathcal{O}_S))$, we let $g: X \to X$ be the corresponding isometry. We choose a neighborhood $N(\Gamma)^g \subseteq X$ of the set $N(\Gamma)$ from Lemma 8.3, such that $N(\Gamma) \subseteq g(N(\Gamma)^g)$.

Define vol_F to be Lebesgue measure on F, and let ε be as in Lemma 8.3. There is a Weyl chamber $\mathfrak{C} \subseteq X$ such that for any $g \in \mathrm{Aut}_{\mathrm{Hd}}(G;\mathbf{G}(\mathcal{O}_S))$, for any flat $F \subseteq X$ that contains $\mathfrak{C}$ up to Hausdorff equivalence, and for any point $x \in F$, we have

$$\lim_{r\to\infty} \frac{\mathrm{vol}_F\left(\left[F \cap N\left(\Gamma\right)^g\right] \cap B_x(r)\right)}{\mathrm{vol}_F\left(B_x(r)\right)} < 1 - \varepsilon.$$

Let $F' \subseteq X$ be a flat containing $g(\mathfrak{C})$ up to Hausdorff equivalence. Then, by replacing F with $g^{-1}(F')$ in the preceding inequality, it follows that for any point $y \in F'$:

$$\lim_{r\to\infty} \frac{\mathrm{vol}_{F'}\left(\left[F' \cap N\left(\Gamma\right)\right] \cap B_y(r)\right)}{\mathrm{vol}_{F'}\left(B_y(r)\right)} < 1 - \varepsilon.$$

Therefore, $F' \notin \mathfrak{U}$. Hence, if $c \subset \mathcal{B}(G)$ is the chamber representing the equivalence class of $\mathfrak{C}$, then $\mathrm{Aut}_{\mathrm{Hd}}(G;\mathbf{G}(\mathcal{O}_S)) \cdot c \subseteq \mathcal{B}(G) - U_\partial$.

The lemma follows from Fubini's theorem since U_∂ is co-null in $\widehat{X}$. □

The number field case for isotropic groups. The proof of the following proposition was indicated to me by Nimish Shah, and it completes the proof of Theorem 1.1(i).

Proposition 7.2 *If* K *is a number field and* $\mathbf{G}$ *is* K*-isotropic, then*

$$\mathrm{Aut}_{\mathrm{Hd}}\left(G;\mathbf{G}(\mathcal{O}_S)\right) = \mathrm{Comm}\left(\mathbf{G}(\mathcal{O}_S)\right).$$

Proof Let $\varphi \in \mathrm{Aut}_{\mathrm{Hd}}(G, \mathbf{G}(\mathcal{O}_S))$. We have to show that $\varphi \in \mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$.

To simplify notation we let $\Lambda = \mathbf{G}(\mathcal{O}_S)$ and $\Lambda^\varphi = \varphi(\mathbf{G}(\mathcal{O}_S))$. By replacing Λ with a finite index subgroup, we can assume that Λ and Λ^φ are contained in the group

$$G^+ = \prod_{v \in S} \mathbf{G}(K_v)^+.$$

By Ratner's theorem on unipotent flows ([Ra] Theorem 6.4), the orbit of the point $(\Lambda, \Lambda^\varphi)$ in $G^+/\Lambda \times G^+/\Lambda^\varphi$ under the diagonal action of G^+ is homogeneous. If we denote the diagonal embedding of G^+ into $G^+ \times G^+$ by ΔG^+, then the previous sentence says that

$$\overline{\Delta G^+(\Lambda, \Lambda^\varphi)} = L(\Lambda, \Lambda^\varphi),$$

where L is a closed subgroup of $G^+ \times G^+$ which contains ΔG^+.

We claim that either $L = \Delta G^+$ or there is some $v \in S$ such that

$$1 \times \mathbf{G}(K_v)^+ \leq (1 \times G^+) \cap L.$$

Indeed, if $\Delta G^+ < L$, then there are group elements $g_1, g_2 \in G^+$ such that $(g_1, g_2) \in L$ and $g_1 \neq g_2$. Hence, there is some $g \in G^+$ with $g \neq 1$ and $(1, g) \in L$. That is to say, $(1 \times G^+) \cap L$ is nontrivial. Note that if $(1, h) \in L$, then for any $g \in G^+$, we have $(1, ghg^{-1}) = (g, g)(1, h)(g^{-1}, g^{-1}) \in L$ since $\Delta G^+ < L$. Thus, $(1 \times G^+) \cap L$ is a normal subgroup of $1 \times G^+$. Now a theorem of Tits' ([Ti 1] Main Theorem) tells us that that each group $\mathbf{G}(K_v)^+$ is simple since $\mathbf{G}$ has a trivial center. Therefore, $1 \times \mathbf{G}(K_v)^+ \leq L$ for some $v \in S$, since $(1 \times G^+) \cap L$ is a nontrivial normal subgroup of $1 \times G^+$. Thus, our claim is proved.

If it is the case that $1 \times \mathbf{G}(K_v)^+ \leq L$, then $\overline{\Delta G^+(\Lambda, \Lambda^\varphi)}$ contains $\{\Lambda\} \times G^+/\Lambda^\varphi$, as Λ^φ is irreducible. Hence, for any $g \in G^+$, there is a sequence $\{g_k\} \subseteq G^+$ such that $\Delta g_k(\Lambda, \Lambda^\varphi) \to (\Lambda, g\Lambda^\varphi)$. Since $g_k\Lambda \to \Lambda$, it follows that there are sequences $\{h_k\} \subseteq G^+$ and $\{\lambda_k\} \in \Lambda$, such that $g_k = h_k\lambda_k$ and $h_k \to 1$. Therefore, $h_k^{-1}g_k \in \Lambda$ and $h_k^{-1}g_k\Lambda^\varphi \to g\Lambda^\varphi$ which proves that $\overline{\Lambda\Lambda^\varphi} = G^+/\Lambda^\varphi$. Note that our assumption that $\varphi \in \mathrm{Aut}_{\mathrm{Hd}}(G; \Lambda)$ implies that $\Lambda\Lambda^\varphi$ is bounded. Thus, this case is precluded.

We are left to consider the case when $\Delta G^+ = L$. We will show that $\Lambda\Lambda^\varphi \subseteq G^+/\Lambda^\varphi$ is a closed set. To this end, suppose there is a sequence $\{\lambda_k\} \subseteq \Lambda$ and a group element $g \in G^+$ with $\lambda_k\Lambda^\varphi \to g\Lambda^\varphi$. Then $\Delta\lambda_k(\Lambda, \Lambda^\varphi) \to (\Lambda, g\Lambda^\varphi)$. Since $\Delta G^+(\Lambda, \Lambda^\varphi)$ is closed, $(\Lambda, g\Lambda^\varphi) = \Delta h(\Lambda, \Lambda^\varphi)$ for some $h \in G^+$. Therefore, $g\Lambda^\varphi = h\Lambda^\varphi$. Since $h\Lambda = \Lambda$, we have $h \in \Lambda$ which shows that $\Lambda\Lambda^\varphi$ is closed.

Since $\Lambda\Lambda^\varphi$ is bounded, it must be compact which would require it to be finite or perfect. As perfect sets are known to be uncountable, $\Lambda\Lambda^\varphi$ is finite. That is $\varphi \in \mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ as desired. □

Assuming the material from Sections 8 and 9, the proof of Theorem 1.1 is complete. It is the absence of the counterpart to Proposition 7.2 for function fields that leads to the discrepancy between (i) and (ii) of Theorem 1.1 and Corollary 1.2.

The commensurator group. We close this section with a lemma that provides a concrete description of $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$.

Lemma 7.3 *The group* $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ *is an extension of* $\mathbf{Aut}(\mathbf{G})(K)$ *by* $\mathrm{Aut}(K)_{\mathbf{G},S}$*. If* $\mathbf{G}$ *is* K*-split and defined over a subfield of* K *that is fixed pointwise by* $\mathrm{Aut}(K)_{\mathbf{G},S}$*, then*

$$\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)) \cong \Big(\mathbf{G}(K) \rtimes \mathbf{Out}(\mathbf{G})(K)\Big) \rtimes \mathrm{Aut}(K)_{\mathbf{G},S}.$$

Proof Recall that $\mathbf{G}(\mathcal{O}_S)$ is embedded diagonally in G with respect to the simple factors of G. Hence, any group element in $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)) \cap G$ would have to take a finite index diagonal subgroup of $\mathbf{G}(\mathcal{O}_S)$ into the diagonal of G. It follows from the Borel density theorem that any finite index subgroup of $\mathbf{G}(\mathcal{O}_S)$ is a Zariski dense subset in each simple factor of G. Therefore, $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)) \cap G$ is also contained in the diagonal of G.

We have shown that, as an abstract group, $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)) \cap G$ is a subgroup of the group L of inner automorphisms of $\mathbf{G}(K_v)$ which commensurate $\mathbf{G}(\mathcal{O}_S) < \mathbf{G}(K_v)$; the choice of $v \in S - V_K^{\mathbf{G},a}$ is arbitrary.

Borel's well known determination of inner commensurators for arithmetic groups ([Bo 1] Theorem 2) essentially contains a proof that $L = \mathbf{G}(K) < \mathbf{G}(K_v)$. Therefore, $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)) \cap G$ is the diagonal subgroup $\Delta\, \mathbf{G}(K) < G$.

If $\varphi \in \mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$, then $\mathbf{G}(\mathcal{O}_S)$ and $\varphi(\mathbf{G}(\mathcal{O}_S))$ are commensurable. Hence, an inner automorphism of G commensurates $\mathbf{G}(\mathcal{O}_S)$ if and only if it commensurates $\varphi(\mathbf{G}(\mathcal{O}_S))$. Therefore, $\varphi(\Delta\, \mathbf{G}(K)) = \Delta\, \mathbf{G}(K)$.

Conversely, suppose φ is an automorphism of G with $\varphi(\Delta\, \mathbf{G}(K)) = \Delta\, \mathbf{G}(K)$. Then $\varphi(\mathbf{G}(\mathcal{O}_S))$ is a lattice contained in $\Delta\, \mathbf{G}(K)$, so $\varphi(\mathbf{G}(\mathcal{O}_S))$ is commensurable to $\mathbf{G}(\mathcal{O}_S)$ by the proof of the Margulis-Venkataramana arithmeticity theorem (see [Mar] pages 307-311). Therefore, $\varphi \in \mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$.

Hence, finding $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ amounts to finding the subgroup of $\mathrm{Aut}(G)$ that stabilizes $\Delta\, \mathbf{G}(K)$. This is what we shall do.

Suppose $\psi \in \mathrm{Aut}(G)$ and that $\psi(\Delta\, \mathbf{G}(K)) = \Delta\, \mathbf{G}(K)$. By Theorem (A) of [Bo-T], $\psi \in \mathrm{Aut}(G)$ can be uniquely written in the form

$$\prod_{v \in S - V_K^{\mathbf{G},a}} \beta_v \circ \alpha_v^{\circ}$$

for some permutation τ of $S - V_K^{\mathbf{G},a}$, a collection of field isomorphisms $\alpha_v : K_v \to K_{\tau(v)}$, and a collection $\beta_v : {}^{\alpha_v}\mathbf{G} \to \mathbf{G}$ of $K_{\tau(v)}$-isomorphisms of algebraic groups. Since ψ is a homeomorphism, each field isomorphism α_v is a homeomorphism as well.

Since $\Delta\,\mathbf{G}(K)$ is stabilized by ψ,

$$\beta_v \circ \alpha_v^\circ|_{\mathbf{G}(K)} = \beta_w \circ \alpha_w^\circ|_{\mathbf{G}(K)}$$

for all $v, w \in S - V_K^{\mathbf{G},a}$. Again by Theorem (A) of [Bo-T], there exists a unique $\sigma \in \mathrm{Aut}(K)$ and a unique K-isomorphism of algebraic groups $\delta : {}^{\sigma}\mathbf{G} \to \mathbf{G}$, such that $\delta \circ \sigma^0$ is extended by all $\beta_v \circ \alpha_v^\circ$.

Because each α_v is a homeomorphism, $\sigma : K \to K$ is a homeomorphism between K with the v-topology and K with the $\tau(v)$-topology. Therefore, $\tau(v) = \sigma \cdot v$ for all $v \in S - V_K^{\mathbf{G},a}$. That is, $\sigma \in \mathrm{Aut}(K)_{\mathbf{G},S}$.

We have identified an inclusion of $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ into the group of pairs (δ, σ), where $\sigma \in \mathrm{Aut}(K)_{\mathbf{G},S}$ and $\delta : {}^{\sigma}\mathbf{G} \to \mathbf{G}$ is a K-isomorphism. To see that the inclusion is an isomorphism, let (δ, σ) be a given pair as above. For any $v \in S - V_K^{\mathbf{G},a}$, let $\sigma_v : K \to K$ be defined by $\sigma_v(x) = \sigma(x)$. We assume that the domain of σ_v has the v-topology and that the image of σ_v has the $\sigma \cdot v$-topology. Hence, σ_v is continuous, and it may be completed topologically to obtain an isomorphism $\overline{\sigma_v} : K_v \to K_{\sigma \cdot v}$. Then we define a homomorphism $\mathbf{G}(K_v) \to \mathbf{G}(K_{\sigma\cdot v})$ by $\delta \circ \overline{\sigma_v}^\circ$. The product map

$$\prod_{v \in S - V_K^{\mathbf{G},a}} \delta \circ \overline{\sigma_v}^\circ$$

is then an automorphism of G that stabilizes $\Delta\,\mathbf{G}(K)$. Hence, the group of pairs (δ, σ) as above is isomorphic to $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$.

Notice that the group operation on $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$ is given by $(\delta, \sigma)(\delta', \sigma') = (\delta \circ {}^{\sigma}\delta', \sigma\sigma')$, where ${}^{\sigma}\delta' : {}^{\sigma\sigma'}\mathbf{G} \to {}^{\sigma}\mathbf{G}$ is the K-isomorphism obtained by applying σ to the coefficients of the polynomials defining δ'. This is the group structure of an extension:

$$1 \to \mathbf{Aut}(\mathbf{G})(K) \to \mathrm{Comm}(\mathbf{G}(\mathcal{O}_S)) \to \mathrm{Aut}(K)_{\mathbf{G},S} \to 1.$$

The above extension splits if $\mathbf{G}$ is defined over a subfield of K that is fixed pointwise by $\mathrm{Aut}(K)_{\mathbf{G},S}$. Indeed, if $\mathbf{G}$ is defined over such a field, then for any $\sigma \in \mathrm{Aut}(K)$ we have ${}^{\sigma}\mathbf{G} = \mathbf{G}$. It follows that if $id_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}$ is the identity map, then the pairs $(id_{\mathbf{G}}, \sigma)$ exist in $\mathrm{Comm}(\mathbf{G}(\mathcal{O}_S))$. Hence, the extension splits.

For the statement that $\mathbf{G}$ being K-split implies

$$\mathbf{Aut}(\mathbf{G})(K) \cong \mathbf{G}(K) \rtimes \mathbf{Out}(\mathbf{G})(K),$$

see, for example, the discussion in 5.7.2 of [Ti 2]. (Recall that we identify $\mathbf{Out}(\mathbf{G})$ with the automorphism group of the Dynkin diagram of $\mathbf{G}$.) □

8 Constructing a boundary function defined a.e.

Sections 6 and 7 show the conclusion of the proof for Theorem 1.1 once the boundary function $\partial\phi : U_\partial \to \mathcal{B}(G)$ is created. In Section 8, we outline the construction of $\partial\phi$. We will refer to [Es] for most of the details of the construction.

Replacing the word metric. Let $\widetilde{\mathbf{G}}$ be the algebraic simply connected cover of $\mathbf{G}$. We define

$$H = \prod_{v \in S - V_K^{\mathbf{G},a}} \widetilde{\mathbf{G}}(K_v)$$

and

$$\Gamma = \widetilde{\mathbf{G}}(\mathcal{O}_S).$$

Note that Γ and $\mathbf{G}(\mathcal{O}_S)$ are commensurable up to finite kernels (see e.g. [Mar] I.3.1.1.iv).

Let $\mathfrak{K}$ be a maximal compact subgroup of H, and let $\varepsilon' > 0$ be given. Let μ be the probability measure on $\Gamma\backslash H$ which is derived from Haar measure on H. We choose a compact set $D \subseteq \Gamma\backslash H$ which contains the coset Γ, and such that $\mu(D) \geq 1 - \varepsilon'$.

We denote by $N(\Gamma)^\circ \subseteq H/\mathfrak{K}$ the set of all cosets with a representative in H that maps into D under the quotient map $H \to \Gamma\backslash H$. In symbols,

$$N(\Gamma)^0 = \{\, h\mathfrak{K} \in H/\mathfrak{K} \mid \Gamma h \in D \,\}.$$

Since $\mathfrak{K}$ is the isotropy group of a point in X, we can identify $H/\mathfrak{K}$ as a subset of X. For each $h\mathfrak{K} \in H/\mathfrak{K}$, we let $P(h\mathfrak{K})$ be the set of points in X that are at least as close to $h\mathfrak{K} \in X$ as to any other point of $H/\mathfrak{K} \subseteq X$. Precisely:

$$P(h\mathfrak{K}) = \{\, x \in X \mid d(x, h\mathfrak{K}) \leq d(x, g\mathfrak{K}) \text{ for all } g \in H \,\}.$$

Let

$$N(\Gamma) = \bigcup_{h\mathfrak{K} \in N(\Gamma)^\circ} P(h\mathfrak{K}).$$

Notice that $N(\Gamma) \subseteq X$ contains the orbit $\Gamma\mathfrak{K}$. Since $\Gamma\backslash N(\Gamma)^\circ = D$ and $P(\mathfrak{K})$ are compact, $\Gamma\backslash N(\Gamma) = \Gamma\backslash[N(\Gamma)^\circ P(\mathfrak{K})]$ is compact. Thus, Γ is quasi-isometric to $N(\Gamma) \subseteq X$ with the path metric.

The geometry of $N(\Gamma) \subseteq X$ with the path metric is more convenient to work with than the word metric on Γ. More convenient still, would be working with the geometry of $N(\Gamma)$ under the restricted metric from X.

In general, a lattice is not quasi-isometric to its orbit with the restricted metric, but with our standing assumption that $\widetilde{\mathbf{G}}$ is placewise not rank one, we can apply the theorem below from [L-M-R]

Theorem 8.1 (Lubotzky-Mozes-Raghunathan) *The word metric on Γ is quasi-isometric to $N(\Gamma) \subseteq X$ with the restricted metric.*

Using Theorem 8.1, the fact that Γ and $\mathbf{G}(\mathcal{O}_S)$ are commensurable up to finite kernels, and the fact that the inclusion of $N(\Gamma)$ with the restricted metric into X is isometric, we can realize a given quasi-isometry

$$\phi : \mathbf{G}(\mathcal{O}_S) \to \mathbf{G}(\mathcal{O}_S)$$

by a quasi isometric embedding

$$N(\Gamma) \to X.$$

The resulting embedding is a finite distance in the sup norm from ϕ, so we will also denote it by ϕ. We will assume that

$$\phi : N(\Gamma) \to X$$

is a (κ, C) quasi-isometric embedding.

Ergodic actions of abelian groups. For each $v \in S - V_K^{\mathbf{G},a}$, let $\mathbf{A_v}$ be a maximal K_v-split torus in $\widetilde{\mathbf{G}}$. We define the group

$$A = \prod_{v \in S - V_K^{\mathbf{G},a}} \mathbf{A_v}(K_v) < H.$$

We denote the flat corresponding to A by $\mathcal{A} \subseteq X$. We may assume that $\mathfrak{K} \in \mathcal{A}$.

We introduce a pseudometric d_A on A by setting $d_A(a_1, a_2)$ to be equal to $d(a_1\mathfrak{K}, a_2\mathfrak{K})$ for $a_1\mathfrak{K}, a_2\mathfrak{K} \in X$.

There is also a Haar measure on A which we denote by da. We denote Lebesgue measure on $\mathcal{A}$ by $\mathrm{vol}_{\mathcal{A}}$. Then, after a normalization, we have for any measurable set $Y \subseteq A$:

$$da\Big(A \cap \Big(\bigcup_{a \in Y} a\mathfrak{K}a^{-1}\Big)\Big) = \mathrm{vol}_{\mathcal{A}}\Big(\mathcal{A} \cap \Big(\bigcup_{a \in Y} aP(\mathfrak{K})\Big)\Big).$$

The Birkhoff ergodic theorem is usually stated for an ergodic action of $\mathbb{Z}$. However, a careful reading of the proof of the Birkhoff ergodic theorem shows that it applies to ergodic actions of our pseudometric group A as well (see e.g. [Bl] Theorem 3.2). That is, if we let $B_1^A(r) \subseteq A$ be the ball of radius r centered at the identity element of A, then we have the following

Proposition 8.2 (Birkhoff ergodic theorem) *If Y is a finite volume right ergodic A-space and $f \in L^1(Y)$, then for a.e. $y \in Y$:*

$$\lim_{r\to\infty} \frac{1}{da(B_1^A(r))} \int_{B_1^A(r)} f(ya)da = \int_Y f.$$

Prasad's proof of the strong approximation theorem for simply connected semisimple Lie groups contains a proof of the ergodicity of the A-action on $\Gamma\backslash H$ (see [Pr 3] Lemma 2.9). Hence, we can apply the Birkhoff ergodic theorem to the action of A on $\Gamma\backslash H$.

Generic flats have most of their volume near Γ. Following Eskin, we are now prepared to show that a generic flat in X has most of its volume contained in $N(\Gamma) \subseteq X$.

For any group element $h \in H$, define $\mathrm{vol}_{h\mathcal{A}}$ to be Lebesgue measure on the flat $h\mathcal{A} \subseteq X$. That is, for any measurable set $Y \subseteq h\mathcal{A}$, we let

$$\mathrm{vol}_{h\mathcal{A}}(Y) = \mathrm{vol}_{\mathcal{A}}(h^{-1}Y).$$

Thus, the measure $\mathrm{vol}_{h\mathcal{A}}$ is compatible da in a natural way.

We denote by $B_x^{h\mathcal{A}}(r) \subseteq h\mathcal{A}$ the metric ball centered at the point $x \in h\mathcal{A}$ with radius $r > 0$. Denote the characteristic functions of $N(\Gamma) \subseteq X$ and $D \subseteq \Gamma\backslash N(\Gamma)$ by $\chi_{N(\Gamma)}$ and χ_D respectively.

By Proposition 8.2, we have that for μ a.e. $\Gamma h \in \Gamma\backslash H$:

$$\begin{aligned}
\lim_{r\to\infty} & \frac{1}{\mathrm{vol}_{h\mathcal{A}}(B_{h\mathfrak{K}}^{h\mathcal{A}}(r))} \int_{B_{h\mathfrak{K}}^{h\mathcal{A}}(r)} \chi_{N(\Gamma)} \,\mathrm{vol}_{h\mathcal{A}} \\
&= \lim_{r\to\infty} \frac{1}{da(B_1^A(r))} \int_{B_1^A(r)} \chi_{N(\Gamma)}(ha\mathfrak{K})da \\
&\geq \lim_{r\to\infty} \frac{1}{da(B_1^A(r))} \int_{B_1^A(r)} \chi_D(\Gamma ha)da \\
&= \int_{\Gamma\backslash H} \chi_D \\
&= \mu(D) \\
&\geq 1-\varepsilon'.
\end{aligned}$$

The inequality shows that for a.e. $\Gamma h \in \Gamma\backslash H$, any $\gamma \in \Gamma$, and any point $x \in \gamma h\mathcal{A}$:

$$\lim_{r\to\infty} \frac{\mathrm{vol}_{\gamma h\mathcal{A}}\left(\left[\gamma h\mathcal{A} \cap N\big(\Gamma\big)\right] \cap B_x^{\gamma h\mathcal{A}}(r)\right)}{\mathrm{vol}_{\gamma h\mathcal{A}}\left(B_x^{\gamma h\mathcal{A}}(r)\right)} \geq 1-\varepsilon'.$$

Hence, the generic flat has much of its volume contained in $N(\Gamma)$.

The above argument is the basic idea behind Lemma 8.3 below. Refining the argument will yield more precise information about how much of a generic flat is contained in $N(\Gamma)$. Then we will be in a position to apply the quasiflats with holes theorem from [W1] to begin constructing a map on $\mathcal{B}(G)$.

More on the position of a generic flat with respect to Γ. Let $h \in H$. For a set $W \subseteq X$ contained in the flat $h\mathcal{A}$, we let

$$W_{(\varepsilon,\rho)} = \{\, x \in W \mid B_y^{h\mathcal{A}}\big(\varepsilon d(x,y)\big) \cap W \neq \emptyset \text{ for all } y \in h\mathcal{A} - B_x^{h\mathcal{A}}(\rho) \,\}.$$

Hence, $W_{(\varepsilon,\rho)}$ is the set of all points $x \in W$ which can serve as an observation point from which all points in $h\mathcal{A}$ (that are a sufficient distance from x) have a distance from W that is proportional to their distance from x.

We denote the metric r-neighborhood of a set $Y \subseteq X$ by $\mathrm{Nbhd}_r(Y)$. We denote the Hausdorff distance between two sets $P, Q \subseteq X$ by $\mathrm{Hd}(P,Q)$.

Recall the definition of a *wall* $L \subseteq X$ as a codimension 1 affine subspace of a flat, that is contained in at least two distinct flats.

Lemma 8.3 below is an amalgam of Lemmas 2.2, 3.2, and 5.2 from [Es]. We omit the proof of the lemma as it is nearly identical to those in [Es]. We note that the proof follows the principle shown above using the Birkhoff ergodic theorem.

We will assume throughout that $\varepsilon > 0$ is a sufficiently small number depending on κ and X.

Lemma 8.3 *There are constants $\rho > 0$, and $\rho' > 0$ depending on ε and X; constants $\lambda_0 > 1$, $\lambda_1 > 1$, $N' > 0$, $m > 0$, and $1 > b > 0$ depending on X; and a Γ-invariant co-null set $\mathcal{U} \subseteq H$ such that for any $h \in \mathcal{U}$ there are sets*

$$\Omega^*_{h\mathcal{A}} \subseteq \Omega_{h\mathcal{A}} \subseteq \Omega'_{h\mathcal{A}} \subseteq h\mathcal{A} \cap N(\Gamma)$$

which satisfy the following properties:

(i) For any point $x \in h\mathcal{A}$:

$$\lim_{r\to\infty} \frac{\mathrm{vol}_{h\mathcal{A}}\Big(\big[h\mathcal{A} \cap N\big(\Gamma\big)\big] \cap B_x^{h\mathcal{A}}(r)\Big)}{\mathrm{vol}_{h\mathcal{A}}\Big(B_x^{h\mathcal{A}}(r)\Big)} \geq 1 - \varepsilon/4.$$

(ii) $\Omega'_{h\mathcal{A}} \subseteq \left(h\mathcal{A} \cap N(\Gamma)\right)_{(\varepsilon,\rho)}$ *and for any point* $x \in h\mathcal{A}$*:*

$$\lim_{r\to\infty} \frac{\mathrm{vol}_{h\mathcal{A}}\left(\Omega'_{h\mathcal{A}} \cap B_x^{h\mathcal{A}}(r)\right)}{\mathrm{vol}_{h\mathcal{A}}\left(B_x^{h\mathcal{A}}(r)\right)} \geq 1 - \varepsilon/2.$$

(iii) $\Omega_{h\mathcal{A}} \subseteq (\Omega'_{h\mathcal{A}})_{(\varepsilon,\rho')}$ *and for any point* $x \in h\mathcal{A}$*:*

$$\lim_{r\to\infty} \frac{\mathrm{vol}_{h\mathcal{A}}\left(\Omega_{h\mathcal{A}} \cap B_x^{h\mathcal{A}}(r)\right)}{\mathrm{vol}_{h\mathcal{A}}\left(B_x^{h\mathcal{A}}(r)\right)} \geq 1 - \varepsilon/2.$$

(iv) For any point $x \in h\mathcal{A}$*:*

$$\lim_{r\to\infty} \frac{\mathrm{vol}_{h\mathcal{A}}\left(\Omega^*_{h\mathcal{A}} \cap B_x^{h\mathcal{A}}(r)\right)}{\mathrm{vol}_{h\mathcal{A}}\left(B_x^{h\mathcal{A}}(r)\right)} \geq 1 - m\varepsilon.$$

(v) If $y \in \Omega^*_{h\mathcal{A}}$ *and* $L \subset h\mathcal{A}$ *is a wall with* $d(y,L) < N'$ *then there is a group element* $h' \in \mathcal{U}$ *such that*

$$h'\mathcal{A} \cap h\mathcal{A} \subseteq \mathrm{Nbhd}_{2N'}(L)$$

and

$$\mathrm{Hd}\left(h\mathcal{A} \cap \mathrm{Nbhd}_r(h'\mathcal{A})\ ,\ L\right) \leq \lambda_1 r$$

for any $r > \lambda_0$*.*

(vi) For any wall $L \subset \mathcal{A}$ *and any point* $x \in h\mathcal{A}$*:*

$$\lim_{r\to\infty} \frac{\mathrm{vol}_{h\mathcal{A}}\left(\Omega^*_{h\mathcal{A}} \cap B_x^{h\mathcal{A}}(r) \cap hL\right)}{\mathrm{vol}_{h\mathcal{A}}\left(B_x^{h\mathcal{A}}(r)\right)} \geq b.$$

(vii) There is a Γ *invariant set* $E \subseteq \mathcal{U}$ *such that* $\mu(\Gamma\backslash E) > 1 - \varepsilon/2$*, and* $h\mathfrak{K} \in \Omega_{h\mathcal{A}}$ *for any* $h \in E$*.*

Remarks. There are some differences in this lemma with Lemmas 2.2, 3.2, and 5.2 in [Es]. In particular, the transverse flats in part (v) do not necessarily intersect in a wall for the general space X, as can be arranged if X is a symmetric space. Take for example a regular trivalent tree which is the Euclidean building for $\mathbf{SL_2}(\mathbb{Q}_2)$. The walls in this example are vertices; the flats are lines, and there is no pair of lines which intersect in a single point.

Also, the constant b in part (vi) is shown in [Es] to be nearly one. This discrepancy is essentially due to the fact that if X is a Euclidean building, then the orbit of $P(\mathfrak{K})$ under the action of the $\mathfrak{p}$-adic group that stabilizes a wall containing $\mathfrak{K}$ may not contain all of L. Take for example the building for $\mathbf{SL_3}(\mathbb{Q}_p)$. However, Eskin's proof only uses that the constant is greater than 0, and that is all we shall need as well.

A collection of useful flats. Lemma 8.3 provides us with a collection of flats in X that have most of their volume, and a substantial portion of the volume of their walls, contained in $N(\Gamma)$. We denote this collection of flats by $\mathfrak{U}$. That is,

$$\mathfrak{U} = \{h\mathcal{A} | h \in \mathcal{U}\}.$$

Since any flat $F \in \mathfrak{U}$ has most of its volume contained in $N(\Gamma)$, we can restrict $\phi : N(\Gamma) \to X$ to $F \cap N(\Gamma)$ and begin to analyze the image using Theorem 1.2 of [W1]. We state this theorem as

Theorem 8.4 (Quasiflats with holes) *Let $\varphi : \Omega \to X$ be a (κ, C) quasi-isometric embedding of a set $\Omega \subseteq \mathbb{E}^n$. There are constants $M = M(\kappa, X)$ and $\delta_0 = \delta_0(\kappa, X)$ such that if $\delta < \delta_0$, then there exists flats $F_1, F_2, ..., F_M \subseteq X$ such that*

$$\varphi\big(\Omega_{(\delta,R)}\big) \subseteq \mathrm{Nbhd}_N\left(\bigcup_{i=1}^{M} F_i\right),$$

where $N = N(\kappa, C, R, X)$.

Theorem 8.4, and the fact that a generic flat $F \subseteq X$ is contained in $\mathfrak{U}$, positions us to begin constructing the function $\partial\phi : U_\partial \to \mathcal{B}(G)$ where the set $U_\partial \subseteq \mathcal{B}(G)$ has full measure in $\widehat{X}$.

Weyl chambers are mapped to Weyl chambers. For points $x, z, w \in X$ and a number $\rho \geq 0$, we let

$$D_x(\rho; z, w) = \max\{\rho, d(x, z), d(x, w)\}.$$

Define a function $\phi : X \to Y$ to be a $(\kappa, \rho, \varepsilon)$ *graded quasi-isometric embedding based at* $x \in X$, if for all $z, w \in X$:

$$\frac{1}{\kappa}d(z,w) - \varepsilon D_x(\rho; z,w) \leq d(\phi(z),\phi(w)) \leq \kappa d(z,w) + \varepsilon D_x(\rho; z,w).$$

If $F \in \mathfrak{U}$ we let $p : F \to \Omega'_F$ be a closest point projection and define

$$\phi_F : F \to X$$

by $\phi_F = \phi \circ p$.

If $x \in \Omega_F$, then using Lemma 8.3(ii), ϕ_F is a $(\kappa, \rho, 2\kappa\varepsilon)$ graded quasi-isometric embedding based at x. Also note that by Theorem 8.4, $\phi_F(F)$ is contained in a neighborhood of finitely many flats since

$$\Omega'_F \subseteq \big(F \cap N(\Gamma)\big)_{(\varepsilon,\rho)}.$$

We fix a Weyl chamber $\mathcal{A}^+ \subseteq \mathcal{A}$ based at $\mathfrak{K} \in X$. For any $h \in H$, let $h\mathcal{A}^+(\infty)$ be the equivalence class of $h\mathcal{A}^+$ in $\widehat{X}$.

For two subsets A and C of X, any point $x \in X$, and a small number $\delta > 0$, we write $A \sim_\delta C$ if

$$\mathrm{Hd}\left(A \cap B_x(r)\, ,\, C \cap B_x(r)\right) \leq \delta r$$

for all sufficiently large numbers $r > 0$.

At this point in [Es], a detailed argument is used to show the analogue of the lemma below (Lemma 3.14 in [Es]) for the case when X is a symmetric space.

Lemma 8.5 *Suppose $h\mathcal{A} \in \mathfrak{U}$ for some $h \in H$. There exists a constant λ depending on κ and X, and some $k \in \mathfrak{K}$ depending on h, such that*

$$\phi_{h\mathcal{A}}\big(h\mathcal{A}^+\big) \sim_{\lambda \sqrt[n]{\varepsilon}} k\mathcal{A}^+.$$

Eskin's proof proceeds by first showing that if L is a wall of a flat $F \in \mathfrak{U}$, then ϕ_F maps L into a "graded neighborhood" of a wall $L' \subseteq X$. (For a definition of a graded neighborhood see below, before the proof of Lemma 8.7.) This is shown using the Eskin-Farb quasiflats with holes theorem and the characterization of walls of flats in $\mathfrak{U}$ as "coarse intersections" of flats in $\mathfrak{U}$ (see Lemma 8.3(v)). A key ingredient for this step is Eskin's "no turns" lemma about quasi-isometries of Euclidean space which respect a family of hyperplanes. (In this case the Euclidean spaces are our flats, and the hyperplanes are the walls of the flats.)

Since Weyl chambers are defined by the set of walls that bound them, Eskin uses the information about the images of walls to deduce the lemma above for symmetric spaces.

Eskin's proof of the symmetric space version of Lemma 8.5 uses the geometry of symmetric spaces mostly to supply foundational tools for the main argument. We will replace these tools with analogues that hold for products of symmetric spaces and Euclidean buildings.

The first of the foundational tools needed is Lemma 8.3 – even here Eskin's proof applied to the general case. The second tool is Theorem 8.4 which was proved in [W1]. The last two tools needed are Lemmas 8.6 and 8.7 below. They are direct analogues of Lemmas B.1 and B.7 of [Es] respectively. After proving Lemmas 8.6 and 8.7, the foundation to carry out Eskin's proof for the general space X will be in place. Then Eskin's proof applies to establish Lemma 8.5.

Coarse intersections of convex polyhedra. Any wall, L, in a flat $F \subseteq X$, divides F into two components. The closure of any such component is called a *half-space*. We define a *convex polyhedron* in X as an intersection of a flat, F, with a (possibly empty) finite collection of half-spaces contained in F. Note that flats are convex polyhedra, as are walls.

The following lemma is an analogue of Lemma B.1 in [Es]. It allows us to replace coarse intersections of flats, walls, or convex polyhedra with a convex polyhedron.

Lemma 8.6 *There are constants, λ_2 and λ_3, such that if Q_1 and Q_2 are convex polyhedra in X, and if $r > \lambda_2(1 + d(Q_1, Q_2))$, then there is a convex polyhedron $P \subseteq Q_1$ such that*

$$\mathrm{Hd}\left(Q_1 \cap \mathrm{Nbhd}_r(Q_2)\ ,\ P\right) \leq \lambda_3 r.$$

Proof If $Q_k \subseteq X$ is a convex polyhedron in the flat $F_k \subseteq X$, and if $F_{k,\infty} \subseteq X_\infty$ and $F_{k,\mathfrak{p}} \subseteq X_\mathfrak{p}$ are flats such that $F_k = F_{k,\infty} \times F_{k,\mathfrak{p}}$, then

$$Q_k = F_k \cap \bigcap_i \left(H_{k,\infty,i} \times F_{k,\mathfrak{p}}\right) \ \cap \ \bigcap_i \left(F_{k,\infty} \times H_{k,\mathfrak{p},i}\right),$$

where each $H_{k,\infty,i} \subseteq F_{k,\infty}$ and each $H_{k,\mathfrak{p},i} \subseteq F_{k,\mathfrak{p}}$ is a half-space.

Hence, if $Q_{k,\infty} \subseteq F_{k,\infty}$ is the convex polyhedron given by

$$Q_{k,\infty} = F_{k,\infty} \cap \bigcap_i H_{k,\infty,i}$$

and $Q_{k,\mathfrak{p}} \subseteq F_{k,\mathfrak{p}}$ is the convex polyhedron given by

$$Q_{k,\mathfrak{p}} = F_{k,\mathfrak{p}} \cap \bigcap_i H_{k,\mathfrak{p},i},$$

then $Q_k = Q_{k,\infty} \times Q_{k,\mathfrak{p}}$.

Note that

$$\Big[Q_{1,\infty} \cap \mathrm{Nbhd}_{r/\sqrt{2}}(Q_{2,\infty})\Big] \times \Big[Q_{1,\mathfrak{p}} \cap \mathrm{Nbhd}_{r/\sqrt{2}}(Q_{2,\mathfrak{p}})\Big]$$
$$\subseteq$$
$$Q_1 \cap \mathrm{Nbhd}_r(Q_2)$$
$$\subseteq$$
$$\Big[Q_{1,\infty} \cap \mathrm{Nbhd}_r(Q_{2,\infty})\Big] \times \Big[Q_{1,\mathfrak{p}} \cap \mathrm{Nbhd}_r(Q_{2,\mathfrak{p}})\Big],$$

so we can reduce the proof of this lemma to the separate cases of $X = X_\infty$ and $X = X_\mathfrak{p}$. The former case is Lemma B.1 of [Es]. We will prove the lemma for the latter case.

Let Q_1 and Q_2 be convex polyhedron in a Euclidean building $X_\mathfrak{p}$. Let $F \subseteq X_\mathfrak{p}$ be an apartment (flat) containing Q_1.

Define

$$P_{d(Q_1,Q_2)} = Q_1 \cap \overline{\mathrm{Nbhd}_{d(Q_1,Q_2)}(Q_2)}$$

Since Q_2 is convex, $\overline{\mathrm{Nbhd}_{d(Q_1,Q_2)}(Q_2)}$ is convex as well ([Bri-H] Cor. II.2.5(1)). Therefore $P_{d(Q_1,Q_2)}$ is convex. In fact, $P_{d(Q_1,Q_2)}$ is a convex polyhedron. Indeed, if $\mathfrak{c} \subseteq F$ is a chamber, let

$$\rho_{F,\mathfrak{c}} : X_\mathfrak{p} \to F$$

be the retraction corresponding to F and $\mathfrak{c}$. Then $d(x,y) = d(x, \rho_{F,\mathfrak{c}}(y))$ for all $x \in \mathfrak{c}$ and all $y \in Q_2$. (For a good reference for retractions, and for buildings in general, see [Bro].) Therefore, points in $\partial P_{d(Q_1,Q_2)}$ are determined by translating the region $\rho_{F,\mathfrak{c}}(Q_2)$ a distance of $d(Q_1, Q_2)$. Hence, $P_{d(Q_1,Q_2)}$ is bounded by walls which are translates of the walls bounding $\rho_{F,\mathfrak{c}}(Q_2)$. Since $P_{d(Q_1,Q_2)}$ is convex, and since there are finitely many parallel families of walls in F, $P_{d(Q_1,Q_2)}$ is bounded by finitely many walls.

We let each $H_i \subseteq F$ be a half-space such that

$$P_{d(Q_1,Q_2)} = F \cap \bigcap_i H_i.$$

For any number $r \geq 0$, let $H_i^{r+} \subseteq F$ be the half-space that contains H_i, and with the additional property that

$$\mathrm{Hd}\left(H_i\, ,\, H_i^{r+}\right) = r + d(Q_1, Q_2).$$

Define the convex polyhedron P_r^+ by

$$P_r^+ = Q_1 \cap \Big(\bigcap_i H_i^{r+}\Big).$$

We claim that if $r \geq 0$, then

$$Q_1 \cap \mathrm{Nbhd}_r(Q_2) \subseteq P_r^+.$$

That is, we want to prove that

$$Q_1 \cap \mathrm{Nbhd}_r(Q_2) \subseteq H_i^{r+}$$

for all i. To this end, let $\mathfrak{c}_i \subseteq F$ be a chamber that is separated from $P_{d(Q_1,Q_2)}$ by ∂H_i^{r+}. Let

$$\rho_{F,\mathfrak{c}_i} : X_{\mathfrak{p}} \to F$$

be the retraction corresponding to $\mathfrak{c}_i$ and F. Since $\rho_{F,\mathfrak{c}_i}$ is distance nonincreasing, we have that

$$d\big(\rho_{F,\mathfrak{c}_i}(P_{d(Q_1,Q_2)})\,,\,\rho_{F,\mathfrak{c}_i}(Q_2)\big) \leq d\big(P_{d(Q_1,Q_2)}\,,\,Q_2\big) = d\big(Q_1\,,\,Q_2\big).$$

Therefore, if $x \in Q_2$:

$$\begin{aligned} d\big(\partial H_i^{r+}\,,\,x\big) &\geq d\big(\partial H_i^{r+}\,,\,\rho_{F,\mathfrak{c}_i}(x)\big) \\ &\geq d\big(\partial H_i^{r+}\,,\,P_{d(Q_1,Q_2)}\big) - d\big(P_{d(Q_1,Q_2)}\,,\,\rho_{F,\mathfrak{c}_i}(x)\big) \\ &= r + d(Q_1,Q_2) - d\big(\rho_{F,\mathfrak{c}_i}(P_{d(Q_1,Q_2)})\,,\,\rho_{F,\mathfrak{c}_i}(x)\big) \\ &\geq r. \end{aligned}$$

Hence,

$$Q_1 \cap \mathrm{Nbhd}_r(Q_2) \subseteq H_i^{r+}$$

as desired.

We have shown that $Q_1 \cap \mathrm{Nbhd}_r(Q_2)$ is contained in a convex polyhedron created by pushing out the walls of $P_{d(Q_1,Q_2)}$ by a uniform distance that is linear in r. Next we observe that $Q_1 \cap \mathrm{Nbhd}_r(Q_2)$ also contains a convex polyhedron created by pushing out the walls of $P_{d(Q_1,Q_2)}$ by a uniform distance that is linear in r.

Indeed, since there are only finitely many walls in any flat F' up to translation, there exists a positive constant $\beta < 1$ depending only on X, such that if $Q \subseteq F'$ is a convex polyhedron, $s \geq 0$, and $Q(s) \subseteq F'$ is the convex polyhedron obtained by pushing out the walls that bound Q by a distance of βs, then

$$Q(s) \subseteq \mathrm{Nbhd}_s(Q) \cap F'.$$

Thus for any number $r \geq d(Q_1, Q_2)$, and for the set of half-spaces $\{ H_i \}$ that define $P_{d(Q_1,Q_2)}$, we let $H_i^{r-} \subseteq F$ be the half-space containing H_i and such that

$$\mathrm{Hd}\left(H_i\,,\, H_i^{r-}\right) = \beta\big(r - d(Q_1, Q_2)\big).$$

And we define the convex polyhedron P_r^- by

$$P_r^- = Q_1 \cap \Big(\bigcap_i H_i^{r-}\Big),$$

so that

$$\begin{aligned} P_r^- &\subseteq \mathrm{Nbhd}_{(r-d(Q_1,Q_2))}(P_{d(Q_1,Q_2)}) \cap F \\ &\subseteq Q_1 \cap \mathrm{Nbhd}_r(Q_2) \end{aligned}$$

In summary, we have shown that for $r \geq d(Q_1, Q_2)$

$$P_r^- \subseteq Q_1 \cap \mathrm{Nbhd}_r(Q_2) \subseteq P_r^+$$

The lemma follows since there clearly exists a constant λ' depending only on $X_\mathfrak{p}$ such that

$$\begin{aligned} \mathrm{Hd}\left(P_r^-\,,\, P_r^+\right) &< \lambda'[r + d(Q_1, Q_2) - \beta(r - d(Q_1, Q_2))] \\ &< \lambda'[r + 2d(Q_1, Q_2)] \\ &\leq \lambda'[3r]. \end{aligned}$$

□

Graded equivalence implies Hausdorff equivalence for Weyl chambers. Let $\mathcal{A}_\alpha \subseteq \mathcal{A}$ be a wall containing $\mathfrak{K}$. For any collection of such walls $\{ \mathcal{A}_\alpha \}_{\alpha\in\sigma}$, let

$$\mathcal{A}_\sigma^+ = \mathcal{A}^+ \cap \bigcap_{\alpha\in\sigma} \mathcal{A}_\alpha.$$

For any set $A \subseteq X$ and any $t > 0$, we define the *graded t-neighborhood of A* as the set

$$A[t] = \{ x \in X \mid \text{there is an } a \in A \text{ with } d(x, a) < td(x, \mathfrak{K}) \}.$$

The following lemma is a generalization of Lemma B.7 in [Es].

Lemma 8.7 *Assume there are three group elements $h, h_1, h_2 \in H$ and that, outside of some metric ball,*

$$h\mathcal{A}_\sigma^+ \subseteq h_1\mathcal{A}^+[\lambda\sqrt[n]{\varepsilon}] \cap h_2\mathcal{A}^+[\lambda\sqrt[n]{\varepsilon}].$$

If $k_1, k_2 \in \mathfrak{K}$ satisfy the condition

$$\mathrm{Hd}(h_i\mathcal{A}^+ \,,\, k_i\mathcal{A}^+) < \infty,$$

then

$$k_1\mathcal{A}^+_\sigma = k_2\mathcal{A}^+_\sigma.$$

Proof A Weyl chamber $\mathfrak{C} \subseteq X$ is a product of Weyl chambers $\mathfrak{C}_\infty \subseteq X_\infty$ and $\mathfrak{C}_\mathfrak{p} \subseteq X_\mathfrak{p}$. Note that $\mathfrak{C}_\infty \times \mathfrak{C}_\mathfrak{p} \subseteq (\mathfrak{C}'_\infty \times \mathfrak{C}'_\mathfrak{p})[t]$ implies that, outside of a ball, $\mathfrak{C}_\infty \subseteq \mathfrak{C}'_\infty[t']$ and $\mathfrak{C}_\infty \subseteq \mathfrak{C}'_\infty[t']$ for $t' > t$. Hence, we only need to show the case of a building since symmetric spaces are covered by Lemma B.7 of [Es].

We can replace $h\mathcal{A}^+_\sigma$ by $k\mathcal{A}^+_\sigma$ for some $k \in \mathfrak{K}$ such that $\mathrm{Hd}(h\mathcal{A}^+_\sigma, k\mathcal{A}^+_\sigma) < \infty$. Then

$$k\mathcal{A}^+_\sigma \subseteq k_1\mathcal{A}^+[\lambda\sqrt[n]{\varepsilon}] \cap k_2\mathcal{A}^+[\lambda\sqrt[n]{\varepsilon}]$$

outside of a large ball.

For any $r > 0$, let $a_r \in \mathcal{A}^+_\sigma$ be such that $d(a_r, \mathcal{A}^+_\alpha) > r$ for all $\alpha \notin \sigma$. By the preceding inclusion, there exist points $a_1, a_2 \in \mathcal{A}^+$ such that $d(ka_r, k_ia_i) \leq \lambda\sqrt[n]{\varepsilon}r$ for all sufficiently large numbers r. Therefore, $d(k_1a_1, k_2a_2) \leq 2\lambda\sqrt[n]{\varepsilon}r$.

There is an apartment $\mathcal{A}' \subseteq X_\mathfrak{p}$ such that, outside of a ball, $k_i\mathcal{A}^+_\sigma \subseteq \mathcal{A}'$ for $i = 1, 2$. If $k_1\mathcal{A}^+_\sigma \neq k_2\mathcal{A}^+_\sigma$, then for all sufficiently large r, we have $k_ia_i \in \mathcal{A}' \cap k_i\mathcal{A}^+_\sigma$ and $d(k_1a_1, k_2a_2) > \alpha r$ for some constant α depending only on $X_\mathfrak{p}$. This is a contradiction. □

The proof of Lemma 8.5 only requires the case of Lemma 8.7 for $\sigma = \emptyset$. However, the full form of Lemma 8.7 is needed for the construction of $\partial\phi$.

The a.e. defined boundary function. Let $N < H$ be the normalizer of $A < H$. Let $\mathcal{B}(G)$ be the Tits building for X. We define U_∂ as the simplicial subcomplex of $\mathcal{B}(G)$ given by

$$U_\partial = \bigcup_{h\in\mathcal{U}} \bigcup_{n\in N} hn\mathcal{A}^+(\infty).$$

We are prepared to define

$$\partial\phi : U_\partial \to \mathcal{B}(G)$$

using Lemma 8.5. We let $\partial\phi(h\mathcal{A}^+(\infty)) = k\mathcal{A}^+(\infty)$ where $k \in \mathfrak{K}$ is such that $\phi_{h\mathcal{A}}(h\mathcal{A}^+) \sim_{\lambda\sqrt[n]{\varepsilon}} (k\mathcal{A}^+)$.

That $\partial\phi$ is well-defined, and restricts to an isomorphism of U_∂ onto its image, follows from Step 4 of [Es] using our Lemma 8.7 in place of Lemma B.7 in [Es].

Flats are preserved. In Section 6, we complete $\partial\phi$ to an automorphism of $\mathcal{B}(G)$. In Lemma 6.7, we use that apartments in $\mathcal{B}(G)$ that are contained in U_∂, are mapped to apartments by $\partial\phi$. This is the content of the lemma below. The proof is from Proposition 3.3 [Es], but we include it here as it is brief.

Lemma 8.8 *If $F \in \mathfrak{U}$, then there is a flat $F' \subseteq X$ such that $\phi_F(F) \subseteq \mathrm{Nbhd}_N(F')$.*

Proof Corresponding to $\phi_F(F) \subseteq X$ there is a finite set $\mathcal{L}(\phi_F) \subseteq \widehat{X}$ of limit points (see [W1]). Intuitively $\mathcal{L}(\phi_F)$ is a set of equivalence classes for finitely many Weyl chambers $\mathfrak{C}_1, \ldots \mathfrak{C}_k \subseteq X$ such that

$$\mathrm{Hd}\left(\phi_F(F)\,,\, \cup_i \mathfrak{C}_i\right) < \infty.$$

Choose $x, y \in \mathcal{L}(\phi_F)$ that are opposite chambers in $\mathcal{B}(G)$. (That such chambers exist is shown in [W1].) Since $\partial\phi$ preserves incidence relations, $\partial\phi$ is Tits distance nonincreasing. Therefore, $\partial\phi^{-1}(x)$ and $\partial\phi^{-1}(y)$ are opposite.

Any chamber $c \subset F(\infty)$ is contained in a minimal gallery between $\partial\phi^{-1}(x)$ and $\partial\phi^{-1}(y)$. Hence, $\partial\phi(c)$ is contained in a minimal gallery from x to y. That is, $\partial\phi(c)$ is a chamber in the unique apartment containing x and y. Now let $F' \subseteq X$ be the unique flat such that $F(\infty)$ contains x and y. □

Countable subcomplexes. In Section 6 we use the following lemma to find a "global sub-building" of $\mathcal{B}(G)$ contained in U_∂.

Lemma 8.9 *If V is a countable collection of chambers in U_∂, then there is some $h \in H$ such that $V \subseteq hU_\partial$.*

Proof For each number $i \in \mathbb{N}$, we choose a chamber $c_i \subset \mathcal{B}(G)$ such that $V = \{c_i\}_{i=1}^{\infty}$. Define the set

$$\mathcal{U}_i = \{\, g \in H \mid gc_i \subseteq U_\partial \}.$$

Note that $\mathcal{U}_i \subseteq H$ is co-null, so $\cap_{i=1}^{\infty} \mathcal{U}_i$ is co-null. Hence, there exists some $h^{-1} \in \cap_{i=1}^{\infty} \mathcal{U}_i$, and h satisfies the lemma. □

9 Continuity of the boundary function on neighborhoods of faces

To complete $\partial\phi$ to an automorphism of $\mathcal{B}(G)$ in Section 6, we use that $\partial\phi$ restricts to a continuous map on simplicial neighborhoods of $(n-2)$-dimensional simplices. Precisely, we use Lemma 9.4 below.

As with Lemma 8.5 in the previous section, our Lemma 9.4 follows from the proof of the analogous Lemma 5.3 in [Es] once a few foundational lemmas are provided for products of symmetric spaces and Euclidean buildings. What we require are replacements for Lemmas B.4, B.6, and B.8 in [Es]. Their analogues are listed below as Lemmas 9.1, 9.2, and 9.3 respectively.

Recall that we defined a metric on $\widehat{X}$ in the early portion of Section 6. We can assume that the metric is invariant under the action of $\mathfrak{K}$. Equivalently, we assume that the basepoint used to define the metric $\widehat{d}$ is the coset $\mathfrak{K} \in H/\mathfrak{K} \subseteq X$.

Lemma 9.1 *There are constants ν_1, ν_2, and ν_3 depending on X, such that if $k_i \in \mathfrak{K}$, $z_i \in k_i\mathcal{A}^+$ with $d(z_1, z_2) \leq \nu_1 r$, and $d(z_i, k_i\partial\mathcal{A}^+) \geq \nu_2 r$ where r is sufficiently large, then*

$$\widehat{d}(k_1\mathcal{A}^+(\infty), k_2\mathcal{A}^+(\infty)) \leq \exp(-\nu_3 r).$$

Proof The hypotheses imply the analogous hypotheses on each factor, X_∞ and $X_\mathfrak{p}$. On the symmetric space factor the result is implied by Lemma B.4 of [Es], and since we have endowed $\widehat{X}$ with the box metric, the result follows once we establish the lemma for the case that X is a Euclidean building.

Supposing $X_\mathfrak{p}$ is a Euclidean building, we let $\nu_1 = 1/2$ and $\nu_2 = 1$. For the Weyl chamber $\mathcal{A}^+ \subseteq X$, we let $\alpha > 1$ be the constant such that the basepoints of the sectors $\mathcal{A}^+$ and $\mathcal{A}^+ - \text{Nbhd}_r(\partial\mathcal{A}^+)$ are at distance αr from each other for all $r > 0$. Clearly α depends only on $X_\mathfrak{p}$. We let $\nu_3 = \alpha/2$.

We can assume, by repositioning the direction of the geodesic rays used to define $\widehat{d}$, that $\gamma_{\mathcal{A}^+}$ contains the point that the sector $\mathcal{A}^+ - \text{Nbhd}_r(\partial\mathcal{A}^+)$ is based at. Indeed, our choice that $\gamma_{\mathcal{A}^+}(\infty) \in \mathcal{A}^+(\infty)$ is the center of mass was completely arbitrary and any point in the interior of $\mathcal{A}^+(\infty)$ would suffice.

Now we proceed by forcing a contradiction. That is we assume that $\widehat{d}(k_1\mathcal{A}^+(\infty), k_2\mathcal{A}^+(\infty)) > \exp(-\alpha r/2)$. Then $\gamma_{k_1\mathcal{A}^+} \cap \gamma_{k_2\mathcal{A}^+}$ is a geodesic segment with distinct endpoints $\mathfrak{K}, x \in X_\mathfrak{p}$, that satisfy the inequality $d(\mathfrak{K}, x) < \alpha r/2$.

Let $W_x \subseteq k_1\mathcal{A}$ be a wall containing x and such that the closure of the component of $k_1\mathcal{A} - W_x$ containing $\mathfrak{K}$ also contains $k_1\mathcal{A}^+ \cap k_2\mathcal{A}^+$. Note that the point $z_1 \in k_1\mathcal{A}^+ - \text{Nbhd}_r(k_1\partial\mathcal{A}^+)$ is in the opposite component of $k_1\mathcal{A} - W_x$ by our choice of α. Also by our choice of α,

$$d(z_1, W_x) > r/2.$$

If $\mathfrak{c} \subseteq k_1\mathcal{A}^+$ is a chamber containing x, but not contained in $k_2\mathcal{A}^+$, then the retraction

$$\rho_{k_1\mathcal{A},\mathfrak{c}} : X_\mathfrak{p} \to k_1\mathcal{A}$$

corresponding to the apartment $k_1\mathcal{A}$ and to the chamber $\mathfrak{c}$, maps z_2 to the component of $k_1\mathcal{A} - W_x$ containing $\mathfrak{K}$.

Therefore, the geodesic segment from z_1 to $\rho_{k_1\mathcal{A},\mathfrak{c}}(z_2)$ passes through W_x. Hence,

$$\begin{aligned} d(z_1, z_2) &\geq d(z_1, \rho_{k_1\mathcal{A},\mathfrak{c}}(z_2)) \\ &\geq d(z_1, W_x) \\ &> r/2. \end{aligned}$$

This completes our contradiction. □

Lemma 9.2 *There is a constant ν_4 depending on X such that for sufficiently large numbers Q and any $k_1, k_2 \in \mathfrak{K}$, there are $z_i \in k_i\mathcal{A}^+$ satisfying:*

(*i*) $d(z_1, z_2) \leq Q$
(*ii*) $d(z_i, e) \leq \nu_4 |\log\left(\widehat{d}(k_1\mathcal{A}^+(\infty), k_2\mathcal{A}^+(\infty))\right)|$, *and*
(*iii*) $d(z_i, k_i\partial\mathcal{A}^+) \geq \nu_5 |\log\left(\widehat{d}(k_1\mathcal{A}^+(\infty), k_2\mathcal{A}^+(\infty))\right)|$

for some constant ν_5 which depends on Q and on X.

Proof Again we prove the lemma for the case $X = X_\mathfrak{p}$. The case $X = X_\infty$ is Lemma B.6 of [Es], and the Lemma 9.2 follows from the lemmas for each case.

If $X_\mathfrak{p}$ is a Euclidean building, and if $k_1\mathcal{A}^+ \cap k_2\mathcal{A}^+$ does not contain a chamber of $X_\mathfrak{p}$, then choose $z_1 \in \gamma_{k_1\mathcal{A}^+}$ and $z_2 \in \gamma_{k_2\mathcal{A}^+}$ to be distance 1 away from $\mathfrak{K}$. Then the conclusion of the lemma is satisfied for all $Q > 0$ by ν_4=1 and some ν_5 which depends only on the angle between $\gamma^+_\mathcal{A}$ and $\partial\mathcal{A}^+$.

If $k_1\mathcal{A}^+ \cap k_2\mathcal{A}^+$ does contain a chamber of X, then let $z_1 = z_2 \in k_1\mathcal{A}^+ \cap k_2\mathcal{A}^+$ be the endpoint of $\gamma_{k_1\mathcal{A}^+} \cap \gamma_{k_2\mathcal{A}^+}$. Now the lemma holds for any $Q > 0$, $\nu_4 = 1$, and some ν_5 that depends only on the angle between $\gamma_{\mathcal{A}^+}$ and $\partial\mathcal{A}^+$. □

Lemma 9.3 *Let $x, y \in X$. For any Weyl chamber $\mathfrak{C}_x \subseteq X$ based at x, there is a Weyl chamber $\mathfrak{C}_y \subseteq X$ based at y such that*

$$\mathrm{Hd}(\mathfrak{C}_x\,,\ \mathfrak{C}_y) < \lambda' d(x, y)$$

for some constant λ'.

Proof The lemma follows from Lemma B.8 of [Es], and from Lemma 4.3 of [W1]. □

Recall that n is the rank of X and that for any $(n-2)$-dimensional simplex $f \subset U_\partial$, we defined $\mathcal{N}_U(f)$ as the set of all chambers in U_∂ that contain f.

We can apply the proof of Lemma 5.3 in [Es] by replacing Lemmas B.4, B.6, and B.8 of [Es] with the three lemmas above to show:

Lemma 9.4 *If $f \subset U_\partial$ is a simplex of dimension $n-2$, then $\partial\phi|_{\mathcal{N}_U(f)}$ is continuous in the Furstenberg metric.*

Note that Lemma 5.3 of [Es] claims that $\partial\phi|_{\mathcal{N}_U(c)}$ is bi-Hölder. We only require $\partial\phi|_{\mathcal{N}_U(c)}$ to be continuous as our method for completing $\partial\phi$ is more algebraic, and less topological, than Eskin's.

The condition that chambers share a wall in the above lemma is needed so that two Weyl chambers can be simultaneously slid along a common wall until they are based at points in $N(\Gamma)$ – the set our quasi-isometry is defined on. The sliding technique does not change their Furstenberg distance.

References

[Abl] **Abels, H.,** *Finiteness properties of certain arithmetic groups in the function field case.* Israel J. Math., **76** (1991), 113-128.

[Abr] **Abramenko, P.,** *Finiteness properties of Chevalley groups over $F_q[t]$.* Israel J. Math., **87** (1994), 203-223.

[Be] **Behr, H.,** $\mathrm{SL}_3(F_q[t])$ *is not finitely presentable.* Proc. Sympos. "Homological group theory" (Durham 1977). London Math. Soc., Lecture Notes Ser., **36**, 213-224.

[Bl] **Blume, F.,** *Ergodic theory.* in *Handbook of measure theory* Vol. II. North-Holland, Amsterdam (2002), 1185-1235.

[Bo 1] **Borel, A.,** *Density and maximality of arithmetic subgroups.* J. Reine Angew. Math., **224** (1966), 78-89.

[Bo 2] **Borel, A.,** *Linear algebraic groups.* Graduate Texts in Mathematics, No. 126, Springer-Verlag, New York (1991).

[Bo-Sp] **Borel, A., and Springer, T. A.,** *Rationality properties of linear algebraic groups II.* Tôhoku Math. Journ., **20** (1968), 443-497.

[Bo-T] **Borel, A., and Tits, J.,** *Homomorphisms "abstraits" de groups algébriques simples.* Ann. Math., **97** (1973), 499-571.

[Bri-H] **Bridson, M., and Haefliger, A.,** *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, Vol. 319. Springer-Verlag, Heidelberg (1999).

[Bro] **Brown, K.,** *Buildings.* Springer-Verlag, New York (1989).

[Dr] **Druţu, C.,** *Quasi-isometric classification of non-uniform lattices in semisimple groups of higher rank.* Geom. Funct. Anal., **10** (2000), 327-388.

[Es] **Eskin, A.,** *Quasi-isometric rigidity of nonuniform lattices in higher rank symmetric spaces.* J. Amer. Math. Soc., **11** (1998), 321-361.

[E-F 1] **Eskin, A., and Farb, B.,** *Quasi-flats and rigidity in higher rank symmetric spaces.* J. Amer. Math. Soc., **10** (1997), 653-692.

[E-F 2] **Eskin, A., and Farb, B.,** *Quasi-flats and rigidity in* $\mathbb{H}^2 \times \mathbb{H}^2$. Lie Groups and Ergodic Theory, Nari's Publishing House, New Delhi (1998), 75-104.

[Fa] **Farb, B.,** *The quasi-isometry classification of lattices in semisimple Lie groups.* Math. Res. Letta., **4** (1997), 705-717.

[Fa-Sch] **Farb, B., and Schwartz, R.,** *The large-scale geometry of Hilbert modular groups.* J. Diff. Geom., **44** (1996), 435-478.

[Har] **Harder, G.,** *Über die Galoiskohomologie halbeinfacher algebraischer Gruppen, III.* J. Reine Angew. Mat., **274/275** (1975), 125-138.

[K-L] **Kleiner, B., Leeb, B.,** *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings.* Inst. Hautes Études Sci. Publ. Math., **86** (1997), 115-197.

[L-M-R] **Lubotzky, A., Mozes, S., and Raghunathan, M. S.,** *The word and Riemannian metrics on lattices of semisimple groups.* Inst. Hautes Études Sci. Publ. Math., **91** (2000), 5-53.

[Mar] **Margulis, G. A.,** *Discrete subgroups of semisimple Lie groups.* Ergebnisse der Mathematik und ihrer Grenzgebeite, Springer-Verlag, Berlin-Heidelberg-New York (1991).

[Mo] **Mostow, G. D.,** *Strong rigidity of locally symmetric spaces.* Princeton University Press, Princeton (1973).

[Pl-Ra] **Platonov, V., and Rapinchuk, A.,** *Algebraic groups and number theory.* Pure and Applied Mathematics, No. 139, Academic Press, Boston, (1994).

[Pr 1] **Prasad, G.,** *Strong rigidity of **Q**-rank 1 lattices.*, Invent. Math., **21** (1973), 255-86.

[Pr 2] **Prasad, G.,** *Lattices in semisimple groups over local fields.* in *Studies in algebra and number theory.* Academic Press, New York (1979), 285-354.

[Pr 3] **Prasad, G.,** *Strong approximation for semi-simple groups over function fields.* Ann. of Math., **105** (1977), 553-572.

[Ra] **Ratner, M.,** *On the p-adic and S-arithmetic generalizations of Raghunathan's conjectures.* Lie Groups and ergodic theory, Narosa Publishing House, New Delhi (1998), 167-202.

[Sch 1] **Schwartz, R.,** *The quasi-isometry classification of rank one lattices.* Inst. Hautes Études Sci. Publ. Math., **82** (1995), 133-168.

[Sch 2] **Schwartz, R.,** *Quasi-isometric rigidity and Diophantine approximation.* Acta Math., **177** (1996), 75-112.

[Ta] **Taback, J.,** *Quasi-isometric rigidity for $PSL_2(\mathbb{Z}[1/p])$.* Duke Math. J., **101** (2000), 335-357.

[Ti 1] **Tits, J.,** *Algebraic and abstract simple groups.* Ann. of Math., **80** (1964), 313-329.

[Ti 2] **Tits, J.,** *Buildings of spherical type and finite BN-pairs.* Lecture Notes in Math., vol. 386, Springer-Verlag, New York (1974).

[Ve] **Venkataramana, T. N.,** *On superrigidity and arithmeticity of lattices in semisimple groups over local fields of arbitrary characteristic.* Invent. Math., **92** (1988), 255-306.

[W1] **Wortman, K.,** *Quasiflats with holes in reductive groups.* Algebr. Geom. Topol. **6** (2006), 91-117.

[W2] **Wortman, K.,** *Quasi-isometries of* $\mathbf{SL_n}(\mathbb{F}_q[t])$. In preparation.

Yale University, Mathematics Dept., PO Box 208283, New Haven, CT 06520-8283, USA

kevin.wortman@yale.edu