Math 6510

University of Utah

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These lecture notes take from the following sources:

Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry.

Gallot-Hulin-Lafontaine, Riemannian Geometry.

Guillemin-Pollack, Differential Topology.

Milnor, Topology from the Differentiable Viewpoint.

Spivak, A comprehensive Introduction to Differential Geometry, Vol. 1.

Thurston, The geometry and Topology of Three-Manifolds.

Warner, Foundations of Differentiable Manifolds and Lie Groups.

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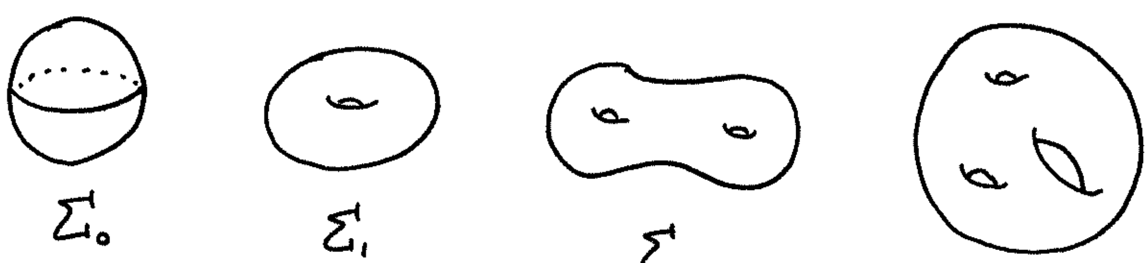
§ Surfaces

A <u>surface</u> Σ, or <u>2-manifold</u>, is a Hausdorff topological space that is locally homeomorphic to $\mathbb{R}^2$. That is, for any $p \in \Sigma$ there is an open set $U \subseteq \Sigma$ containing p that is homeomorphic to $\{x \in \mathbb{R}^2 \mid \|x\| < 1\}$.

A surface is <u>closed</u> if it is connected, compact, and has no boundary.

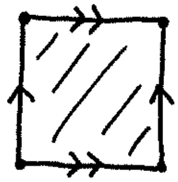
Any closed, orientable surface is homeomorphic to some Σ_n, where $n \geq 0$. Σ_n is the closed, orientable surface of <u>genus</u> n.



Σ_0 sphere, Σ_1 torus, Σ_2 two holed torus, Σ_3 three holed torus, ...

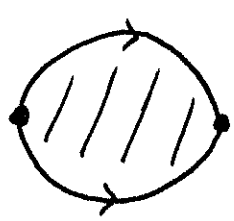
Lie group:

For $(x,y) \in [0,1]^2$, define $(x_1,y_1) \sim (x_2,y_2)$ if $|x_1 - x_2|$, $|y_1 - y_2| \in \{0,1\}$. Then

$(\mathbb{R}/\mathbb{Z})^2 = [0,1]^2/\sim =$  $=$  $= \Sigma_1$

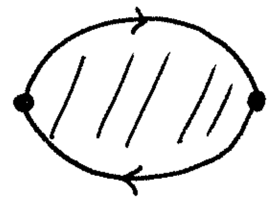
$(\mathbb{R}/\mathbb{Z})^2$ is a group, so Σ_1 is a group. Thus, Σ_1 is a surface and a group, which makes Σ_1 a Lie group. We'll see later that Σ_n is not a Lie group if $n \neq 1$.

Visualization exercise:

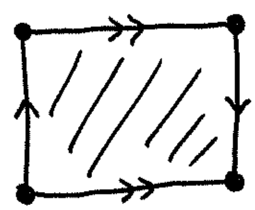
 $= \Sigma_0$

Non-orientable surfaces:

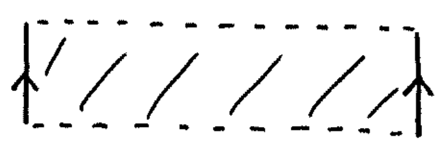
(•) $\mathbb{P}^2(\mathbb{R})$ is the real projective plane.



(•) The Klein bottle



(•) Any nonorientable surface contains an open subset that is homeomorphic to a Moebius strip.



If a Möbius band is embedded in $\mathbb{R}^3$, how many sides would it have?

Euler Characteristic:

A <u>triangulation</u> of a closed surface is a way to disect the surface into a family of triangles where each triangle consists of 3 distinct vertices, 3 edges with pairwise disjoint interiors, and is homeomorphic to $\{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$. The interior of an edge of a triangle occuring in the triangulation cannot contain a vertex of a triangle. The interior of a triangle cannot contain a vertex or any point of an edge from another triangle.

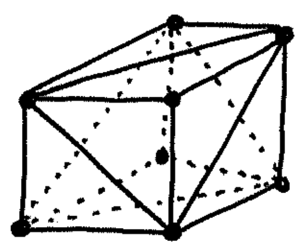
Examples of triangulations of Σ_0:

tetrahedron

4 vertices

6 edges

4 triangles, or faces



cube, with each square cut into two triangles

8 vertices

18 edges

12 faces

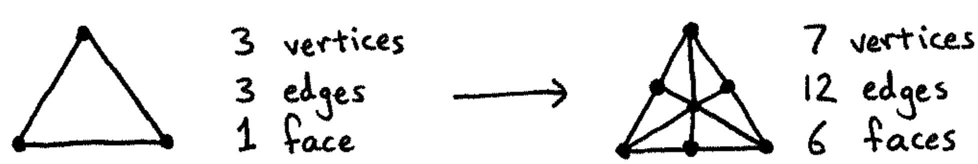
Given a triangulation of Σ_n – let's call it Δ – we let V be the number of vertices in Δ, E be the number of edges, and F be the number of faces.

The <u>Euler characteristic</u> of Δ is

$$\chi(\Delta) = V - E + F$$

<u>Examples</u>: The Euler characteristic of the triangulations of Σ_0 from the previous page are $4-6+4=2$ and $8-18+12=2$.

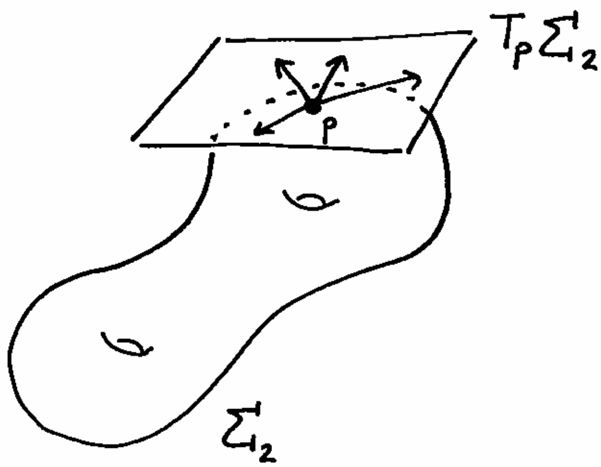
<u>Subdivision</u>: If Δ is a triangulation of Σ_n, we can make a new triangulation, Δ', each of whose edges and faces are smaller than those of Δ and such that $\chi(\Delta') = \chi(\Delta)$.



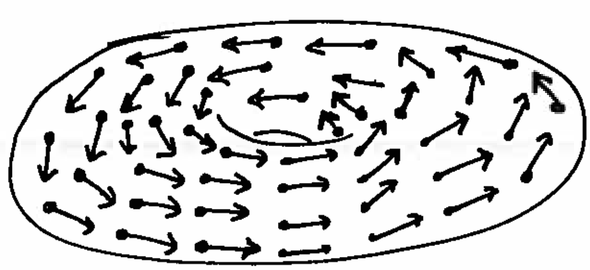
Net change in Euler characteristic: $4-9+5=0$.
This process is called subdivision.

Vector fields: A tangent vector at a point $p \in \Sigma_n$ is a choice of direction and magnitude at the point p.

The set of all tangent vectors at p forms a 2-dimensional real vector space that we denote by $T_p\Sigma_n$. We call $T_p\Sigma_n$ the tangent space of Σ_n at p.



A vector field is a continuous choice of vectors $v(p) \in T_p\Sigma_n$ for each $p \in \Sigma_n$.



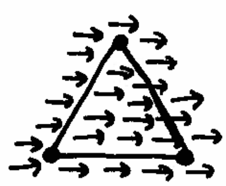
vector field on Σ_1

Theorem: Let v be a vector field on Σ_n such that $v(p) \neq 0$ for all $p \in \Sigma_n$. Then for any triangulation Δ of Σ_n we have $\chi(\Delta) = 0$.

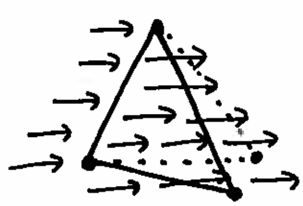
(v as in this theorem is called a nonvanishing vector field.)

proof:

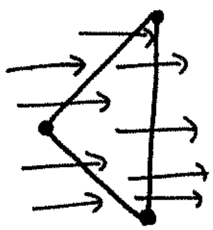
(I) Subdivide Δ many times to make a triangulation Δ' of Σ_n with $\chi(\Delta') = \chi(\Delta)$ and such that we may assume Δ' has straight edges and that v is constant on faces.



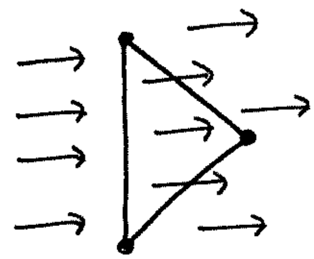
(II) Jiggle Δ' so that no vector in v is tangent to an edge.



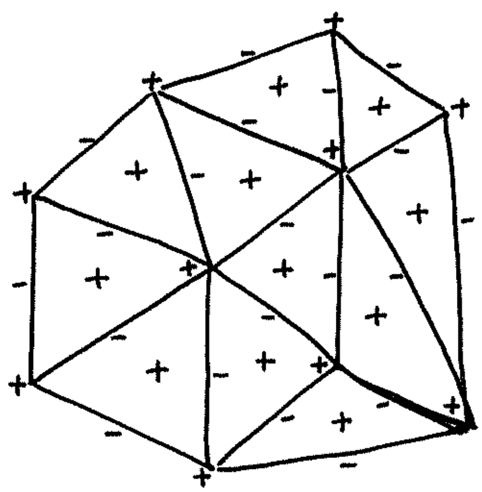
There are now two cases for each face in Δ'. Either the vectors cross into the face on 2 edges, and cross out on 1 edge, or else the vectors cross in on 1 and out on 2.



or



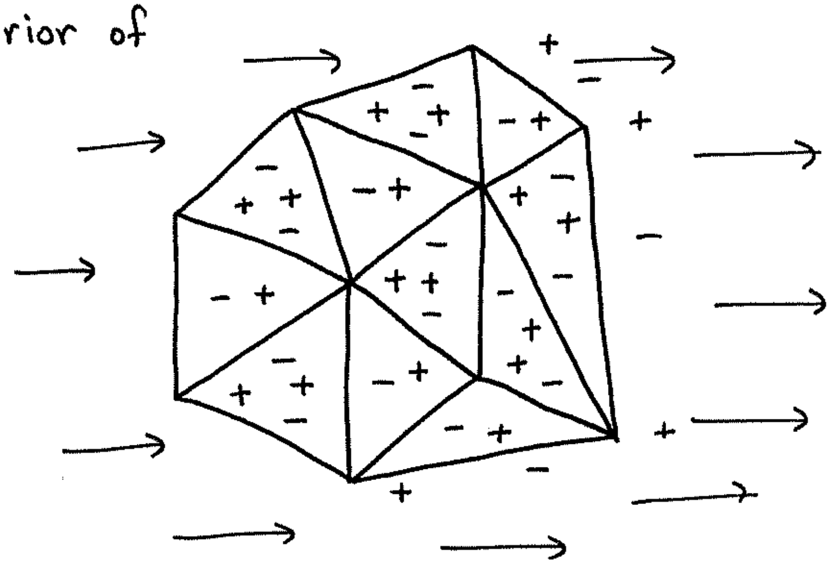
(III) Place a $+1$ on every vertex of Δ', a $+1$ in the middle of each face, and a -1 in the interior of each edge.



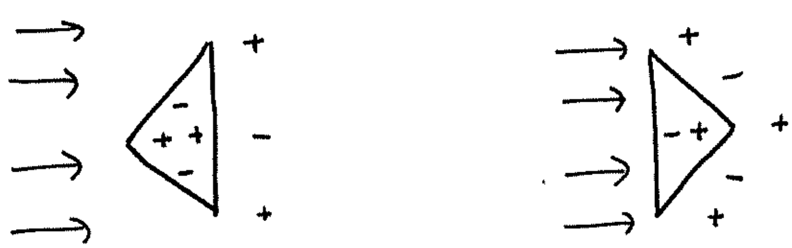
A portion of the triangulation

Notice that the sum of all ± 1's as described above is $V - E + F = \chi(\Delta')$.

(IV) Let each $+1$ and -1 travel a little bit in the direction of v, until each is in the interior of a face.



Notice now that the interior of each face contains either one $+1$ and one -1, or else two $+1^s$ and two -1^s. Indeed there are two cases:



Thus, the sum of every $+1$ and -1 on Σ'_n, that is $\chi(\Delta') = \chi(\Delta)$ is a sum of one 0 for each face of Δ'. That is $\chi(\Delta) = 0$.

END

<u>Corollary:</u> There is no nonvanishing vector field on Σ_0.

<u>proof:</u> If there were, then by Theorem 1 the Euler characteristic of a tetrahedron would equal 0. But we saw that it equalled 2.

$\boxed{\text{END}}$

<u>Corollary:</u> If Δ is a triangulation of Σ_1, then $\chi(\Delta) = 0$. Therefore, we may write $\chi(\Sigma_1) = 0$.

<u>proof:</u> Apply Theorem 1 to the vector field drawn on Σ_1 on page 10.

$\boxed{\text{END}}$

Exercises:

① Suppose Δ is a triangulation of Σ_n, and that Δ' is a triangulation of Σ_n obtained by subdividing each face of Δ simultaneously. Prove that $\chi(\Delta) = \chi(\Delta')$.

② Draw a vector field on Σ_n that vanishes at only finitely many points of Σ_n. Let Δ be a triangulation of Σ_n. Use your vector field to prove $\chi(\Delta) = 2-2n$.

Thus, the Euler characteristic of Σ_n is independent of the triangulation chosen, so we write $\chi(\Sigma_n) = 2-2n$.

§ Manifolds

An <u>n-manifold</u>, sometimes called a <u>topological n-manifold</u> is a Hausdorff topological space that is locally homeomorphic to $\mathbb{R}^n$.

More precisely, M is an n-manifold if for each $p \in M$ there is an open set $U \subseteq M$ containing p such that U is homeomorphic to an open set in $\mathbb{R}^n$.

We can make the above definition seem more precise still by adding some more notation and some Roman numerals. Parts (i)-(iii) are a rephrasing of the paragraph above, and part (iv) follows from (iii) since compositions of homeomorphisms are homeomorphisms:

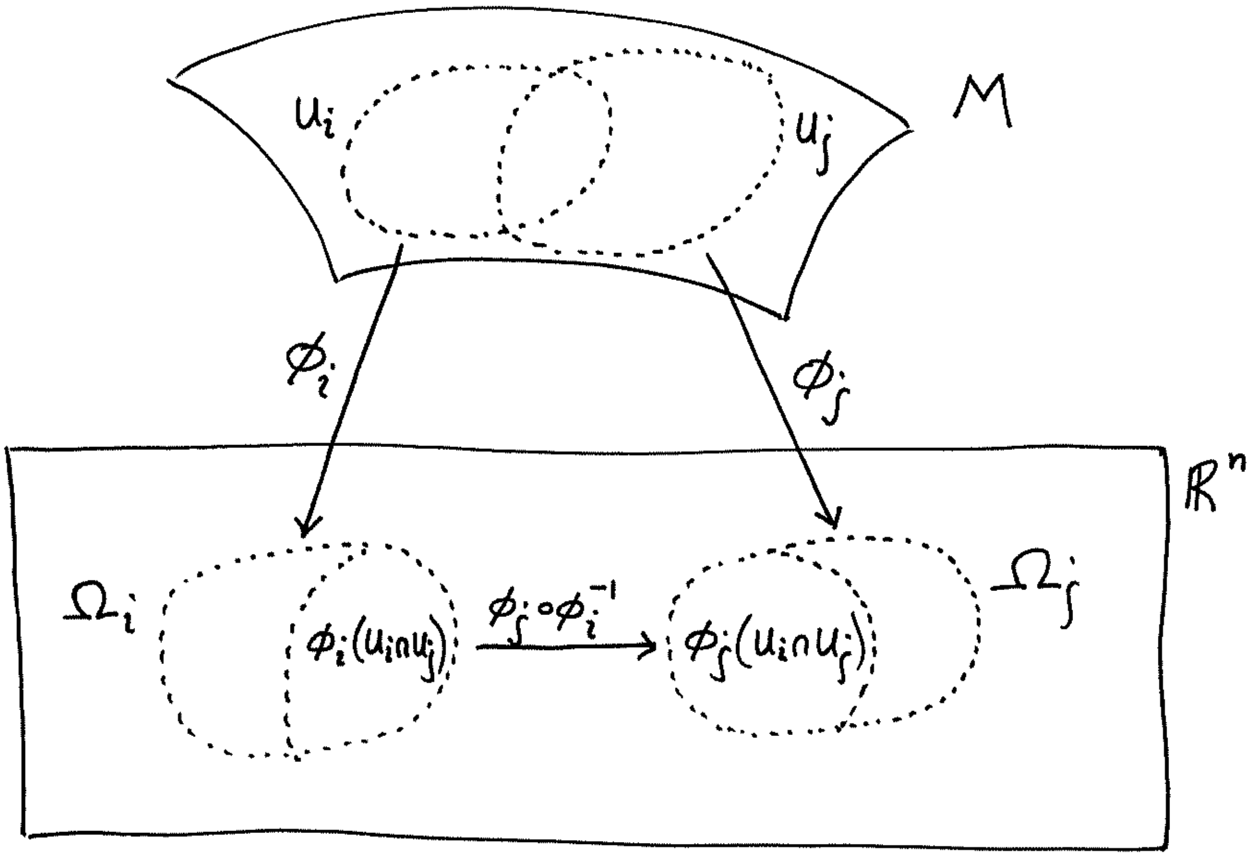
M, a Hausdorff space, is an <u>n-manifold</u> if

(i) M is covered by open subsets U_i, for $i \in I$.

(ii) There are open sets $\Omega_i \subseteq \mathbb{R}^n$, for $i \in I$.

(iii) There are homeomorphisms $\phi_i : U_i \to \Omega_i$ for $i \in I$.

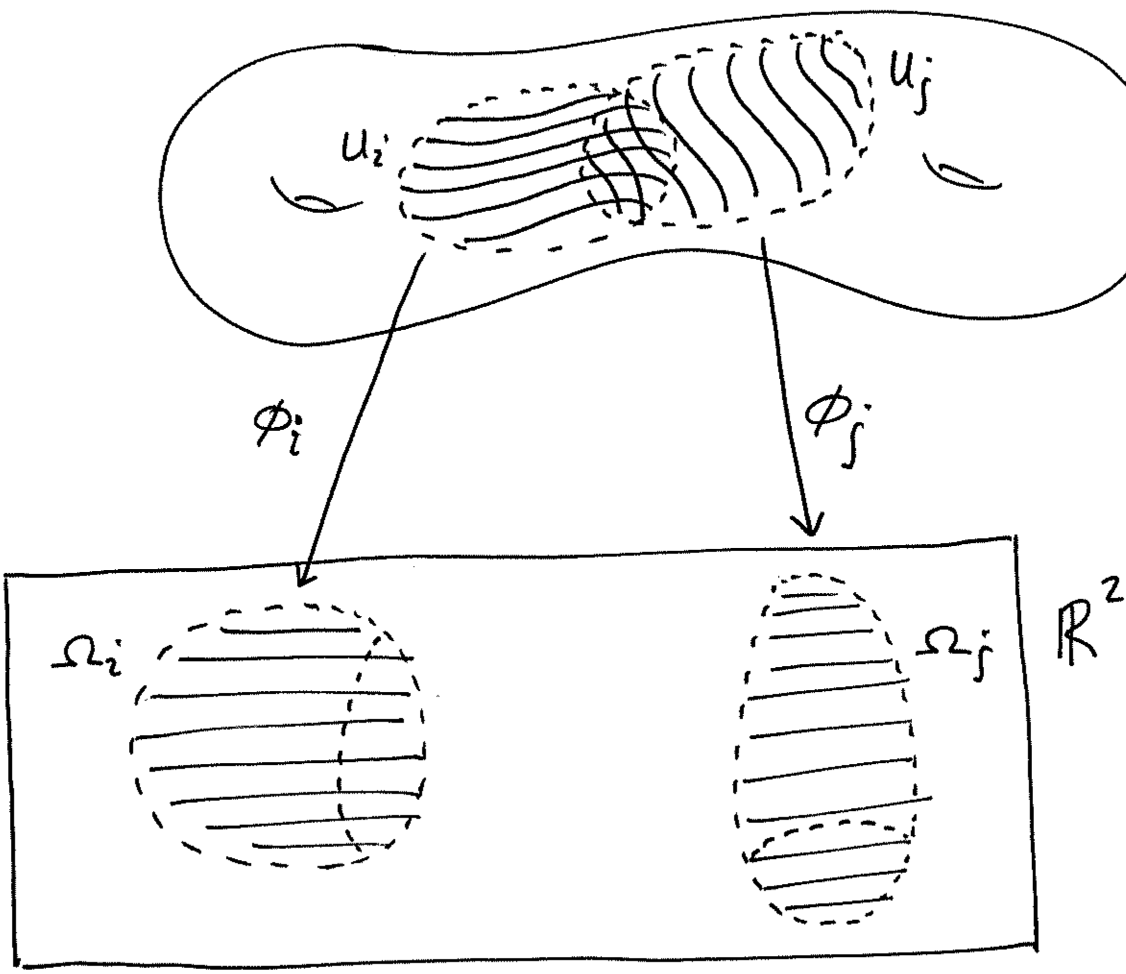
(iv) For any $i, j \in I$, $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a homeomorphism.



The maps $\phi_j \circ \phi_i^{-1}$ are called <u>transition functions</u>. For an n-manifold, transition functions are homeomorphisms, which are maps that equate topological spaces. If you'd like your manifold to display more structure than that of a topological space, then you'll need the transition functions to equate the structure that you want to display.

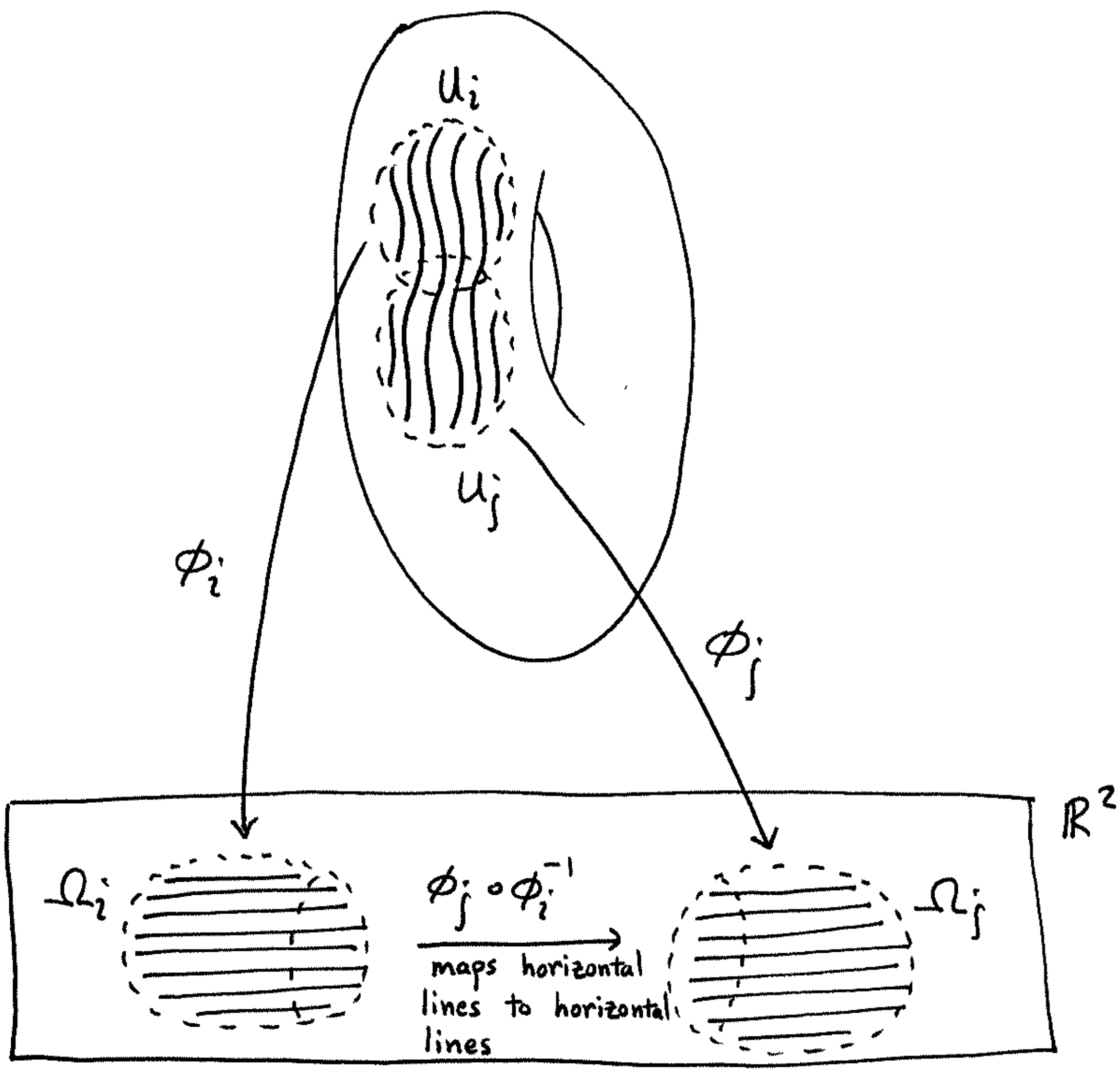
Examples

(I) Let's look at a 2-manifold. It's a topological space. If we looked at horizontal lines in the sets $\Omega_i \subseteq \mathbb{R}^2$, and if we drew those lines in U_i using ϕ_i^{-1}, then the "horizontal lines" for U_i might not match up with the "horizontal lines" in U_j.



So it doesn't make sense to speak of the structure of horizontal lines on a 2-manifold.

However, if the transition functions on your surface map horizontal lines to horizontal lines in $\mathbb{R}^2$, then the induced "horizontal lines" in the sets U_i will agree, and then we could talk about the horizontal lines on the surface.



A 2-manifold whose transition functions map horizontal lines to horizontal lines is called a <u>foliated 2-manifold.</u>

(II) If you wanted to do some geometry on your manifold, you could ask that the transition functions preserve distance in $\mathbb{R}^n$. That is, for all $i,j \in I$, and any $x,y \in \phi_i(U_i \cap U_j)$ we have $d(x,y) = d\left(\phi_j \circ \phi_i^{-1}(x), \phi_j \circ \phi_i^{-1}(y)\right)$ where d is the usual Euclidean distance in $\mathbb{R}^n$. Such a manifold is called a <u>Euclidean manifold</u>.

(III) If you had a surface, and you wanted to do some complex analysis on that surface, then you could identify $\mathbb{R}^2$ with $\mathbb{C}$ and ask that your transition functions equate those structures that allow you to do complex analysis. That is, you'd ask your transition functions to be holomorphic. Such surfaces are called <u>Riemann surfaces</u>.

<u>Functions that equate sets for purposes of calculus</u>

For open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$, we say $f: U \to V$ is a <u>smooth function</u>, or a <u>C^∞ function</u>, if f is continuous and all its derivatives exist. A <u>diffeomorphism</u> is a smooth bijective function whose inverse is smooth.

<u>Smooth manifolds</u>

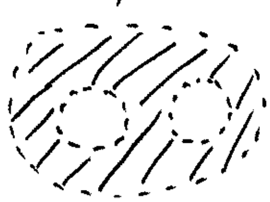
An n-manifold is a <u>smooth n-manifold</u>, or a <u>C^∞ n-manifold</u>, if its transition functions are diffeomorphisms.

<u>Examples</u>:

The collection $\{(U_i, \phi_i)\}_{i \in I}$ from the definition of a manifold are called the <u>charts</u> of the manifold.

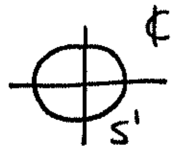
(I) $\mathbb{R}^n$ with the single chart $(\mathbb{R}^n, id)$.

(II) Any open set $U \subseteq \mathbb{R}^n$ with the chart (U, id)

 or $\{x \in \mathbb{R}^n \mid \|x\| < 1\}$ or $GL_k(\mathbb{R})$

where $GL_k(\mathbb{R})$ is the set of $k \times k$ matrices with nonzero determinant. (It's a Lie group.)

III $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$



<u>charts:</u> $U_1 = \{e^{i\theta} \mid -\pi/2 < \theta < \pi\} \subseteq S^1$

$\Omega_1 = (-\pi/2, \pi) \subseteq \mathbb{R}$

$\phi_1 : U_1 \to \Omega_1$ where $\phi_1(e^{i\theta}) = \theta$

$U_2 = \{e^{i\theta} \mid \pi/2 < \theta < 2\pi\} \subseteq S^1$

$\Omega_2 = (\pi/2, 2\pi) \subseteq \mathbb{R}$

$\phi_2 : U_2 \to \Omega_2$ where $\phi_2(e^{i\theta}) = \theta$.

Note that $U_1, U_2 \subseteq S^1$ are open, as are $\Omega_1, \Omega_2 \subseteq \mathbb{R}$. Furthermore, $U_1 \cup U_2 = S^1$ and ϕ_1 and ϕ_2 are homeomorphisms.

<u>transition functions:</u>

$$\phi_2 \circ \phi_1^{-1} : (\pi/2, \pi) \cup (-\pi/2, 0) \longrightarrow (\pi/2, \pi) \cup (3\pi/2, 2\pi)$$

$$\phi_2 \circ \phi_1^{-1}(x) = \begin{cases} x & \text{if } x \in (\pi/2, \pi) \\ x + 2\pi & \text{if } x \in (-\pi/2, 0) \end{cases}$$

This is a diffeomorphism, and inverses of diffeomorphisms are diffeomorphisms, so $\phi_1 \circ \phi_2^{-1}$ is a diffeomorphism too. Thus, S^1 is a smooth 1-manifold.

(IV) If M is a smooth n-manifold and N is a smooth k-manifold, then $M \times N$ is a smooth $(n+k)$-manifold.

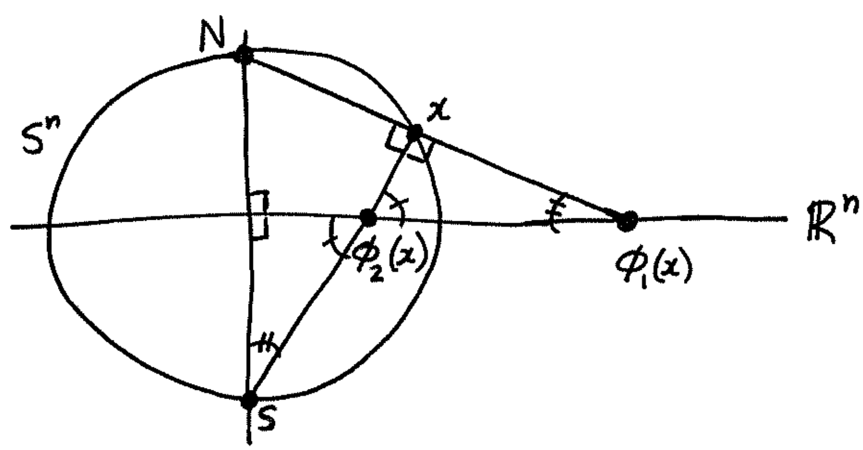
(V) $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$ is called an n-torus. $\left(T^2 = \Sigma_1\right)$

(VI) The n-sphere $S^n = \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} x_i^2 = 1 \right\}$ is a smooth n-manifold:

Let $N = (1, 0, \dots, 0)$ and $S = (-1, 0, \dots, 0)$.

Let $\phi_1 : S^n - \{N\} \longrightarrow \mathbb{R}^n$ be $\phi_1(x_0, \dots, x_n) = \dfrac{(x_1, \dots, x_n)}{1 - x_0}$

and $\phi_2 : S^n - \{S\} \longrightarrow \mathbb{R}^n$ be $\phi_2(x_0, \dots, x_n) = \dfrac{(x_1, \dots, x_n)}{1 + x_0}$



so $\dfrac{\|\phi_2(x)\|}{\|S\|} = \dfrac{\|N\|}{\|\phi_1(x)\|}$. That is, $\|\phi_2(x)\| = \dfrac{1}{\|\phi_1(x)\|}$.

Note that 0, $\phi_1(x)$, and $\phi_2(x)$ are colinear, so there is some $\lambda_x > 0$ such that $\phi_2(x) = \lambda_x \phi_1(x)$. Therefore,

$$\frac{1}{\|\phi_1(x)\|} = \|\phi_2(x)\| = \|\lambda_x \phi_1(x)\| = \lambda_x \|\phi_1(x)\|$$

Thus, $\lambda_x = \dfrac{1}{\|\phi_1(x)\|^2}$ so $\phi_2(x) = \dfrac{\phi_1(x)}{\|\phi_1(x)\|^2}$. Hence,

$$\phi_2 \circ \phi_1^{-1}(y) = \frac{\phi_1(\phi_1^{-1}(y))}{\|\phi_1(\phi_1^{-1}(y))\|^2} = \frac{y}{\|y\|^2}$$

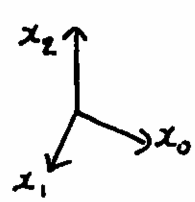
This is a diffeomorphism since $0 \notin \phi_1(S^n - \{N, S\})$.

Similarly, $\phi_1 \circ \phi_2^{-1}$ is a diffeomorphism.

VII Let $\mathbb{P}^n(\mathbb{R})$ be the set of lines in $\mathbb{R}^{n+1}$ that contain 0.

For each $0 \le i \le n$, let $U_i \subseteq \mathbb{P}^n(\mathbb{R})$ be the set of lines whose i^{th} coordinates do not equal 0.

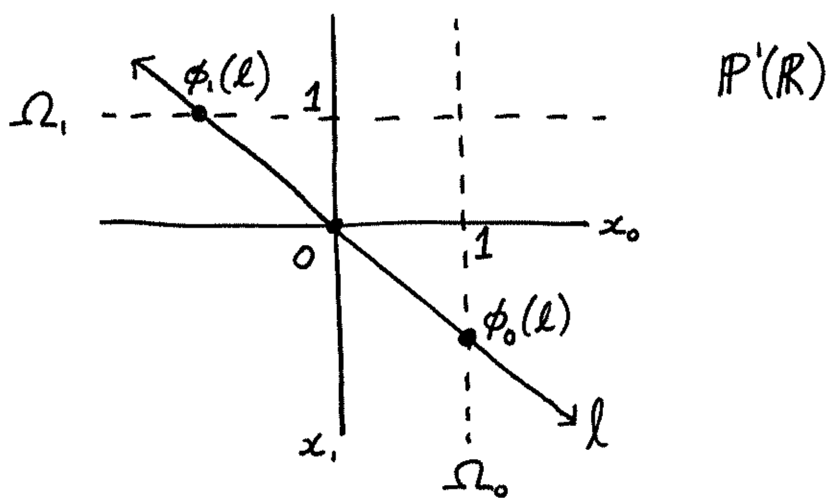
For example, $U_2 \subseteq \mathbb{P}^2(\mathbb{R})$ is the set of non horizontal lines in $\mathbb{R}^3$.



Let $\Omega_i \subseteq \mathbb{R}^n$ be the subset of $\mathbb{R}^{n+1}$ of the points $(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$ where $x_k \in \mathbb{R}$.

Let $\varphi_i : U_i \longrightarrow \Omega_i$ be

$$\varphi_i\Big(\mathbb{R}(x_0, \ldots, x_n)\Big) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_i}{x_i} = 1, \ldots, \frac{x_n}{x_i}\right)$$



Note that

$$\varphi_k \circ \varphi_i^{-1}(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = \varphi_k\Big(\mathbb{R}(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)\Big)$$

$$= \left(\frac{x_0}{x_k}, \ldots, \frac{x_{i-1}}{x_k}, \frac{1}{x_k}, \frac{x_{i+1}}{x_k}, \ldots, \frac{x_n}{x_k}\right)$$

is a diffeomorphism since any line in $U_i \cap U_k \subseteq U_k$ cannot have a coordinate x_k equal to zero.

$\mathbb{P}^1(\mathbb{R}) = S^1$. We've also seen $\mathbb{P}^2(\mathbb{R})$. $\mathbb{P}^n(\mathbb{R})$ is compact since any line contains a unit vector, and thus, the compact S^n surjects onto $\mathbb{P}^n(\mathbb{R})$.

Ridding pathologies:

Throughout this course, we'll assume that for any manifold M, there is a sequence of compact sets $K_n \subseteq M$ such that $\bigcup_{n=0}^{\infty} K_n = M$.

You can ignore this for now. You don't have to remember it. You'll be reminded once it becomes important.

Notation: Unless otherwise noted, in this text M and N will generally refer to smooth manifolds.

Exercises:

① Let p be a point contained in a chart (U_i, ϕ_i) of a foliated manifold. Define the horizontal line through p to be $\phi_i^{-1}(L \cap \phi_i(U_i))$ where L is the horizontal line in $\mathbb{R}^2$ containing $\phi_i(p)$.

Prove that the horizontal line through p is well-defined, that is that it doesn't depend on the choice of (U_i, ϕ_i).

② Show that the distance between any two points in a common chart is well-defined for a Euclidean manifold.

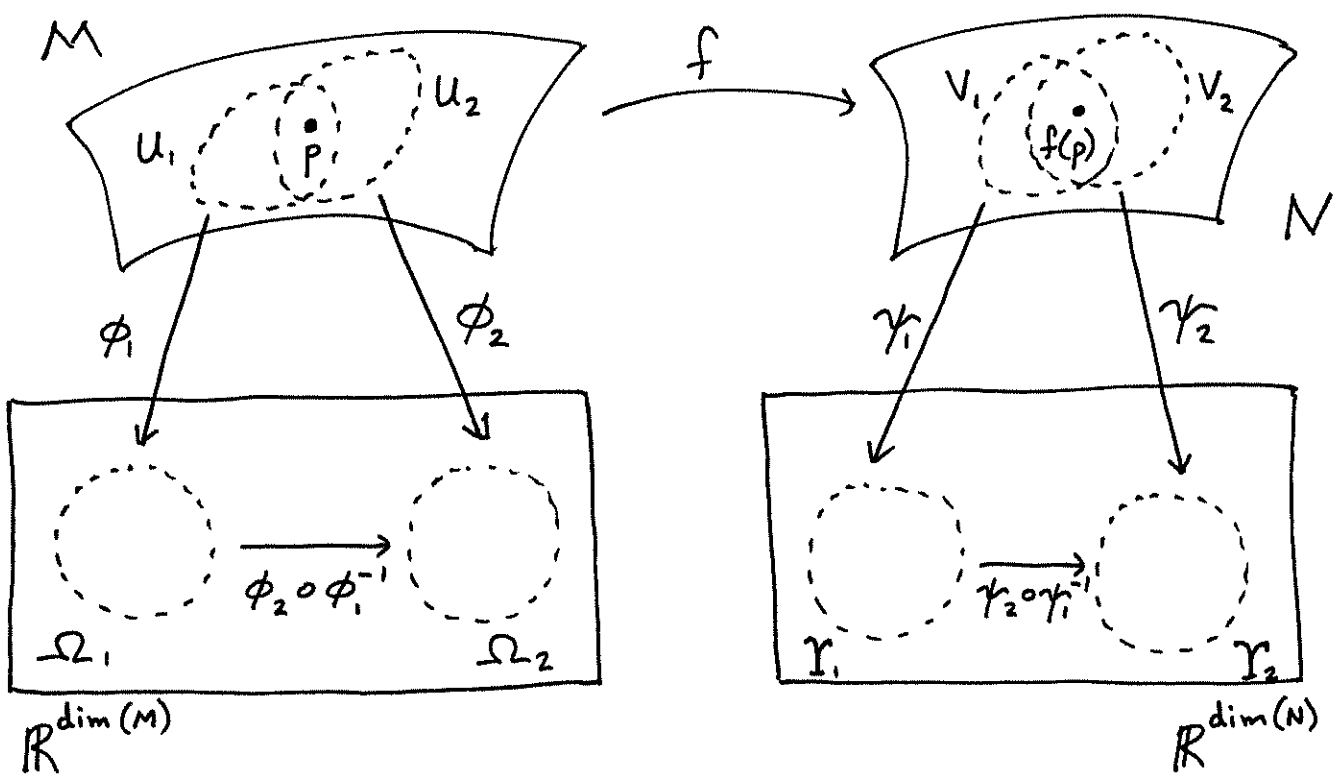
③ Let $M = \mathbb{R} \times [0,1]/\sim$ where $(t,0) \sim (-t,1)$. M is the Möbius band Find charts for M that make it into a smooth 2-manifold.

④ If M is an m-manifold, we say the <u>dimension</u> of M is m, and we write $\dim(M) = m$.
Prove that if M and N are smooth manifolds, then so is $M \times N$, and $\dim(M \times N) = \dim(M) + \dim(N)$.

§ Functions between manifolds

Let M, N be smooth manifolds. A function $f: M \to N$ is <u>smooth</u> if it is continuous, and for each $p \in M$, there is a chart of M, (U, ϕ), containing p, and a chart of N, (V, ψ), containing $f(p)$, such that $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth. (It's assumed that $f(U) \subseteq V$.)

This definition is well-defined because smooth manifolds have smooth transition functions:



$f : M \longrightarrow N$ is a <u>diffeomorphism</u> if f is smooth, bijective, and f^{-1} is smooth.

Note that if f is a diffeomorphism, so too is f^{-1}.

Diffeomorphic manifolds are thought of as being equal in the study of smooth manifolds.

Exercises:

① Suppose $M, N,$ and P are smooth manifolds and that $f: M \to N$ and $g: N \to P$ are smooth maps. Prove that $g \circ f: M \to P$ is a smooth map.

② If M is a smooth manifold, prove that the identity $id: M \to M$ is smooth.

③ Let (U, φ) be a chart for the smooth manifold M. Prove that $\varphi: U \to \varphi(U)$ is a diffeomorphism.

§ Tangent Spaces

Suppose $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open, and that $f: U \longrightarrow V$ is smooth.

Any $x \in U$ and $y \in V$ can be expressed as $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$. Let $\pi_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ be $\pi_i(y) = y_i$.

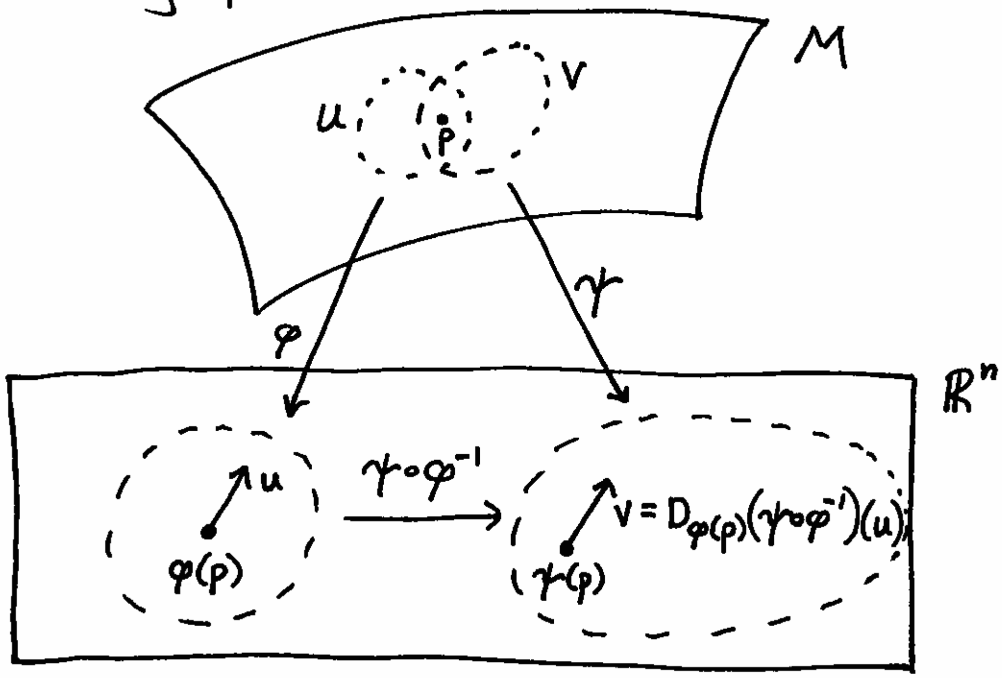
Let $D_x f$ be the $m \times n$ matrix

$$\left(D_x f\right)_{ij} = \left.\frac{\partial(\pi_i \circ f)}{\partial x_j}\right|_x$$

$D_x f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear approximation of f near x called the <u>differential of f at x</u>.

The differential satisfies the following <u>chain rule</u>: If $g: V \longrightarrow \mathbb{R}^k$ is smooth, then for any $x \in U$ we have $\left(D_{f(x)} g\right) \circ \left(D_x f\right) = D_x(g \circ f)$.

Let M be a smooth n-manifold. Let $p \in M$ and choose charts (U, φ) and (V, ψ) containing p.



By the chain rule, $\left(D_{\varphi(p)}(\psi \circ \varphi^{-1})\right)^{-1} = D_{\psi(p)}(\varphi \circ \psi^{-1})$.

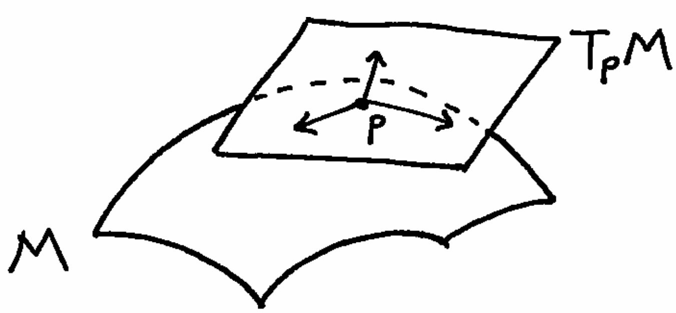
That is, $D_{\varphi(p)}(\psi \circ \varphi^{-1})$ is a linear isomorphism, so it identifies vectors in $\mathbb{R}^n$ based at $\varphi(p)$ with vectors in $\mathbb{R}^n$ based at $\psi(p)$.

The <u>tangent space to M at p</u> is defined as

$$T_pM = \left\{ (\mathcal{U}, \varphi, u) \;\middle|\; \begin{array}{l} (\mathcal{U}, \varphi) \text{ is a chart, } p \in \mathcal{U}, \text{ and } u \in \mathbb{R}^n \text{ is a} \\ \text{vector at } \varphi(p) \end{array} \right\} \Big/ \sim$$

where $(\mathcal{U}, \varphi, u) \sim (\mathcal{V}, \psi, v)$ if $D_{\varphi(p)}(\psi \circ \varphi^{-1})(u) = v$.

T_pM is a vector space. Vectors in T_pM are called <u>tangent vectors to M at p</u>.



Exercises:

① Let S^2 be the smooth manifold with charts

$$\varphi_1 : S^2 - \{N\} \longrightarrow \mathbb{R}^2 \quad \text{and} \quad \varphi_2 : S^2 - \{S\} \longrightarrow \mathbb{R}^2$$

where $N = (1,0,0)$, $S = (-1,0,0)$ and

$$\varphi_i (x_0, x_1, x_2) = \frac{(x_1, x_2)}{1 + (-1)^i x_0}$$

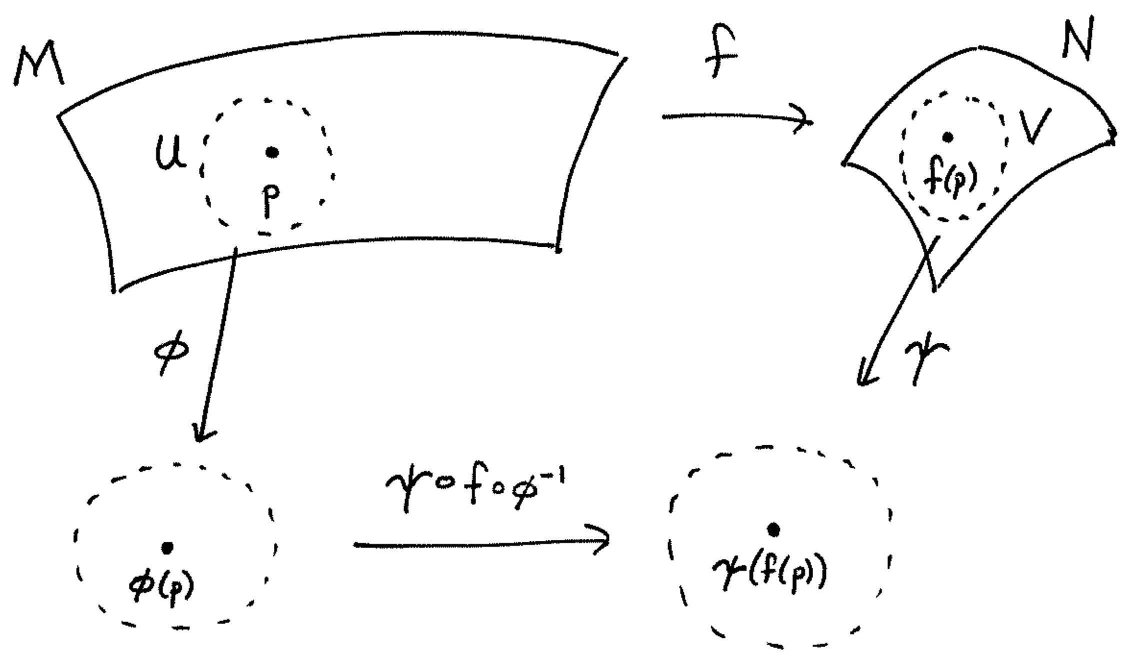
Suppose $u \in \mathbb{R}^2$ so that $(S^2 - \{N\}, \varphi_1, u)$ represents a tangent vector to S^2 at some point $p \in S^2 - \{N, S\}$. Find $v \in \mathbb{R}^2$ such that $(S^2 - \{N\}, \varphi_1, u) \sim (S^2 - \{S\}, \varphi_2, v)$.

② Prove if M is a smooth manifold and $p \in M$, then T_pM is an $\mathbb{R}$-vector space. That is, show that if $(U, \varphi, u_1) \sim (V, \psi, v_1)$ and $(U, \varphi, u_2) \sim (V, \psi, v_2)$, then $(U, \varphi, u_1 + u_2) \sim (V, \psi, v_1 + v_2)$ and $(U, \varphi, \lambda u_1) \sim (V, \psi, \lambda v_1)$ for any $\lambda \in \mathbb{R}$.

③ For $(m,n) \in M \times N$, prove $T_{(m,n)}(M \times N) \cong T_mM \times T_nN$ using $[(U \times V, \varphi \times \psi, (u,v))] \longmapsto ([(U, \varphi, u)], [(V, \psi, v)])$.

§ Differentials for manifolds

Let $f: M \longrightarrow N$ be a smooth map between smooth manifolds. For $p \in M$, choose a chart (U, ϕ) containing p, and a chart (V, ψ) containing $f(p)$.



The <u>differential of f at p</u> is the linear map $D_p f : T_p M \longrightarrow T_{f(p)} N$ defined as

$$D_p f\left[(U, \phi, u)\right] = \left[\left(V, \psi, D_{\phi(p)}(\psi \circ f \circ \phi^{-1})(u)\right)\right]$$

The differential is well-defined. Indeed, if $(\hat{U},\hat{\phi})$ and $(\hat{V},\hat{\psi})$ are alternatives to (U,ϕ) and (V,ψ) respectively, and if $[(U,\phi,u)]=[(\hat{U},\hat{\phi},\hat{u})]$ so that $D_{\hat{\phi}(p)}(\phi\circ\hat{\phi}^{-1})(\hat{u})=u$, then

$$[(\hat{V},\hat{\psi},D_{\hat{\phi}(p)}(\hat{\psi}\circ f\circ\hat{\phi}^{-1})(\hat{u}))]$$

$$=[(\hat{V},\hat{\psi},D_{\hat{\phi}(p)}(\hat{\psi}\circ\psi^{-1}\circ\psi\circ f\circ\phi^{-1}\circ\phi\circ\hat{\phi}^{-1})(\hat{u}))]$$

$$=[(\hat{V},\hat{\psi},D_{\psi(f(p))}(\hat{\psi}\circ\psi^{-1})\,D_{\phi(p)}(\psi\circ f\circ\phi^{-1})\,D_{\hat{\phi}(p)}(\phi\circ\hat{\phi}^{-1})(\hat{u}))]$$

$$=[(V,\psi,D_{\phi(p)}(\psi\circ f\circ\phi^{-1})\,D_{\hat{\phi}(p)}(\phi\circ\hat{\phi}^{-1})(\hat{u}))]$$

$$=[(V,\psi,D_{\phi(p)}(\psi\circ f\circ\phi^{-1})(u))]$$

Note that $D_p f : T_p M \longrightarrow T_{f(p)} N$ is linear since $D_{\phi(p)}(\gamma \circ f \circ \phi^{-1})$ is linear.

We have the following lemma, the Chain Rule for manifolds:

<u>Lemma</u>: Let M, N, P be smooth manifolds and $f : M \longrightarrow N$ and $g : N \longrightarrow P$ be smooth maps. Then, for any $p \in M$,

$$D_p(g \circ f) = D_{f(p)}(g) \circ D_p(f).$$

Exercises:

① Prove the Chain Rule for manifolds.

② Prove that for any $p \in M$, $D_p id : T_pM \to T_pM$ is the identity map where $id : M \to M$ is the identity map.

③ Let $f : M \to N$ be a diffeomorphism of smooth manifolds. Use ① and ② to prove that if $p \in M$, then $D_p f$ is an isomorphism of vector spaces.

④ Use ③ to prove that diffeomorphic smooth manifolds have the same dimension.

§ Manifolds from group actions

Given a smooth manifold M, there is a group

$$\mathrm{Diff}(M) = \{ f : M \to M \mid f \text{ is a diffeomorphism} \}$$

$\Gamma \leq \mathrm{Diff}(M)$ acts <u>freely</u> on M if for any $p \in M$ and $\gamma \in \Gamma$, the equation $\gamma(p) = p$ implies $\gamma = 1$.

$\Gamma \leq \mathrm{Diff}(M)$ acts <u>properly discontinuously</u> if for any compact $K \subseteq M$,

$$\left| \{ \gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset \} \right| < \infty$$

<u>Proposition:</u> If $\Gamma \leq \mathrm{Diff}(M)$ acts freely and properly discontinuosly, then $\Gamma \backslash M$ is a smooth manifold, and $\dim(\Gamma \backslash M) = \dim(M)$.

<u>Examples:</u>

① For $n \in \mathbb{Z}$, let $\gamma_n \in \mathrm{Diff}(\mathbb{R})$ be $\gamma_n(x) = x + n$. Then $\{\gamma_n\}_{n \in \mathbb{Z}} \cong \mathbb{Z}$ and $\mathbb{Z} \backslash \mathbb{R} = S^1$.

② Let $n \in \mathbb{Z}^k$ and define $\gamma_n \in \mathrm{Diff}(\mathbb{R}^k)$ by $\gamma_n(x) = x + n$.
Then $\{\gamma_n\}_{n \in \mathbb{Z}^k} \cong \mathbb{Z}^k$ and $\mathbb{Z}^k \backslash \mathbb{R}^k = T^k$.

③ Let $\alpha, \beta \in \mathrm{Diff}(\mathbb{R}^2)$ be $\alpha(x,y) = (x, y+1)$ and $\beta(x,y) = (x+1, 1-y)$. Then $\langle \alpha, \beta \rangle \backslash \mathbb{R}^2$ is a Klein bottle.

④ For $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$, let $a \in \mathrm{Diff}(S^n)$ be $a(x) = -x$. Then $\langle a \rangle \cong \mathbb{Z}/2\mathbb{Z}$ and $\langle a \rangle \backslash S^n = \mathbb{P}^n(\mathbb{R})$, the <u>n-dimensional real projective space</u>.

⑤ Let $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ be $\gamma(x) = 2x$. Then $\langle \gamma \rangle \cong \mathbb{Z}$, $\langle \gamma \rangle \leq \mathrm{Diff}(\mathbb{R}^n - \{0\})$, and $\langle \gamma \rangle \backslash \mathbb{R}^n - \{0\} = S^{n-1} \times S^1$.

⑥ Let $p, q \in \mathbb{N}$ with $(p,q) = 1$. Let $\gamma \in \mathrm{Diff}(\mathbb{C}^2)$ be $\gamma(z,w) = \left(e^{2\pi i/p} z, e^{2\pi i q/p} w\right)$. Then $\langle \gamma \rangle \cong \mathbb{Z}/p\mathbb{Z}$ and $\langle \gamma \rangle \leq \mathrm{Diff}(S^3)$ where

$$S^3 = \{(z,w) \in \mathbb{C}^2 \mid z = z_1 + i z_2,\ w = w_1 + i w_2,\ z_1^2 + z_2^2 + w_1^2 + w_2^2 = 1\}$$

Then $\mathbb{Z}/p\mathbb{Z} \backslash S^3$ is called a <u>(p,q)-Lens space</u> and is denoted $L_{(p,q)}$.

⑦ Let M be a smooth manifold with $f \in \mathrm{Diff}(M)$. Define $\bar{f} \in \mathrm{Diff}(M \times \mathbb{R})$ by $\bar{f}(m,t) = (f(m), t+1)$ so that $\langle \bar{f} \rangle \cong \mathbb{Z}$. Let $M_f = {}^{M \times \mathbb{R}}\!/\!_{\langle \bar{f} \rangle}$.

An important method for constructing 3-manifolds is creating $(\Sigma_{1n})_f$ for $f \in \mathrm{Diff}(\Sigma_{1n})$.

Note $S^1_{id} = \Sigma_{10}$, while $S^1_{x \mapsto -x}$ is a Klein bottle.

Proof of Proposition:

Step 1 : $\Gamma \backslash M$ is Hausdorff

(•) Suppose $\Gamma p \neq \Gamma q$ for $p, q \in M$.

(•) Choose disjoint nbhds $p \in U_p \subseteq M$ and $q \in U_q \subseteq M$ with $\overline{U_p}$ and $\overline{U_q}$ compact.

(•) Let $V_p = U_p - \bigcup_{\gamma \in \Gamma - 1} \gamma(\overline{U_p} \cup \overline{U_q})$ and

$$V_q = U_q - \bigcup_{\gamma \in \Gamma - 1} \gamma(\overline{U_p} \cup \overline{U_q}).$$

(•) By proper discontinuity, the relevant portions of the above unions are finite.

(•) By freeness, and the finiteness above, we may replace U_p and U_q with smaller subsets of themselves such that $p \in V_p$ and $q \in V_q$.

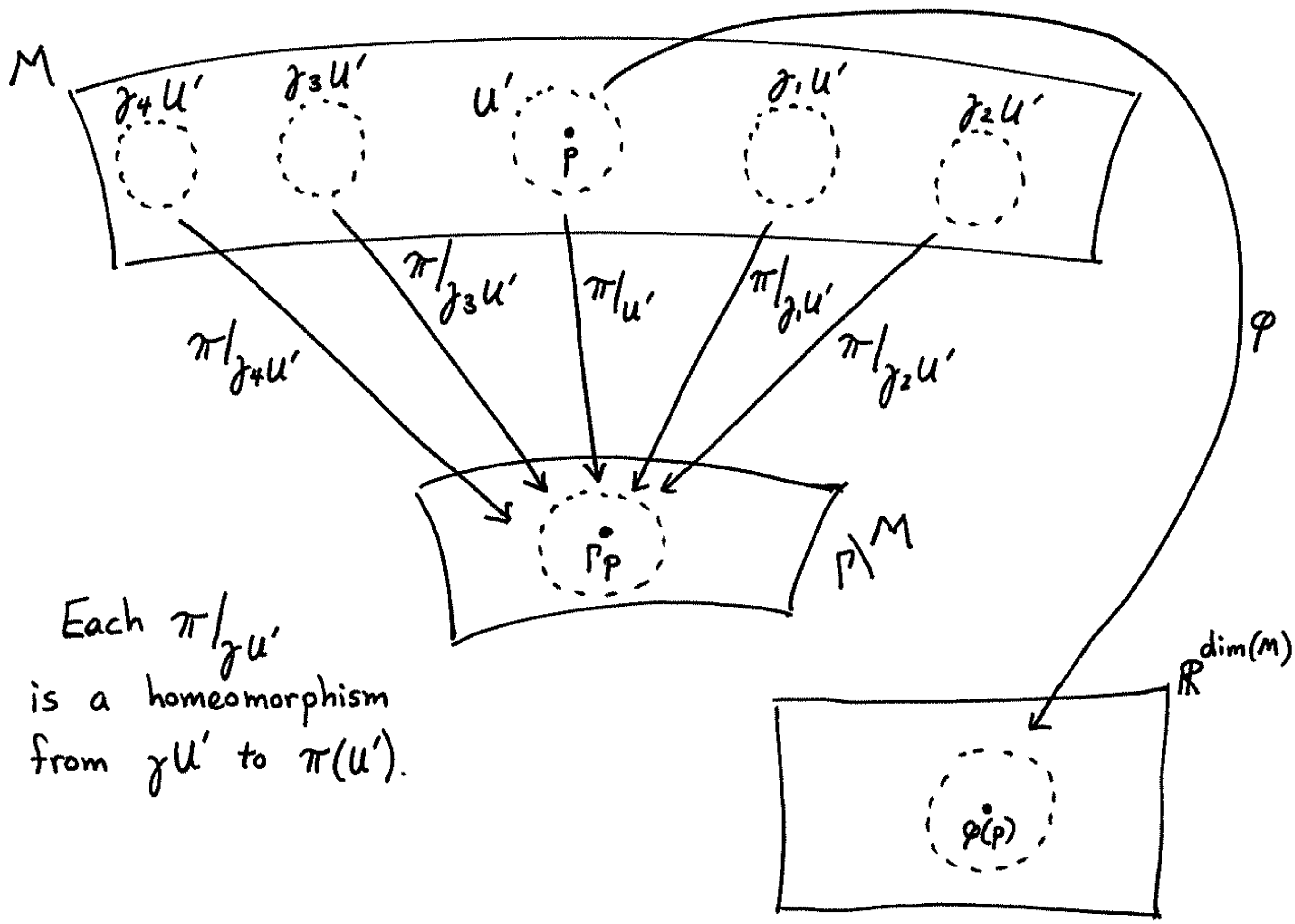
(•) Note that $V_p \cap \Gamma V_q \subseteq V_p \cap \Gamma U_q = \emptyset$ and $V_q \cap \Gamma V_p = \emptyset$. Therefore, ΓV_p and ΓV_q are disjoint open sets in $\Gamma \backslash M$ that contain Γp and Γq, respectively.

<u>Step 2:</u> $M = \bigcup_{i=1}^{\infty} K_i$ for compact sets $K_i \subseteq K_{i+1}$ so $\Gamma \backslash M = \bigcup_{i=1}^{\infty} \Gamma K_i$ where ΓK_i is the image of K_i in $\Gamma \backslash M$, and thus is compact.

<u>Step 3:</u> Charts.

(•) Let $\Gamma p \in \Gamma \backslash M$ for $p \in M$.

- Let $(\mathcal{U},\varphi)$ be a chart containing p with $\overline{\mathcal{U}}$ compact.
- $\{\gamma\in\Gamma \mid \gamma\overline{\mathcal{U}}\cap\overline{\mathcal{U}}\neq\emptyset\}$ is finite, and Γ acts freely, so we may choose $\mathcal{U}'\subseteq\mathcal{U}$ such that $\gamma\mathcal{U}'\cap\mathcal{U}'=\emptyset$ if $\gamma\neq 1$.
- Let $\pi: M\longrightarrow \Gamma\backslash M$ be the quotient map. By the above, $\pi|_{\mathcal{U}'}$ is a homeomorphism. Let $\left(\pi(\mathcal{U}'),\ \varphi\circ(\pi|_{\mathcal{U}'})^{-1}\right)$ be a chart containing Γp.

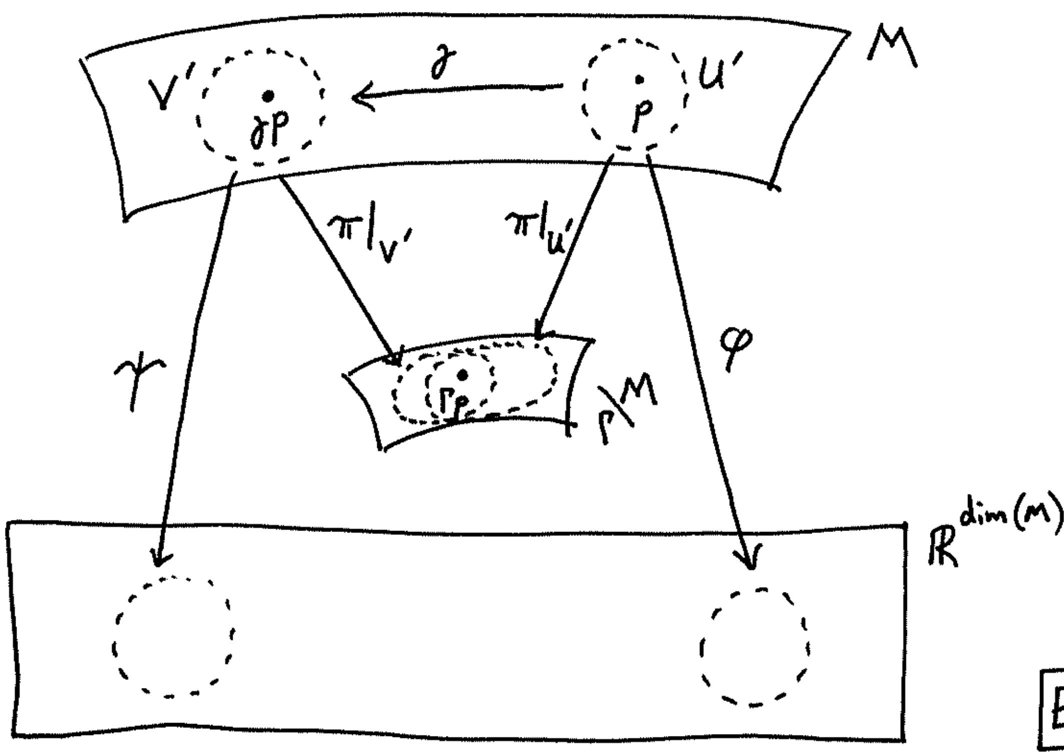


Each $\pi|_{\gamma\mathcal{U}'}$ is a homeomorphism from $\gamma\mathcal{U}'$ to $\pi(\mathcal{U}')$.

(•) If V' is a set similar to U' as above, containing some $\gamma p \in \Gamma_p$ with chart $\left(\pi(V'), \psi \circ (\pi|_{V'})^{-1}\right)$ for $\Gamma\backslash M$, then the transition function

$$\left(\psi \circ (\pi|_{V'})^{-1}\right) \circ \left(\varphi \circ (\pi|_{U'})^{-1}\right)^{-1}$$
$$=$$
$$\psi \circ (\pi|_{V'})^{-1} \circ \pi|_{U'} \circ \varphi^{-1}$$
$$=$$
$$\psi \circ \gamma \circ \varphi^{-1}$$

is smooth since $\gamma \in \mathrm{Diff}(M)$ is smooth.

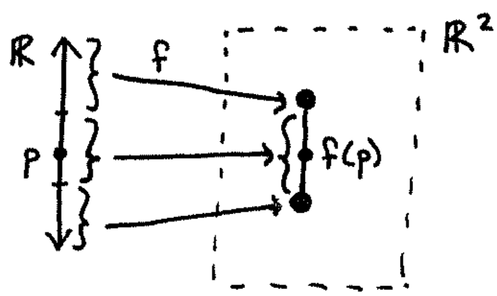


$\boxed{\text{END}}$

§ Types of smooth maps

Let $f: M \longrightarrow N$ be a smooth map between smooth manifolds.

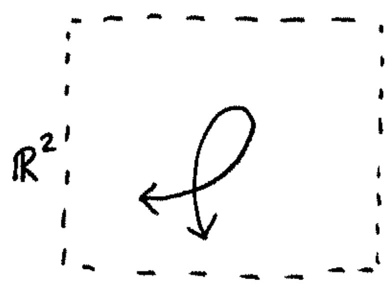
(•) f is an <u>immersion at $p \in M$</u> if $D_p f$ is injective.



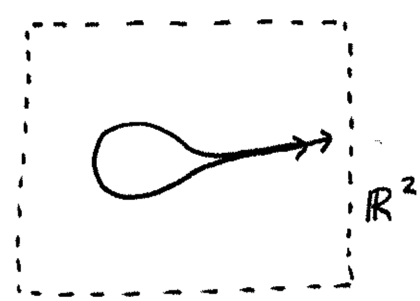
$f: \mathbb{R} \longrightarrow \mathbb{R}^2$

(•) f is an <u>immersion</u> if $D_p f$ is injective for all $p \in M$.

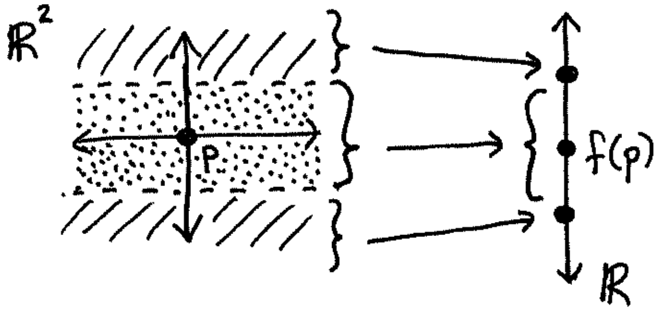
$f: \mathbb{R} \longrightarrow \mathbb{R}^2$



or

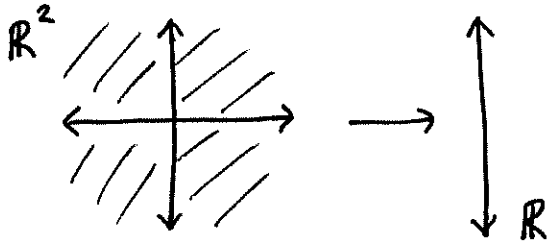


(•) f is a <u>submersion at $p \in M$</u> if $D_p f$ is surjective.



$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

(•) f is a <u>submersion</u> if $D_p f$ is surjective for all $p \in M$.



$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$f(x,y) = y$

(•) f is a <u>local diffeomorphism</u> if f is both an immersion and a submersion.

$f : \mathbb{R} \longrightarrow S^1$

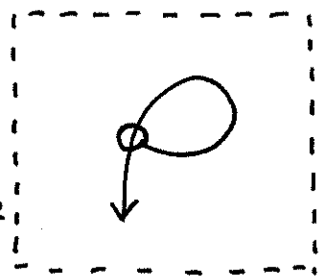
$f(\theta) = e^{i\theta}$

or

$\pi : M \longrightarrow \Gamma \backslash M$

Given any of the types of maps defined above, what can you say about $\dim(M)$ compared to $\dim(N)$?

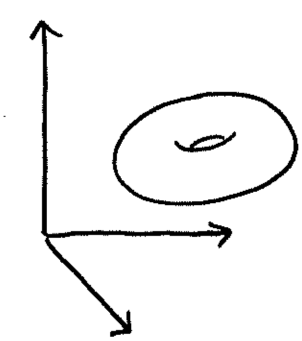
(•) An <u>immersed submanifold</u> is an injective immersion $i_M : M \hookrightarrow N$. For any $p \in M$, we identify T_pM with $D_p i_M(T_pM) \leq T_pN$, so that $T_pM = D_p i_M(T_pM)$. We also will at times identify M with $i_M(M) \subseteq N$, though this should be done with caution as the topologies on M and $i_M(M) \subseteq N$ can be quite different.



$\mathbb{R}^2$

$$i_{(0,\infty)} : (0,\infty) \longrightarrow \mathbb{R}^2$$

(•) An <u>embedding</u> is an injective immersion $i_M : M \hookrightarrow N$ such that $i_M : M \longrightarrow i_M(M)$ is a homeomorphism. In this case, we call M an <u>embedded submanifold</u>, and we usually write that $M \subseteq N$.



$$\Sigma_1 \subseteq \mathbb{R}^3$$

Exercises:

① A continuous map between topological spaces is *proper* if the inverse image of any compact set is compact. Prove that if $f: M \to N$ is continuous, proper, and injective, then $f: M \to f(M)$ is a homeomorphism.

② Let $f: M \to N$ be a smooth, injective, proper immersion. Prove f is an embedding.

③ Let $f: M \to N$ be a smooth map of manifolds. Show that the set of $p \in M$ such that $D_p f$ is bijective (respectively, injective or surjective) is open.

④ Prove the is no immersion of S^1 into $\mathbb{R}$.

⑤ Suppose $f: M \to N$ is an immersion and $\dim(M) = \dim(N)$. Prove f is an open map.

⑥ Prove there are no immersions of S^n, T^n, or $\mathbb{P}^n(\mathbb{R})$ into $\mathbb{R}^n$.

⑦ If $f: M \to N$ is smooth and $Z \subseteq M$ is an embedded submanifold, prove $f|_Z : Z \to N$ is smooth and $D_p(f|_Z) = (D_p f)\big|_{T_p Z}$ for all $p \in Z$. (Hint: $f|_Z = f \circ i_Z$)

⑧ With M, N, f, Z as in ⑦, suppose $Q \subseteq N$ is an embedded submanifold and that $f(Z) \subseteq Q$. Prove that $D_p f(T_p Z) \subseteq T_{f(p)} Q$ for all $p \in Z$.

§ Inverse Function Theorem

Calculus version of IFT:

Suppose $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is smooth and that $D_x f$ is an isomorphism for some $x \in \mathbb{R}^n$.

Then there is a neighborhood $\mathcal{O} \subseteq \mathbb{R}^n$ containing x such that $f(\mathcal{O})$ is open, and $f|_{\mathcal{O}}: \mathcal{O} \longrightarrow f(\mathcal{O})$ is a diffeomorphism.

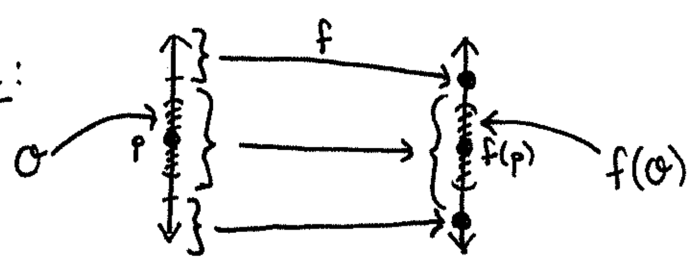
(A good reference for Calculus IFT is Spivak's "Calculus on manifolds.)

Manifold version of IFT:

Suppose $f: M \longrightarrow N$ is a smooth map of smooth manifolds and that $D_p f$ is an isomorphism for some $p \in M$.

Then there is a neighborhood $\mathcal{O} \subseteq M$ containing p such that $f(\mathcal{O})$ is open, and $f|_{\mathcal{O}}: \mathcal{O} \longrightarrow f(\mathcal{O})$ is a diffeomorphism.

Example:

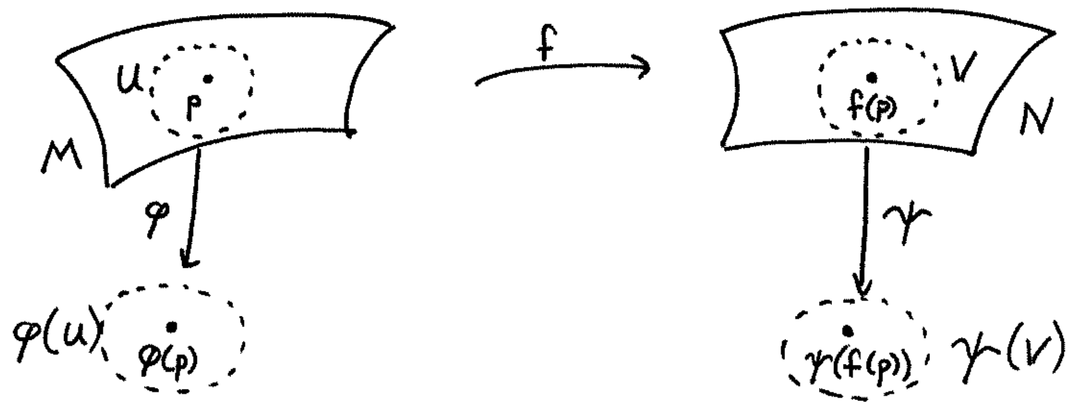


$f: \mathbb{R} \longrightarrow \mathbb{R}$

Proof of MIFT: This is a local result, so using charts reduces to CIFT.

$\boxed{\text{END}}$

Expanded proof of MIFT: Let (U, φ) be a chart on M containing p, (V, ψ) a chart on N containing $f(p)$.



D_pf an isomorphism $\Rightarrow D_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})$ an isomorphism.

$\Rightarrow$ CIFT implies open $\mathcal{O} \subseteq \varphi(U)$ containing $\varphi(p)$ such that $\psi \circ f \circ \varphi^{-1}|_{\mathcal{O}} : \mathcal{O} \to \psi \circ f \circ \varphi^{-1}(\mathcal{O})$ is a diffeomorphism.

$\Rightarrow f|_{\varphi^{-1}(\mathcal{O})}$ is bijective and $\varphi \circ f^{-1} \circ \psi^{-1}|_{\psi \circ f \circ \varphi^{-1}(\mathcal{O})}$ is smooth.

$\Rightarrow \left(f|_{\varphi^{-1}(\mathcal{O})}\right)^{-1}$ is smooth.

$\Rightarrow f|_{\varphi^{-1}(\mathcal{O})}$ is a diffeomorphism.

$\boxed{\text{END}}$

§ Regular Value Theorem

Suppose $f: M \longrightarrow N$ is a smooth map of manifolds. A point $y \in N$ is a <u>regular value</u> if $D_p f$ is surjective for all $p \in f^{-1}(y)$. The point y is a <u>critical value</u> otherwise.

<u>Regular Value Theorem</u>:

If $f: M \longrightarrow N$ is a smooth map of smooth, connected manifolds, and if $y \in N$ is a regular value with $f^{-1}(y) \neq \emptyset$, then $f^{-1}(y)$ is a smooth manifold of dimension $\dim(M) - \dim(N)$. Furthermore, $f^{-1}(y) \subseteq M$ is embedded.

<u>Corollary</u>: S^n is a smooth manifold of dimension n.

<u>proof</u>:

Let $f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be $f(x_0, \ldots, x_n) = \sum_{i=0}^{n} x_i^2$. Since f is a polynomial, it's smooth. Note that $f^{-1}(1) = S^n$ and that $D_p f \neq 0$ if $p \in S^n$ so that $D_p f$ is surjective and 1 is a regular value.

Last, $\dim(\mathbb{R}^{n+1}) - \dim(\mathbb{R}) = n$.

$\boxed{\text{END}}$

<u>Corollary</u>: Let $SL_n(\mathbb{R})$ be the group of $n \times n$ matrices with determinant 1, the <u>special linear group</u>. Then $SL_n(\mathbb{R})$ is a manifold of dimension n^2-1.

<u>proof</u>:

$\det : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$ is a polynomial, $SL_n(\mathbb{R}) = \det^{-1}(1)$, and $D_p \det \neq 0$ if $p \in SL_n(\mathbb{R})$.

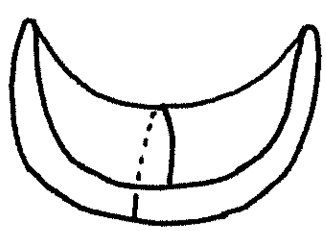
$\boxed{\text{END}}$

<u>Example</u>:

$g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ with $g(x,y,z) = z - xy$ has

$D_p g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)\Big|_p = (-y, -x, 1)\Big|_p$ which has rank 1

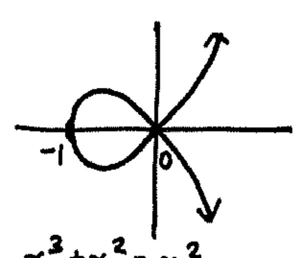
for all $p \in \mathbb{R}^3$. Thus, $g^{-1}(0)$ is a smooth manifold.



$xy = z$

<u>Non example</u>:

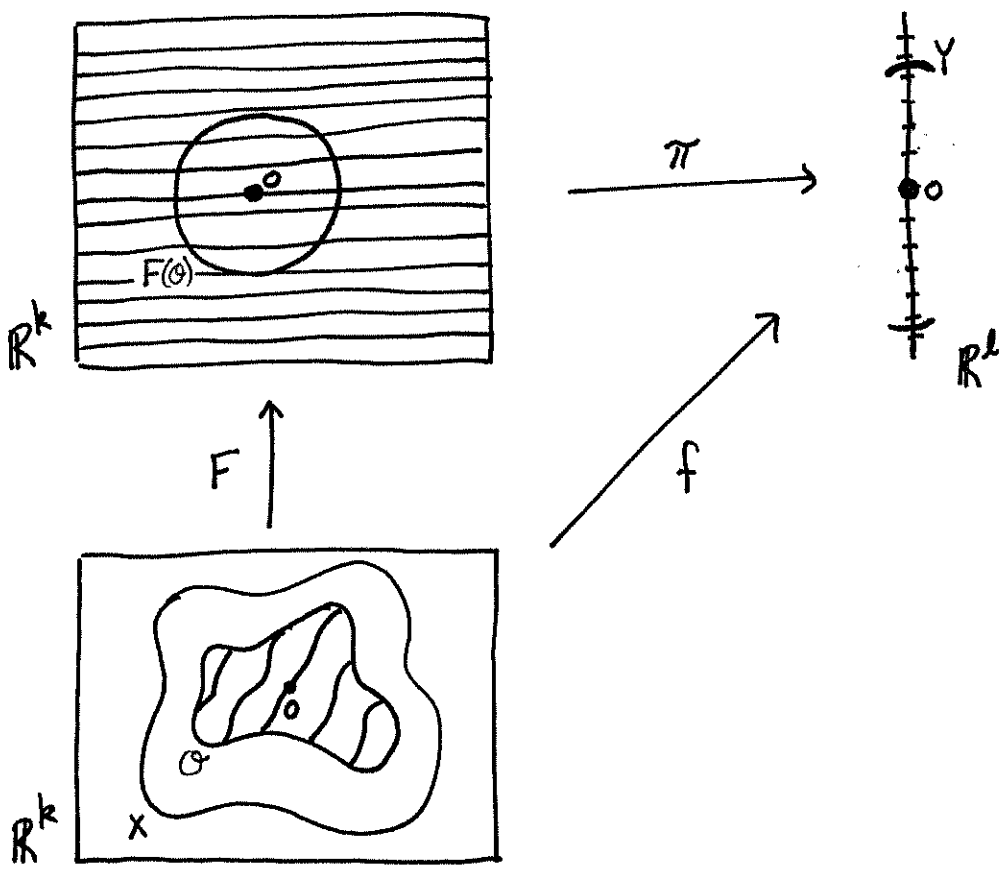
For $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $f(x,y) = x^3 + x^2 - y^2$
0 is a critical value since $(0,0) \in f^{-1}(0)$
and $D_{(0,0)} f = (3x^2 + 2x, -2y)\Big|_{(0,0)} = (0,0)$ is not surjective, so $f^{-1}(0)$ needn't be a manifold.



$x^3 + x^2 = y^2$

<u>Lemma:</u> Suppose $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^\ell$ are open neighborhoods of 0, that $f : X \longrightarrow Y$ is smooth, that $f(0) = 0$, and that f is a submersion at 0.

Then there is a neighborhood $\mathcal{O} \subseteq X$ of 0 and a diffeomorphism $F : \mathcal{O} \longrightarrow F(\mathcal{O}) \subseteq \mathbb{R}^k$ such that $F(\mathcal{O})$ is open, $F(0) = 0$, and $f \circ F^{-1} = \pi$ where $\pi : \mathbb{R}^k \longrightarrow \mathbb{R}^\ell$ is projection onto the first ℓ coordinates.



Proof:

$D_0 f$ is linear and surjective, so after a linear change of basis in $\mathbb{R}^k$, we may assume that

$$D_0 f = \left(Id_{\ell \times \ell} \,\middle|\, O_{\ell \times (k-\ell)} \right)$$

Define $F : X \to \mathbb{R}^k$ by $F(x_1,\dots,x_k) = \left(f(x_1,\dots,x_k), x_{\ell+1},\dots,x_k\right)$ and note that $f = \pi \circ F$.

If e_i is a standard basis element of $\mathbb{R}^k$, and if $1 \le i \le \ell$, then $D_0 F(e_i) = D_0 f(e_i) = e_i$. If $\ell+1 \le i \le k$, then $D_0 F(e_i) = D_0 f(e_i) + D_0 id(e_i) = 0 + e_i = e_i$.

Therefore, $D_0 F : \mathbb{R}^k \to \mathbb{R}^k$ is the identity. Then the IFT implies the existence of $\mathcal{O} \subseteq X$ such that $F : \mathcal{O} \to F(\mathcal{O})$ is a diffeomorphism onto an open set, where $0 \in \mathcal{O}$.

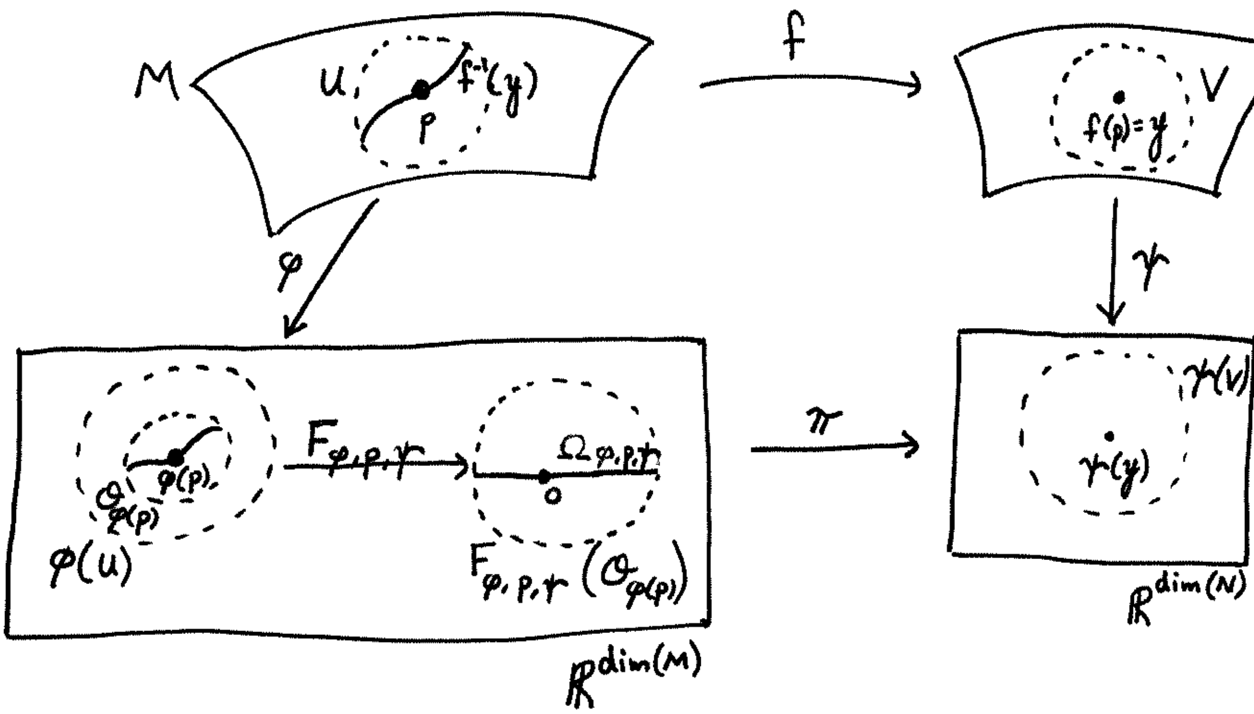
Note that $F(0) = (f(0), 0) = (0,0) = 0$ and that $f \circ F^{-1} = \pi$.

$\boxed{\text{END}}$

Proof of Regular Value Theorem:

Recall $f: M \to N$ is smooth, $y \in f(M)$, M and N are connected, and $D_p f$ is surjective if $p \in f^{-1}(y)$. We want to show that $f^{-1}(y)$ is a manifold.

Since $D_p f$ is surjective, so is $D_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})$.



We may assume $\varphi(p) = 0$ and $\psi(y) = 0$. The previous lemma implies the existence of $\mathcal{O}_{\varphi(p)} \subseteq \varphi(U) \subseteq \mathbb{R}^{\dim(M)}$ and a diffeomorphism $F_{\varphi,p,\psi} : \mathcal{O}_{\varphi(p)} \longrightarrow F_{\varphi,p,\psi}(\mathcal{O}_{\varphi(p)})$ such that $F_{\varphi,p,\psi}(\varphi(p)) = 0$ and $\psi \circ f \circ \varphi^{-1} \circ F^{-1}_{\varphi,p,\psi} = \pi$

Let $\Omega_{\varphi,p,\psi} = F_{\varphi,p,\psi}(\mathcal{O}_{\varphi(p)}) \cap \mathrm{Ker}(\pi)$. Thus:

(i) $\Omega_{\varphi,p,\psi}$ is open in $\mathrm{Ker}(\pi) = \mathbb{R}^{\dim(M)-\dim(N)}$,

(ii) $0 \in \Omega_{\varphi,p,\psi}$, and

(iii) $\psi \circ f \circ \varphi^{-1} \circ F_{\varphi,p,\psi}^{-1}(\Omega_{\varphi,p,\psi}) = \pi(\Omega_{\varphi,p,\psi}) = 0 = \psi(y)$.

Let $U_{\varphi,p,\psi} = \varphi^{-1} \circ F_{\varphi,p,\psi}^{-1}(\Omega_{\varphi,p,\psi}) \subseteq M$. Note that $U_{\varphi,p,\psi}$ is open, and it contains p by (ii). By (iii), $f(U_{\varphi,p,\psi}) = \psi^{-1} \circ \psi \circ f(U_{\varphi,p,\psi}) = \psi^{-1}(\psi(y)) = y$. Therefore $U_{\varphi,p,\psi} \subseteq f^{-1}(y)$, so the set of all $U_{\varphi,p,\psi}$ cover $f^{-1}(y)$.

We claim $\{(U_{\varphi,p,\psi}, F_{\varphi,p,\psi} \circ \varphi)\}_{\varphi,p,\psi}$ is a family of charts for $f^{-1}(y)$. Indeed,

$$(F_{\hat{\varphi},\hat{p},\hat{\psi}} \circ \hat{\varphi}) \circ (F_{\varphi,p,\psi} \circ \varphi)^{-1} = F_{\hat{\varphi},\hat{p},\hat{\psi}} \circ (\hat{\varphi} \circ \varphi^{-1}) \circ F_{\varphi,p,\psi}^{-1}$$

is a diffeomorphism since each of the three maps on the right is a diffeomorphism.

$\boxed{\text{END}}$

Regular values are generic by the following result.

Sard's Theorem:

If $f: M \longrightarrow N$ is smooth, then the set of critical values of f in N has measure 0.

See page 381 for a proof of Sard's Theorem, including what is meant by a measure zero subset of N.

Corollary:

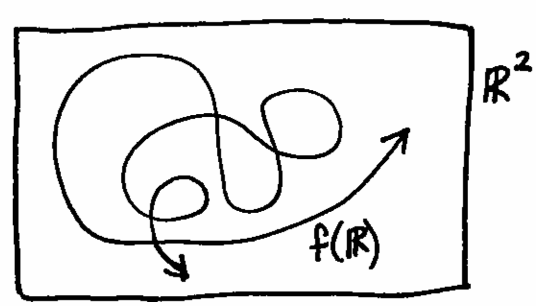
If $f: M \longrightarrow N$ is smooth and $\dim(M) < \dim(N)$ then $f(M) \subseteq N$ has measure 0.

proof:

If $y \in f(M)$ and $p \in f^{-1}(y)$ then f is not a submersion at p by reason of dimension. Thus, y is a critical value, so $f(M)$ is contained in the set of critical values.

END

Example: $f: \mathbb{R} \longrightarrow \mathbb{R}^2$, smooth



Exercises:

① Suppose $f: M \longrightarrow N$ is smooth, M is compact, and N is connected. Prove f is surjective if f is a submersion.

② Suppose $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^l$ are open neighborhoods of 0, that $f: X \longrightarrow Y$ is smooth, that $f(0)=0$, and that f is an immersion at 0.

Prove there is a neighborhood $\mathcal{O} \subseteq Y$ of 0 and a diffeomorphism $F: \mathcal{O} \longrightarrow F(\mathcal{O}) \subseteq \mathbb{R}^l$ such that $F(\mathcal{O})$ is open, $F(0)=0$, and $F^{-1} \circ f = \iota$ where $\iota: \mathbb{R}^k \longrightarrow \mathbb{R}^l$ is $\iota(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$.

③ Suppose M and N are smooth manifolds of the same dimension, that M is compact, and that $f: M \longrightarrow N$ is smooth.

(i) If $y \in N$ is a regular value of f, prove $|f^{-1}(y)| < \infty$. ($|X|$ is the cardinality of X.)

(ii) Suppose f is surjective and that every $y \in N$ is a regular value of f. If M and N are connected, show that $y \longmapsto |f^{-1}(y)|$ is a constant function on N.

§ Transversality

Let's restate what a regular value is. Let $f: M \longrightarrow N$ be smooth and let $y \in N$. Then $\{y\} \subseteq N$ is an embedded submanifold of dimension 0, and $T_y\{y\} = 0$, and y is a regular value of f if

$$D_p f(T_p M) + T_{f(p)}\{y\} = T_{f(p)} N$$

for all $p \in f^{-1}(y)$. That is, $D_p f(T_p M) = T_y N$ for all $p \in f^{-1}(y)$.

A transverse manifold is a direct generalization of a regular value. The Transversality Theorem is the corresponding analogue to the Regular Value Theorem. In what's below, note that the 0-manifold $\{y\}$ is being replaced with another manifold Q.

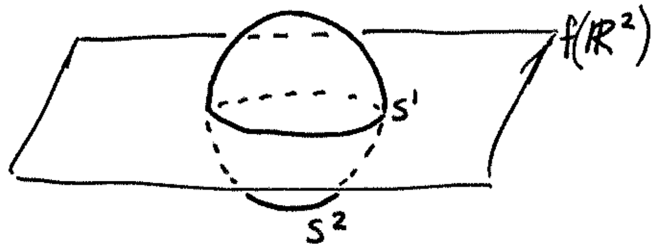
Let $f: M \longrightarrow N$ be smooth, and let $Q \subseteq N$ be an embedded submanifold. Then <u>f is transverse to Q</u> (written $f \pitchfork Q$) if

$$D_p f(T_p M) + T_{f(p)} Q = T_{f(p)} N$$

for all $p \in f^{-1}(Q)$.

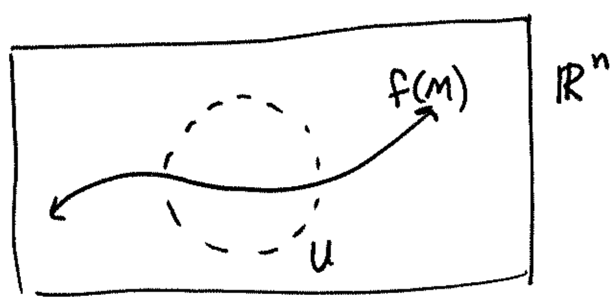
Examples:

① $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ where $f(x,y) = (x,y,0)$ and $S^2 \subseteq \mathbb{R}^3$ where $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$. Then $f \pitchfork S^2$.



Note that $f^{-1}(S^2) = S^1$, a smooth manifold.

② If $f: M \longrightarrow \mathbb{R}^n$ is smooth and $U \subseteq \mathbb{R}^n$ is open, then $f \pitchfork U$ and $f^{-1}(U) \subseteq M$ is a manifold as it's an open subset of a manifold.

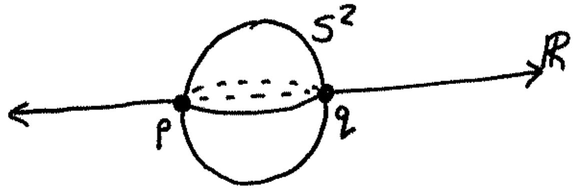


<u>Special case</u>: If $M, Q \subseteq N$ are both embedded submanifolds, we say <u>M is transverse to Q</u> ($M \pitchfork Q$) if $i_M \pitchfork Q$ where $i_M: M \hookrightarrow N$ is the inclusion. That is, if

$$T_pM + T_pQ = T_pN \quad \text{for all } p \in M \cap Q.$$

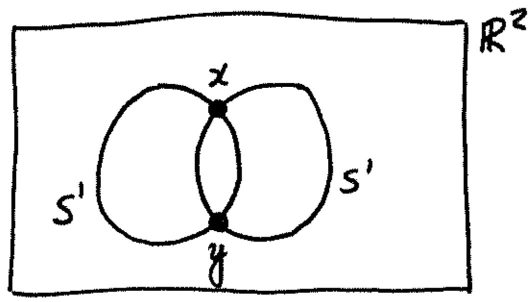
Examples:

③ $\mathbb{R} \pitchfork S^2$ in $\mathbb{R}^3$:



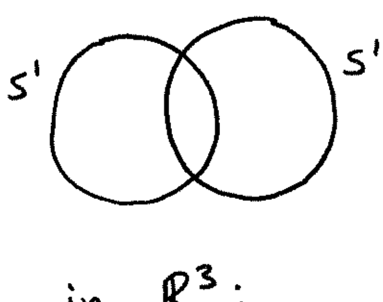
Note $\{p,q\} = \mathbb{R} \cap S^2$ is a 0-submanifold of $\mathbb{R}$.

④ $S^1 \pitchfork S^1$ in $\mathbb{R}^2$:

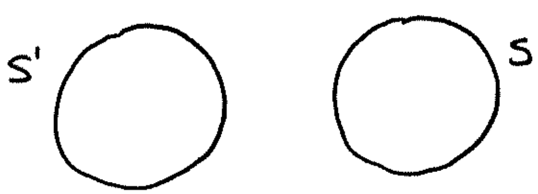


Note $\{x,y\} \subseteq S^1$ is a 0-submanifold in S^1.

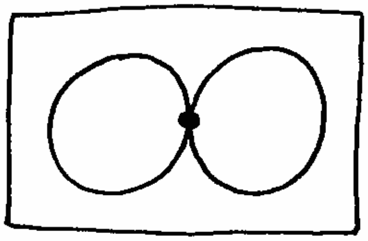
⑤ $S^1 \not\pitchfork S^1$ in $\mathbb{R}^3$:



⑥ $S^1 \pitchfork S^1$ in $\mathbb{R}^3$:



⑦ $S^1 \not\pitchfork S^1$ in $\mathbb{R}^2$:



⑧ We saw on page 52, using the Regular Value Theorem, that $\Sigma_1 = \{(x,y,z) \in \mathbb{R}^3 \mid xy = z\}$ is a surface in $\mathbb{R}^3$. So is the plane $P = \{(x,y,0) \in \mathbb{R}^3\}$.

Let $X = \{(x,0,0) \in \mathbb{R}^3\}$ and $Y = \{(0,y,0) \in \mathbb{R}^3\}$. Since $X, Y \subseteq \Sigma_1$ are lines containing $(0,0,0)$, we have $X, Y \subseteq T_{(0,0,0)}\Sigma_1$, and thus $P = X + Y \leq T_{(0,0,0)}\Sigma_1$. Since $T_{(0,0,0)}\Sigma_1$ is 2-dimensional, we have $T_{(0,0,0)}\Sigma_1 = P = T_{(0,0,0)}P$. Thus,

$$T_{(0,0,0)}\Sigma_1 + T_{(0,0,0)}P = P \neq \mathbb{R}^3 = T_{(0,0,0)}\mathbb{R}^3.$$

Therefore, $\Sigma_1 \not\pitchfork P$ in $\mathbb{R}^3$.

Note that $\Sigma_1 \cap P = X \cup Y$ is not a manifold.

Transversality Theorem:

Suppose $f: M \longrightarrow N$ is smooth, that $Q \subseteq N$ is an embedded submanifold, and that $f \pitchfork Q$. Then $f^{-1}(Q) \subseteq M$ is a smooth manifold and

$$\dim(M) - \dim(f^{-1}(Q)) = \dim(N) - \dim(Q).$$

Furthermore, $f^{-1}(Q)$ is an embedded submanifold of M.

proof:

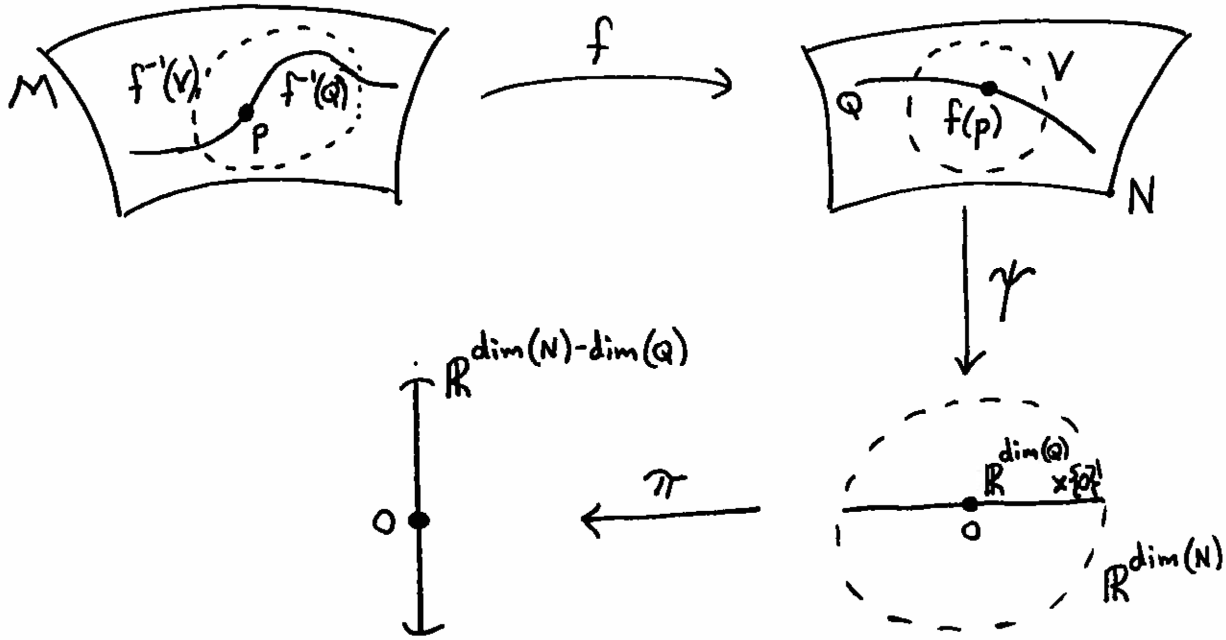
Choose $p \in f^{-1}(Q)$, so that $f(p) \in Q$.

Since $Q \subseteq N$ is an embedded submanifold, we can use Exercise 2 from page 58 to show that there is a neighborhood $V \subseteq N$ of $f(p)$ and a diffeomorphism $\psi: V \longrightarrow \psi(V) \subseteq \mathbb{R}^{\dim(N)}$ such that $\psi(V \cap Q) = \psi(V) \cap (\mathbb{R}^{\dim(Q)} \times \{0\})$.

We will let $\mathbb{R}^{\dim(N)} = \mathbb{R}^{\dim(Q)} \times \mathbb{R}^{\dim(N)-\dim(Q)}$ and $\pi: \mathbb{R}^{\dim(N)} \longrightarrow \mathbb{R}^{\dim(N)-\dim(Q)}$ be projection.

Then $\pi \circ \psi(V \cap Q) = \pi\left(\psi(V) \cap (\mathbb{R}^{\dim(Q)} \times \{0\})\right) = 0$,

and $\psi^{-1} \circ \pi^{-1}(0) = V \cap Q$.



Note that near p, in $f^{-1}(V)$, $f^{-1}(Q)$ is given by $f^{-1}\circ\psi^{-1}\circ\pi^{-1}(0)$, so we only need to show that $f^{-1}\circ\psi^{-1}\circ\pi^{-1}(0)$ is a manifold. We'll do this by showing that 0 is a regular value of $\pi\circ\psi\circ f$ and then applying the Regular Value Theorem.

On the next page we use the Chain Rule, we add 0, we use that $\pi\circ\psi(V\cap Q)=0$, the linearity of differentials, that $f \pitchfork Q$, and finally that $\pi\circ\psi$ is a submersion (since π is) to check that 0 is a regular value of $\pi\circ\psi\circ f$.

$$D_p(\pi\circ\psi\circ f)(T_pM) = D_{f(p)}(\pi\circ\psi)D_pf(T_pM)$$

$$= D_{f(p)}(\pi\circ\psi)D_pf(T_pM) + 0$$

$$= D_{f(p)}(\pi\circ\psi)D_pf(T_pM) + D_{f(p)}(\pi\circ\psi)(T_{f(p)}Q)$$

$$= D_{f(p)}(\pi\circ\psi)\Big(D_pf(T_pM) + T_{f(p)}Q\Big)$$

$$= D_{f(p)}(\pi\circ\psi)\big(T_{f(p)}N\big)$$

$$= \mathbb{R}^{\dim(N)-\dim(Q)}$$

That is, $D_p(\pi\circ\psi\circ f)$ is surjective.

$\boxed{\text{END}}$

Corollary: If $M, Q \subseteq N$ are embedded submanifolds with $M \pitchfork Q$, then $M\cap Q$ is a manifold.

Exercises:

① Let $V, W \leq \mathbb{R}^n$ be vector subspaces. Prove $V \pitchfork W$ in $\mathbb{R}^n$ if and only if $V + W = \mathbb{R}^n$.

② Let $0 \in \mathbb{R}^n$ and let $\Delta\mathbb{R}^n = \{(x,x) \in \mathbb{R}^{2n} \mid x \in \mathbb{R}^n\}$. Prove that $(\mathbb{R}^n \times \{0\}) \pitchfork \Delta\mathbb{R}^n$ in $\mathbb{R}^{2n}$.

③ A matrix is symmetric if $A^t = A$ and skew-symmetric if $A^t = -A$. Prove that the spaces of all symmetric and skew-symmetric matrices are transverse in $M_{n\times n}(\mathbb{R})$. What is their intersection? What are their respective dimensions?

④ Let $y \in N$ be a regular value of a smooth map $f : M \to N$. If $p \in f^{-1}(y)$, prove $T_p f^{-1}(y)$ is the kernel of $D_p f$.

⑤ Suppose $f : M \to N$ is smooth, that $Q \subseteq N$ is an embedded submanifold, and that $f \pitchfork Q$. Prove that $T_p f^{-1}(Q) = (D_p f)^{-1}(T_{f(p)} Q)$ for $p \in f^{-1}(Q)$. (Note that ⑤ implies ④.) (⑧ from page 48 might help.)

⑥ Use ⑤ to prove that if $M, Q \subseteq N$ are embedded submanifolds, and if $M \pitchfork Q$ in N, then $T_p(M \cap Q) = (T_p M) \cap (T_p Q)$ if $p \in M \cap Q$.

⑦ Suppose $f: M \longrightarrow N$ and $g: N \longrightarrow P$ are smooth, $Q \subseteq P$ is an embedded submanifold, and that $g \pitchfork Q$. Prove $f \pitchfork g^{-1}(Q)$ if and only if $g \circ f \pitchfork Q$.

⑧ For all $a \geq 0$, decide when $x^2 + y^2 - z^2 = 1$ and $x^2 + y^2 + z^2 = a$ intersect transversally, and draw the intersection of these two manifolds.

⑨ As in the proof of the Transversality Theorem on page 63, show there is a diffeomorphism $\psi: V \longrightarrow \psi(V) \subseteq \mathbb{R}^{\dim(N)}$ for some neighborhood $V \subseteq N$ of $f(p) \in Q$ such that $\psi(f(p)) = 0$ and $\psi(V \cap Q) = \psi(V) \cap (\mathbb{R}^{\dim(Q)} \times \{0\})$.

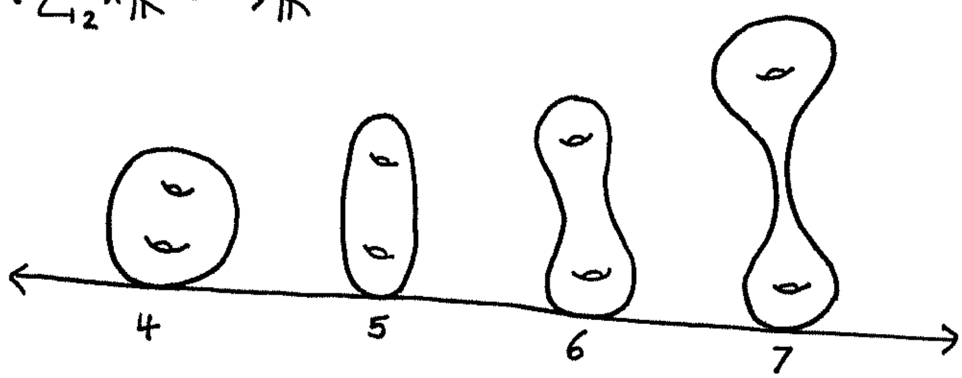
(Use ② from page 58.)

§ Transversality II

For smooth manifolds M, N, Q and a smooth map $F: M \times N \longrightarrow Q$, we fix $n \in N$ and define $f_n: M \longrightarrow Q$ by $f_n(m) = f(m,n)$.

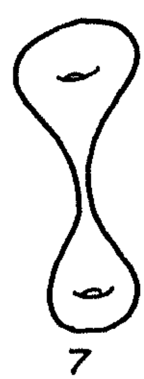
Example:

$F: \Sigma_2 \times \mathbb{R} \longrightarrow \mathbb{R}^3$



$f_4: \Sigma_2 \longrightarrow \mathbb{R}^3$

$f_7: \Sigma_2 \longrightarrow \mathbb{R}^3$



2nd Transversality Theorem:

Let M, N, Q be smooth manifolds, let $Z \subseteq Q$ be an embedded submanifold, let $F: M \times N \to Q$ be smooth, and assume $F \pitchfork Z$. Then for a.e. $n \in N$, $f_n \pitchfork Z$ where $f_n: M \to Q$.

proof:

Let $\pi: M \times N \longrightarrow N$ be projection. Then $D_{(m,n)}\pi: T_m M \times T_n N \longrightarrow T_n N$ is also projection for all $(m,n) \in M \times N$.

Since $F \pitchfork Z$, we also have that $\pi|_{F^{-1}(Z)}: F^{-1}(Z) \longrightarrow N$ is a smooth map of manifolds.

By Sard's Theorem, a.e. $n \in N$ is a regular value of $\pi|_{F^{-1}(Z)}$. We will show that $f_n \pitchfork Z$ for such n. For this, choose $z \in Z$ and $m \in M$ with $f_n(m) = z$. We need to check that $D_m f_n(T_m M) + T_z Z = T_z Q$. That is, for any $v_Q \in T_z Q$ we need to find vectors in $D_m f_n(T_m M)$ and $T_z Z$ that sum to v_Q.

Since $F \pitchfork Z$ and $F(m,n) = f_n(m) = z$, we have $D_{(m,n)}F(T_mM \times T_nN) + T_zZ = T_zQ$. Thus, there are $v_M \in T_mM$ and $v_N \in T_nN$ such that

$$v_Q - D_{(m,n)}F(v_M + v_N) \in T_zZ$$

Since $n \in N$ is a regular value of $\pi|_{F^{-1}(Z)} : F^{-1}(Z) \longrightarrow N$, there is some $w_M + w_N \in T_{(m,n)}(M \times N)$, tangent to $F^{-1}(Z)$, such that

$$w_N = D_{(m,n)}\pi(w_M + w_N) = v_N$$

Note that by Exercise 5 on page 66,

$$D_{(m,n)}F(w_M + w_N) \in T_{F(m,n)}Z = T_zZ$$

Now we can see that

$$\begin{aligned} v_Q - D_mf_n(v_M - w_M) &= v_Q - D_{(m,n)}F(v_M - w_M) \\ &= v_Q - D_{(m,n)}F\big((v_M + v_N) - (w_M + v_N)\big) \end{aligned}$$

$$= v_Q - D_{(m,n)}F\big((v_M + v_N) - (w_M + w_N)\big)$$

$$= \big(v_Q - D_{(m,n)}F(v_M + v_N)\big) - D_{(m,n)}F(w_M + w_N)$$

$$\in T_z Z - T_z Z$$

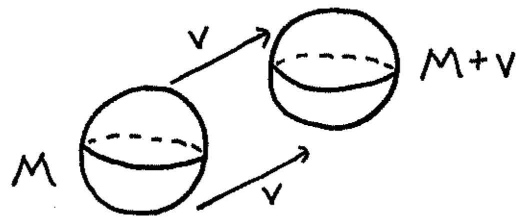
$$= T_z Z$$

Since $D_m f_n(v_M - w_M) \in D_m f_n(T_m M)$, our proof has concluded as $D_m f_n(v_M - w_M)$ and $v_Q - D_m f_n(v_M - w_M)$ sum to v_Q.

$\boxed{\text{END}}$

Example

Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. For any $v \in \mathbb{R}^n$, let $M + v = \{p + v \in \mathbb{R}^n \mid p \in M\}$

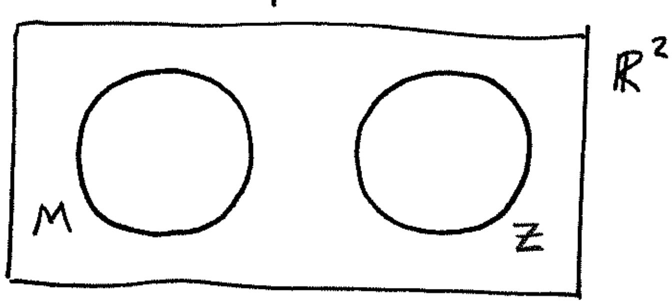


Let $F: M \times \mathbb{R}^n \to \mathbb{R}^n$ be $F(p,v) = p+v$. Then $D_{(p,v)}F(w,v) = w+v$ for all $w \in T_pM$ and $v \in \mathbb{R}^n$. Therefore, $D_{(p,v)}F : T_pM \times \mathbb{R}^n \to \mathbb{R}^n$ is surjective for all $(p,v) \in M \times \mathbb{R}^n$, which implies that $F \pitchfork Z$ for any embedded submanifold $Z \subseteq \mathbb{R}^n$.

Applying the previous theorem, for a.e. $v \in \mathbb{R}^n$ we have that $f_v \pitchfork Z$. That is, $M+v \pitchfork Z$.

<u>Subexample</u>:

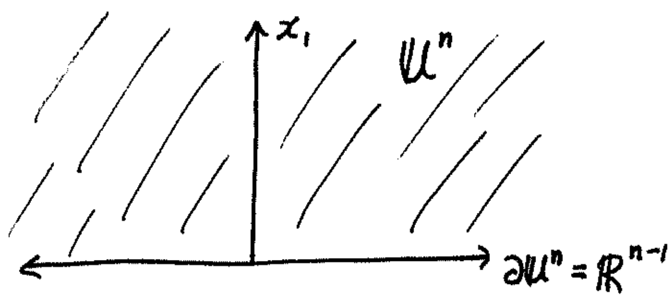
Let M and Z be the following two circles in the plane:



For which $v \in \mathbb{R}^2$ is it that $M+v \not\pitchfork Z$?

§ Boundaries

Let $\mathbb{U}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$. $\mathbb{U}^n$ is the <u>upper half space</u> of $\mathbb{R}^n$.



Let $\partial \mathbb{U}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$ so that $\partial \mathbb{U}^n$ is diffeomorphic to $\mathbb{R}^{n-1}$.

M is a <u>smooth manifold with boundary</u> if:

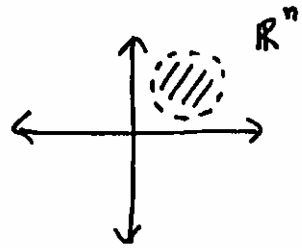
(i) M is Hausdorff;

(ii) $M = \bigcup_{i=1}^{\infty} K_i$ for compact sets $K_i \subseteq K_{i+1}$;

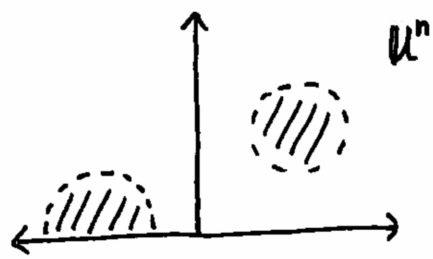
(iii) For any $p \in M$, there is a neighborhood of p $U \subseteq M$ and a homeomorphism $\varphi : U \longrightarrow \Omega$ for some open $\Omega \subseteq \mathbb{U}^n$;

(iv) Given two chart φ, ψ as in (iii), $\psi \circ \varphi^{-1}$ extends to a diffeomorphism of open sets in $\mathbb{R}^n$.

Thus, where a manifold is a union of open sets in $\mathbb{R}^n$



a manifold with boundary is a union of open sets in $\mathbb{U}^n$



The <u>boundary of M</u> (written ∂M) is the space

$$\partial M = \{ p \in M \mid \varphi(p) \in \partial \mathbb{U}^n \text{ for some chart } \varphi \}$$

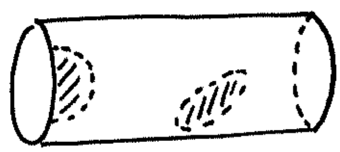
<u>Proposition:</u>

If M is a smooth manifold with boundary, then ∂M is a smooth manifold (without boundary) and either $\partial M = \emptyset$ or $\dim(\partial M) = \dim(M) - 1$. Furthermore, ∂M is embedded.

Examples: ⓪ $\mathbb{H}^n$ is a smooth manifold with boundary and $\partial(\mathbb{H}^n) = \partial\mathbb{H}^n$.

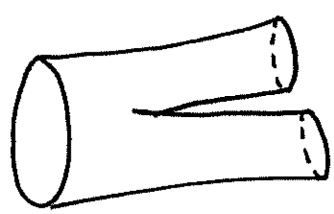
① If M is a smooth manifold, then it is a smooth manifold with boundary and $\partial M = \emptyset$.

② $S^1 \times [0,1]$ is a smooth manifold with boundary and $\partial(S^1 \times [0,1]) = S^1 \amalg S^1$



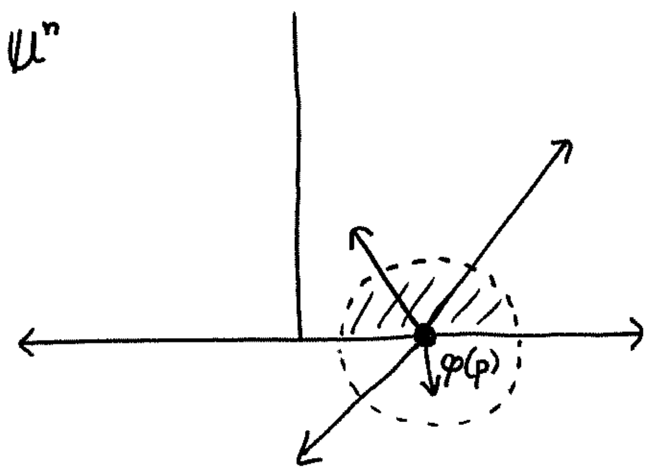
③ More generally, if M is a smooth manifold (with $\partial M = \emptyset$), then $M \times [0,\infty)$ and $M \times [0,1]$ are manifolds with boundaries. Their boundaries are M and $M \amalg M$, respectively.

④ A pair of pants



has boundary $S^1 \amalg S^1 \amalg S^1$.

If M is a smooth manifold with boundary, and if $p \in \partial M$, then $\varphi(p) \in \partial \mathbb{U}^n$ for some chart φ. There is a full n-dimensional vector space of vectors at $\varphi(p) \in \mathbb{R}^n$, and transition functions extend to a neighborhood of $\varphi(p)$, so we can define T_pM with $\dim(T_pM) = \dim(M)$ and differentials of smooth functions as before.



If M and N are smooth manifolds with boundary, then a continuous map $f: M \to N$ is <u>smooth</u> if $\psi \circ f \circ \varphi^{-1}$ can be extended to a smooth map between open sets in Euclidean space for any pair (φ, ψ) of charts on M and N.

Exercise:

① Prove the proposition on page 74.

② Suppose N is a smooth manifold with $\partial N = \emptyset$. Let $f: N \longrightarrow \mathbb{R}$ be smooth and suppose $y \in \mathbb{R}$ is a regular value of f. Prove that $f^{-1}((-\infty, y])$ is a smooth manifold with boundary $f^{-1}(y)$.

For example, $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is a smooth manifold with boundary $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ since $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$ is smooth, $f^{-1}((-\infty, 1]) = D^n$, and $f^{-1}(1) = S^{n-1}$.

For two more examples, let $g: M \times \mathbb{R} \longrightarrow \mathbb{R}$ be $g(p,t) = t$ and let $h: M \times \mathbb{R} \longrightarrow \mathbb{R}$ be $h(p,t) = t(t-1)$. Then $g^{-1}((-\infty, 0]) = M \times (-\infty, 0]$ and $h^{-1}((-\infty, 0]) = M \times [0,1]$.

§ Homotopies

Let $I=[0,1]$ and let M, N be smooth manifolds without boundary.

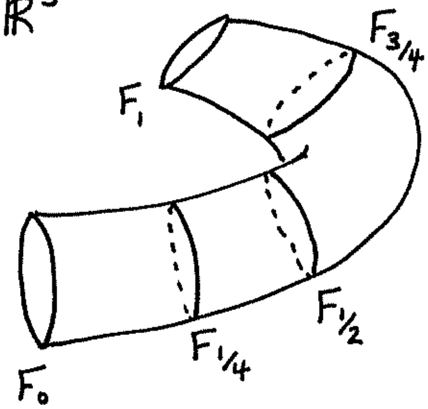
Two smooth maps $f: M \longrightarrow N$ and $g: M \longrightarrow N$ are <u>smoothly homotopic</u> if there is a smooth map $F: M \times I \longrightarrow N$ such that $F(p,0)=f(p)$ and $F(p,1)=g(p)$ for all $p \in M$.

If f and g are smoothly homotopic, we write $f \simeq g$. F is called a <u>homotopy</u> between f and g.

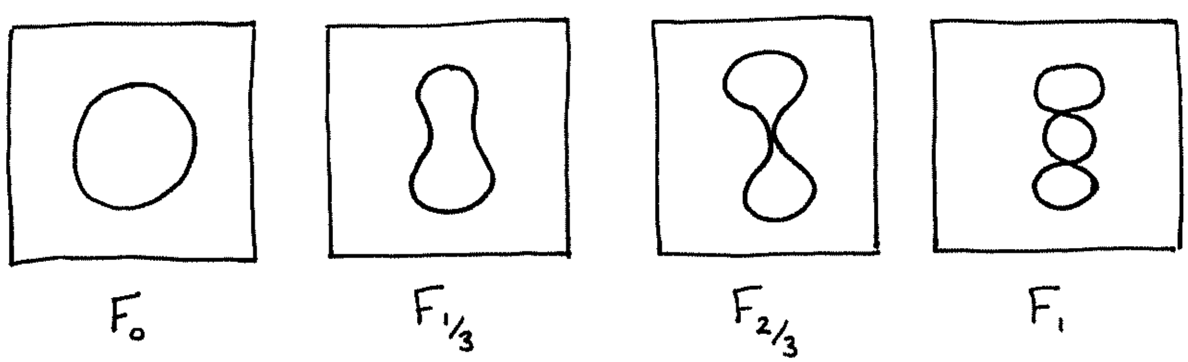
If $t \in I$, we denote the function $F(-,t): M \longrightarrow N$ as $F_t: M \longrightarrow N$. Thus, $F_0 \simeq F_1$, and more generally, $F_t \simeq F_s$ for any $s, t \in I$.

Examples

① $F: S^1 \times I \longrightarrow \mathbb{R}^3$



② In $F: S^1 \times I \longrightarrow \mathbb{R}^2$ below, we visualize the $F_t: S^1 \longrightarrow \mathbb{R}^2$ as a time lapse continuum of maps



③ Let $n \in \mathbb{Z}$ and define $\chi_n: S^1 \longrightarrow S^1$ by $\chi_n(e^{i\theta}) = e^{in\theta}$, where $\theta \in [0, 2\pi)$. Later we'll see that $\chi_n \not\simeq \chi_k$ if $n \neq k$.

Proposition:

Suppose M and N are connected, smooth manifolds, that M is compact, and that $F : M \times I \longrightarrow N$ is smooth. If $F_0 : M \to N$ is $(*)$, then there is some $\varepsilon > 0$ such that $F_t : M \to N$ is $(*)$ for all $t \in [0, \varepsilon)$, where $(*)$ can be

- (·) an immersion
- (·) a submersion
- (·) a local diffeomorphism
- (·) transverse to an embedded $Q \subseteq N$
- (·) an embedding
- (·) a diffeomorphism.

proof:

immersion: Let $p \in M$. Since $D_p F_0$ is injective and $(p,t) \longmapsto D_p F_t$ is continuous, there is an open set $U_p \times [0, \varepsilon_p) \subseteq M \times I$ containing $(p, 0)$ such that if $(q,t) \in U_p \times [0, \varepsilon_p)$ then $D_q F_t$ is injective.

Since M is compact, finitely many of the U_p cover M, and we let ε be the minimum of the corresponding finitely many ε_p. //

<u>Submersion</u>: Similar. //

<u>local diffeomorphism</u>: Immersion + Submersion. //

<u>transverse to $Q \subseteq N$</u>: Similar to the immersion case, noting that if V is a vector space with $W, U, W', U' \leq V$, $W + U = V$, $W \approx W'$, and $U \approx U'$, then $W' + U' = V$. //

<u>embedding</u>: Any continuous map from a compact space M is proper, thus by exercise 2 on page 48, $F_t : M \longrightarrow N$ is an embedding if and only if it is an injective immersion. Therefore, we only need to consider whether F_t is injective for $t \approx 0$.

Our proof proceeds by contradiction. That is, we assume F_0 is injective, but that F_{ε_n} is not injective for $\varepsilon_n \longrightarrow 0$.

Define $G: M\times I \to N\times I$ by $(p,t)\mapsto (F_t(p),t)$.
In local coordinates, $D_{(p,0)}G$ can be represented as

$$\left(\begin{array}{c|c} D_pF_0 & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline 0\cdots 0 & 1 \end{array}\right)$$

Since D_pF_0 is injective for all $p\in M$, we see that $D_{(p,0)}G$ is injective for all $p\in M$, and thus there is a neighborhood $U_p\subseteq M\times I$ of $(p,0)$ such that $G|_{U_p}$ is injective.

We have assumed the F_{ε_n} are not injective, so there are $p_n, q_n\in M$ with $p_n\neq q_n$ and $F_{\varepsilon_n}(p_n)=F_{\varepsilon_n}(q_n)$.

M is compact, so we may assume that $p_n\to p$ and $q_n\to q$ for some $p,q\in M$. Thus,

$$F_0(p)=\lim_{n\to\infty}F_{\varepsilon_n}(p_n)=\lim_{n\to\infty}F_{\varepsilon_n}(q_n)=F_0(q)$$

Because F_0 is injective, $p=q$. Therefore, for $n \gg 0$, $(p_n, \varepsilon_n) \approx (q_n, \varepsilon_n)$, so we may assume all (p_n, ε_n) and (q_n, ε_n) are contained in a common $U_{p_0} \subseteq M \times I$, where we recall here that $\{U_p\}_{p \in M}$ was an open cover of $M \times \{0\}$ by open sets $U_p \subseteq M \times I$.

Because $G|_{U_{p_0}}$ is injective,

$$G(p_n, \varepsilon_n) = \left(F_{\varepsilon_n}(p_n), \varepsilon_n\right) = \left(F_{\varepsilon_n}(q_n), \varepsilon_n\right) = G(q_n, \varepsilon_n)$$

implies that $(p_n, \varepsilon_n) = (q_n, \varepsilon_n)$, and hence that $p_n = q_n$, which is our contradiction.

//

<u>diffeomorphism</u>: By the above, there is some $\varepsilon > 0$ such that F_t is a local diffeomorphism and an embedding for all $t \leq \varepsilon$. Thus, we only have to show that $F_t(M) = N$ for $t \leq \varepsilon$

By IFT, F_t is an open map, so $F_t(M)$ is open in N. Since M is compact, $F_t(M)$ is compact and hence closed in N. By the connectivity of N, $F_t(M) = N$. //

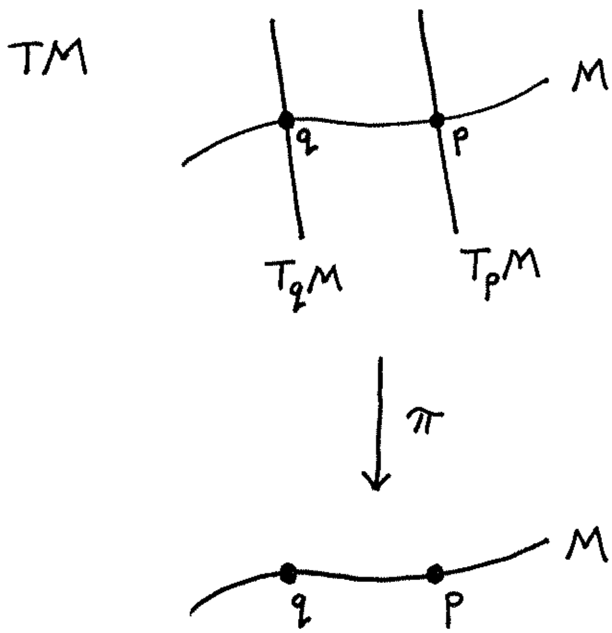
END

Exercise

① Add more detail to the "transverse to $Q \subseteq N$" case of the proposition proved in this chapter.

§ Vector Bundles

Let $TM = \bigcup_{p \in M} T_pM$, the <u>tangent bundle</u> of M.



<u>Proposition</u>: TM is a smooth manifold and $\dim(TM) = 2\dim(M)$.

<u>proof:</u>

Let $\pi : TM \to M$ be defined by $\pi(w) = p$ if $w \in T_pM$.

Let (U,φ) be a chart for M, and let $w \in \pi^{-1}(U)$, so that $w = [(U,\varphi,u)]$ for some $u \in \mathbb{R}^{\dim(M)}$.

Let $\Phi_U : \pi^{-1}(U) \longrightarrow \varphi(U) \times \mathbb{R}^{\dim(M)} \subseteq \mathbb{R}^{2\dim(M)}$ be defined by $\Phi_U(w) = (\varphi \circ \pi(w), u)$.

Note that TM is the union of $\pi^{-1}(U)$ where (U,φ) is a chart for M. We define a topology on TM such that each Φ_U is a homeomorphism. We let the pairs $(\pi^{-1}(U), \Phi_U)$ be charts for TM.

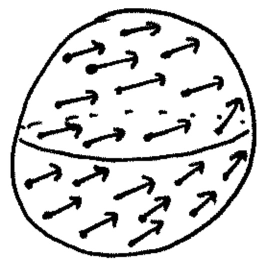
To check transition functions, let (U,φ) and (V,ψ) be charts on M that give rise to the charts $(\pi^{-1}(U), \Phi_U)$ and $(\pi^{-1}(V), \Psi_V)$ as in the previous paragraph. For $(x,u) \in \varphi(U) \times \mathbb{R}^{\dim(M)}$ we let $w \in T_{\varphi^{-1}(x)}M$ be $w = [(U,\varphi,u)]$ so that

$$\Psi_V \circ \Phi_U^{-1}(x,u) = \Psi_V(w) = \left(\psi \circ \varphi^{-1}(x), D_x(\psi \circ \varphi^{-1})(u)\right)$$

which is a diffeomorphism.

$\boxed{\text{END}}$

A <u>smooth vector field</u> on M is a smooth function $s : M \longrightarrow TM$ such that $\pi \circ s = id$.



$$\begin{tikzcd} TM \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$

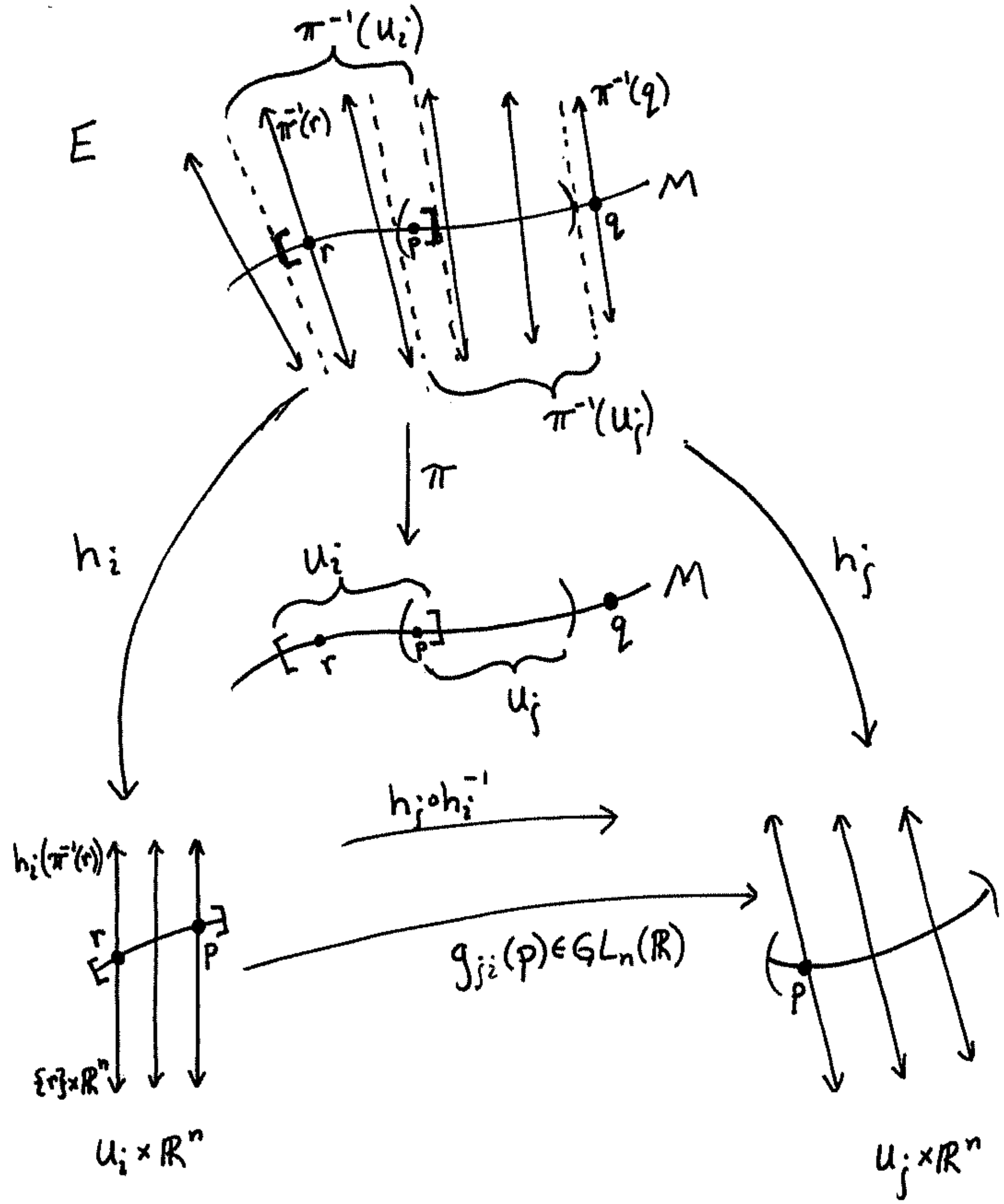
The tangent bundle of M is a certain kind of way of creating a manifold that consists of pairs of points in M and vectors in some vector space. There are other ways too.

Let E and M be smooth manifolds. E is a <u>vector bundle</u> over M if there is a smooth surjection $\pi : E \longrightarrow M$ such that:

(i) M is covered by open sets U_i.

(ii) There are diffeomorphisms $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ such that $p \in U_i$ implies $h_i(\pi^{-1}(p)) = \{p\} \times \mathbb{R}^n$.

(iii) $h_j \circ h_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \longrightarrow (U_i \cap U_j) \times \mathbb{R}^n$ is given by $h_j \circ h_i^{-1}(p, v) = (p, g_{ji}(p)(v))$ for some smooth function $g_{ji} : U_i \cap U_j \longrightarrow GL_n(\mathbb{R})$.



Note that $\{p\} \times \mathbb{R}^n$ is naturally a vector space, and that if $v \in \pi^{-1}(p)$, then $h_j h_i^{-1}(p, h_i(v))$ equals $(p, h_j(v))$. Now because $g_{ji}(p)$ is linear, for $v, w \in \pi^{-1}(p)$ we can define

$$v + w = h_i^{-1}(p, h_i(v) + h_i(w))$$

This addition is well-defined since

$$h_j h_i^{-1}\big(p,\, h_i(v)+h_i(w)\big) = \big(p,\, g_{ji}(p)\big(h_i(v)+h_i(w)\big)\big)$$

$$= \big(p,\, g_{ji}(p)(h_i(v))\big) + \big(p,\, g_{ji}(p)(h_i(w))\big)$$

$$= h_j h_i^{-1}\big(p,\, h_i(v)\big) + h_j h_i^{-1}\big(p,\, h_i(w)\big)$$

$$= \big(p,\, h_j(v)\big) + \big(p,\, h_j(w)\big)$$

$$= \big(p,\, h_j(v) + h_j(w)\big)$$

so $h_i^{-1}\big(p,\, h_i(v)+h_i(w)\big) = h_j^{-1}\big(p,\, h_j(v)+h_j(w)\big)$.

Therefore, each $\pi^{-1}(p)$ is equipped with a vector addition. Similarly, each $\pi^{-1}(p)$ is equipped with a well-defined scalar multiplication, so each $\pi^{-1}(p)$ is a vector space. We call $\pi^{-1}(p)$ the <u>fiber</u> over $p \in M$.

We call n the <u>rank</u> of E, M the <u>base space</u> of E, and h_i's the <u>local trivializations</u>.

Examples:

① $\pi: M \times \mathbb{R}^n \longrightarrow M$ where $\pi(p,x) = p$ is a vector bundle, the product bundle over M of rank n.

② TM is a vector bundle where the $D_x(\psi \circ \varphi^{-1})$ serve as the $g_{ij}(p)$. The rank of TM is $\dim(M)$.

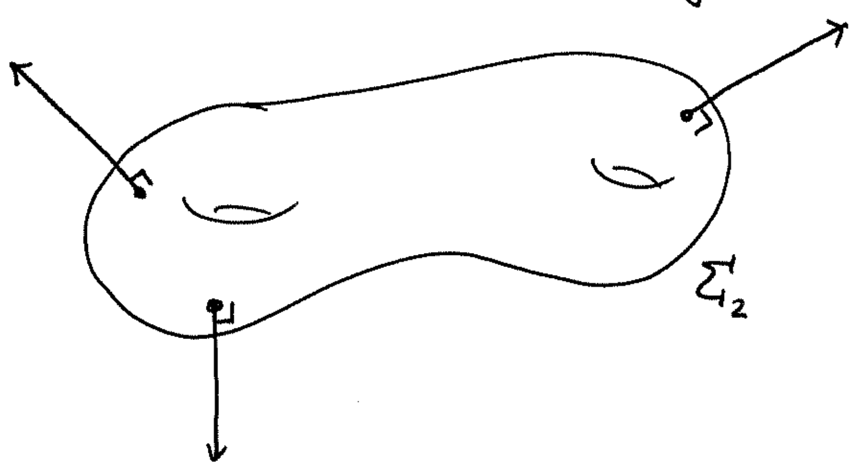
③ Let $\Sigma_g^1 \subseteq \mathbb{R}^3$ be embedded. For each $p \in \Sigma_g^1$, let $L_p \subseteq \mathbb{R}^3$ be the line orthogonal to $T_p\Sigma_g^1$, and passing through p. Then $\bigcup_{p \in \Sigma_g^1} L_p$ is a rank 1 vector bundle over Σ_g^1, called the normal bundle.

Definition: A rank n vector bundle $\pi: E \longrightarrow M$ is trivial if there is a diffeomorphism $f: E \longrightarrow M \times \mathbb{R}^n$ such that for any $p \in M$, $f: \pi^{-1}(p) \longrightarrow \{p\} \times \mathbb{R}^n$ is a vector space isomorphism.

Examples

① Product bundles are trivial.

② The normal bundle of $\Sigma_g \subseteq \mathbb{R}^3$ is trivial. We can match up $\Sigma_g \times [0,\infty)$ with the directions in the normal bundle that point outside of Σ_g. This identification extends to a diffeomorphism with $\Sigma_g \times \mathbb{R}$.



③ The Möebius band is a rank 1 bundle over S^1 that is not trivial.

④ $T\Sigma_g$ is not trivial if $g \neq 1$.

⑤ If M has only one chart, then TM is trivial. So $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ and $TGL_n(\mathbb{R}) = GL_n(\mathbb{R}) \times \mathbb{R}^{n^2}$.

<u>Definition:</u> A <u>smooth section</u> of a vector bundle $\pi: E \to M$ is a smooth map $s: M \to E$ such that $\pi \circ s = id$.

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$

The set of all smooth sections of E is denoted $\Gamma(E)$. Using the operations

$$(s_1 + s_2)(p) = s_1(p) + s_2(p) \quad \text{and} \quad (\alpha s_1)(p) = \alpha(s_1(p))$$

for $s_1, s_2 \in \Gamma(E)$, $\alpha \in \mathbb{R}$, and $p \in M$, we see that $\Gamma(E)$ is a vector space.

An element of $\Gamma(TM)$ is exactly a smooth vector field on M.

<u>Definition:</u> For any vector bundle $\pi: E \to M$, let $0_p \in \pi^{-1}(p)$ be the zero vector. The <u>zero section</u> of E is $s_0 \in \Gamma(E)$ where $s_0(p) = 0_p$ for all $p \in M$.

Note that $s_0: M \to E$ is an embedding. We let $M_0 = s_0(M) \subseteq E$.

Exercises:

① Prove TM is a smooth vector bundle over M.

② The Möbius band is $[0,1]\times\mathbb{R}/\sim$ where $(0,x)\sim(1,-x)$, denoted below as MB. Prove MB is a smooth vector bundle over S^1.

③ Let $a_1,\dots,a_n:\mathbb{R}^n\to\mathbb{R}$ and define $\Delta:\mathbb{R}^n\to T\mathbb{R}^n$ by $\Delta(p)=(p,a_1(p),\dots,a_n(p))$. Prove $\Delta\in\Gamma(T\mathbb{R}^n)$ if and only if each a_i is smooth.

④ Suppose $\pi:E\to M$ is a rank n vector bundle. Show E is trivial if and only if there are sections $\Delta_1,\dots,\Delta_n\in\Gamma(E)$ such that $\{\Delta_1(p),\dots,\Delta_n(p)\}$ is a basis for $\pi^{-1}(p)$ for all $p\in M$.

⑤ Use ④ to determine whether TS^1, MB, and $T\Sigma_g$ are trivial vector bundles.

⑥ Prove $\Gamma(E)$ is a vector space for any smooth vector bundle $\pi:E\to M$.

⑦ Is $(M\times\mathbb{R})-M_0$ connected? Is $MB-(S^1)_0$ connected?

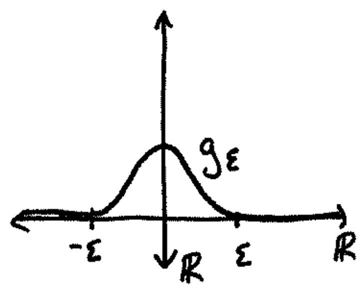
§Extending Smooth Functions

Lemma: For any $\varepsilon > 0$, there is a smooth $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ with

(i) $g_\varepsilon(x) \geq 0$ if $x \in \mathbb{R}$

(ii) $g_\varepsilon(x) = 0$ if $x \notin [-\varepsilon, \varepsilon]$

(iii) $g_\varepsilon(0) > 0$



proof:

Let
$$g_\varepsilon(x) = \begin{cases} e^{-(x/\varepsilon - 1)^{-2}} e^{-(x/\varepsilon + 1)^{-2}} & x \in (-\varepsilon, \varepsilon) \\ 0 & x \notin (-\varepsilon, \varepsilon) \end{cases}$$

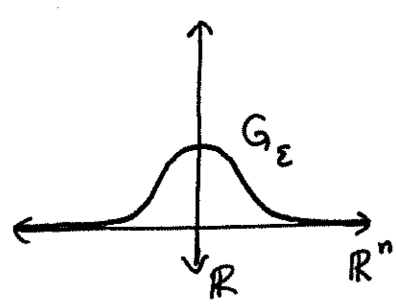
$\boxed{\text{END}}$

Lemma: For any $\varepsilon > 0$, there is a smooth $G_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ with

(i) $G_\varepsilon(x) \geq 0$ if $x \in \mathbb{R}^n$

(ii) $G_\varepsilon(x) = 0$ if $x \notin \prod_{i=1}^{n} [-\varepsilon, \varepsilon]$

(iii) $G_\varepsilon(0) > 0$



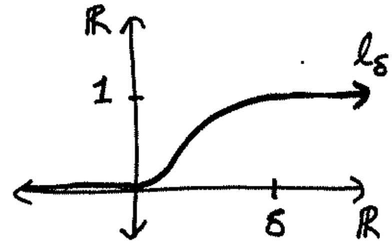
proof: $G_\varepsilon(x) = \prod_{i=1}^{n} g_\varepsilon(x_i)$ $\boxed{\text{END}}$

Lemma: For any $\delta > 0$, there is a smooth $\ell_\delta : \mathbb{R} \to \mathbb{R}$ with

(i) $\ell_\delta(x) \geq 0$ if $x \in \mathbb{R}$

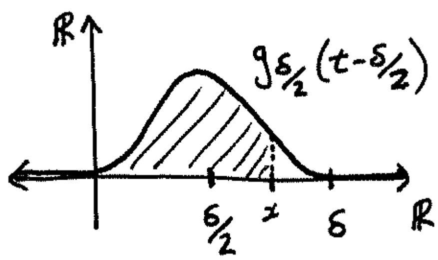
(ii) $\ell_\delta(x) = 0$ if $x \leq 0$

(iii) $\ell_\delta(x) = 1$ if $x \geq \delta$



proof:

Let $$\ell_\delta(x) = \frac{\int_0^x g_{\delta/2}(t - \delta/2)\,dt}{\int_0^\delta g_{\delta/2}(t-\delta/2)\,dt}$$



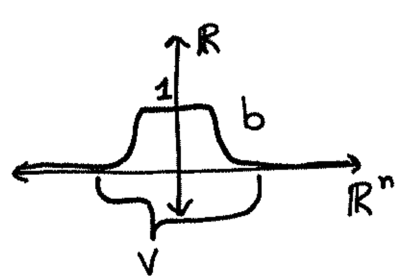
$\boxed{\text{END}}$

Lemma: For any open $V \subseteq \mathbb{R}^n$ containing 0, there is some smooth $b : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $b(x) \geq 0$ if $x \in \mathbb{R}^n$

(ii) $b(x) = 0$ if $x \notin V$

(iii) $b(x) = 1$ if $x \approx 0$



proof: Choose $\varepsilon > 0$ such that $\prod_{i=1}^n [-\varepsilon, \varepsilon] \subseteq V$. Choose $\delta > 0$ such that $G_\varepsilon(x) > \delta$ for $x \approx 0$. Let $b = \ell_\delta \circ G_\varepsilon$. $\boxed{\text{END}}$

<u>Lemma:</u> Let M be a smooth manifold. For any $p \in M$ there is a neighborhood $U \subseteq M$ of p such that for any smooth $f : U \to \mathbb{R}$ there is some smooth $\overline{f} : M \longrightarrow \mathbb{R}$ with $f(x) = \overline{f}(x)$ for all $x \approx p$.

<u>proof:</u>

Choose (U, φ) to be a chart with $p \in U$. We may assume that $\varphi(p) = 0$.

Choose an open $V \subseteq \varphi(U)$ containing 0 with $\overline{V} \subseteq \varphi(U)$, and let $b : \mathbb{R}^n \longrightarrow \mathbb{R}$ be as in the previous lemma.

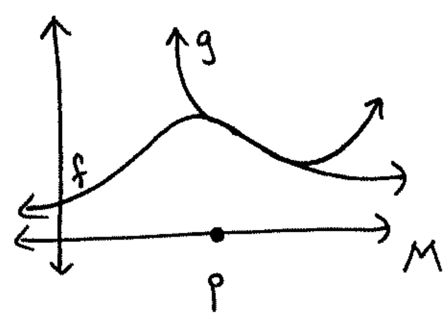
Define $f' = f \cdot (b \circ \varphi) : U \longrightarrow \mathbb{R}$ so that $f' = f$ near p, and $f' = 0$ away from p.

Let $\overline{f}(x) = \begin{cases} f'(x) & x \in U \\ 0 & x \notin U. \end{cases}$

$\boxed{\text{END}}$

§ Derivations at a point

Let M be a smooth manifold, and let $p \in M$. The germs of smooth functions at p is an $\mathbb{R}$-algebra, denoted $C_p^\infty(M)$, and defined as the set of smooth functions $f: U \to \mathbb{R}$ such that U is a neighborhood of p, modulo the equivalence $\sim$, where $f: U \to \mathbb{R} \sim g: V \to \mathbb{R}$ is there is a neighborhood $W \subseteq U \cap V$ of p such that $f|_W = g|_W$



$f \sim g$

Which equivalence classes in $C_p^\infty(M)$ are invertible?

Because M is locally $\mathbb{R}^{\dim(M)}$, $C_p^\infty(M) \cong C_0^\infty(\mathbb{R}^{\dim(M)})$.

A <u>derivation</u> on $C_p^\infty(M)$ is a linear map

$$\delta : C_p^\infty(M) \longrightarrow \mathbb{R}$$

such that if $[f], [g] \in C_p^\infty(M)$, then

$$\delta([fg]) = f(p)\,\delta([g]) + g(p)\,\delta([f])$$

The set of all derivations on $C_p^\infty(M)$ is denoted $D_p(M)$. Note that $D_p(M) \cong D_0(\mathbb{R}^{\dim(M)})$.

<u>Observations</u>:

① If $\alpha \in \mathbb{R}$ and $\delta_1, \delta_2 \in D_p(M)$, then $\alpha\delta_1 + \delta_2 : C_p^\infty(M) \longrightarrow \mathbb{R}$ is linear and it satisfies the "product formula" above, so $\alpha\delta_1 + \delta_2 \in D_p(M)$. That is, $D_p(M)$ is an $\mathbb{R}$-vector space.

② If $\alpha \in \mathbb{R}$ denotes a constant function on M, then $\delta([\alpha]) = 0$. Indeed, $\delta([1]) = \delta([1 \cdot 1]) = \delta([1]) + \delta([1])$ so $\delta([1]) = 0$. Then $\delta([\alpha]) = \delta(\alpha[1]) = \alpha\,\delta([1]) = \alpha \cdot 0 = 0$.

Examples:

① $\frac{\partial}{\partial x_i}\Big|_0 \in D_0(\mathbb{R}^n)$ where $\frac{\partial}{\partial x_i}\Big|_0([f]) = \frac{\partial f}{\partial x_i}\Big|_0$.

This is well-defined since $\frac{\partial f}{\partial x_i}\Big|_0 = \frac{\partial g}{\partial x_i}\Big|_0$ if $f \sim g$.

Note that $\frac{\partial}{\partial x_i}\Big|_0 : C_0^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is linear and $\frac{\partial}{\partial x_i}\Big|_0([fg]) = \frac{\partial (fg)}{\partial x_i}\Big|_0 = f(0)\frac{\partial g}{\partial x_i}\Big|_0 + g(0)\frac{\partial f}{\partial x_i}\Big|_0$

② Since $D_0(\mathbb{R}^n)$ is an $\mathbb{R}$-vector space, $\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}\Big|_0 \in D_0(\mathbb{R}^n)$ for any $\alpha_i \in \mathbb{R}$. Thus, we have a linear map $A : \mathbb{R}^n \longrightarrow D_0(\mathbb{R}^n)$ where $A(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}\Big|_0$.

A is injective. Indeed if $\alpha_k \neq 0$, then $\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}\Big|_0 ([x^k]) = \alpha_k \neq 0$ where $x^k : \mathbb{R}^n \longrightarrow \mathbb{R}$ is projection to the k^{th} coordinate. Thus, $A(\alpha_1, \dots, \alpha_n) \neq 0$.

Theorem:

$A : \mathbb{R}^n \longrightarrow D_0(\mathbb{R}^n)$ is surjective so that $D_0(\mathbb{R}^n) \cong \mathbb{R}^n$. In particular, if $\delta \in D_0(\mathbb{R}^n)$ then $\delta = \sum_{i=1}^{n} \delta([x^i]) \frac{\partial}{\partial x_i}\Big|_0$.

proof:

Let $\delta \in D_0(\mathbb{R}^n)$. For any $[f] \in C_0^\infty(\mathbb{R}^n)$, if $t \in [0,1]$ and $x \approx 0$ then tx is in the domain of f. Hence we can define $h_x : [0,1] \longrightarrow \mathbb{R}$ by $h_x(t) = f(tx)$. Note that if $m_x : [0,1] \longrightarrow \mathbb{R}^n$ is $m_x(t) = tx$, then $h_x = f \circ m_x$. Thus,

$$\begin{aligned}
h_x'(t) &= D_t h_x \\
&= D_t(f \circ m_x) \\
&= D_{tx} f \circ D_t m_x \\
&= \left(\frac{\partial f}{\partial x_1}\Big|_{tx}, \dots, \frac{\partial f}{\partial x_n}\Big|_{tx} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_{tx} x^i(x)
\end{aligned}$$

Hence,

$$
\begin{aligned}
f(x)-f(0) &= h_x(1)-h_x(0) \\
&= \int_0^1 h_x'(t)\,dt \\
&= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_{tx} x^i(x)\,dt \\
&= \sum_{i=1}^n \left(\int_0^1 \frac{\partial f}{\partial x_i}\Big|_{tx} dt\right) x^i(x)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\delta([f]) &= \delta([f]) - 0 \\
&= \delta([f]) - \delta([f(0)]) \\
&= \delta([f-f(0)]) \\
&= \delta\left(\left[\sum_{i=1}^n \left(\int_0^1 \frac{\partial f}{\partial x_i}\Big|_{tx} dt\right) x^i\right]\right) \\
&= \sum_{i=1}^n \left(\int_0^1 \frac{\partial f}{\partial x_i}\Big|_0 dt \; \delta([x^i]) + x^i(0)\,\delta\left(\left[\int_0^1 \frac{\partial f}{\partial x_i}\Big|_{tx} dt\right]\right)\right) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_0 \delta([x^i]) + 0 \\
&= \left(\sum_{i=1}^n \delta([x^i]) \frac{\partial}{\partial x_i}\Big|_0\right)([f]).
\end{aligned}
$$

$\boxed{\text{END}}$

Since $C_p^\infty(M) \cong C_0^\infty(\mathbb{R}^{\dim(M)})$, we have the following

<u>Corollary:</u> $D_p(M) \cong \mathbb{R}^{\dim(M)}$

It will help us in the next chapter to have a coordinate-free way to express derivations. For this, if $v \in T_pM$, then we define the <u>Lie derivative in the direction of v</u> to be $L_v \in D_p(M)$ where

$$L_v([f]) = D_p f\,(v)$$

Thus, for example, if $M = \mathbb{R}^3$ and $p = 0$, then

$$\begin{aligned} L_{(a,b,c)}([f]) &= D_0 f \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= \left(\frac{\partial f}{\partial x_1}\Big|_0 , \frac{\partial f}{\partial x_2}\Big|_0 , \frac{\partial f}{\partial x_3}\Big|_0 \right) \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= a \frac{\partial f}{\partial x_1}\Big|_0 + b \frac{\partial f}{\partial x_2}\Big|_0 + c \frac{\partial f}{\partial x_3}\Big|_0 \end{aligned}$$

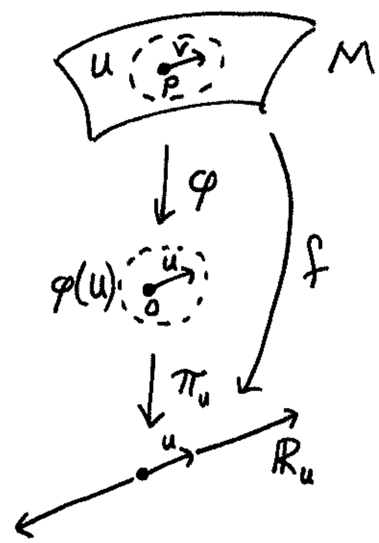
<u>Theorem:</u> $v \longmapsto L_v$ is a vector space isomorphism between T_pM and $D_p(M)$.

<u>proof:</u>

You can check that $v \longmapsto L_v$ is linear, and we know $\dim(T_pM) = \dim(D_p(M))$, so we will show that $v \longmapsto L_v$ is injective.

Suppose $v \neq 0$. We want to show $L_v \neq 0$. Indeed, $v = [(U, \varphi, u)]$ for a chart (U, φ) with $\varphi(p) = 0$. Let $\pi_u : \varphi(U) \to \mathbb{R}u$ be projection onto the line $\mathbb{R}u$. Then if $f = \pi_u \circ \varphi : U \to \mathbb{R}$ we have

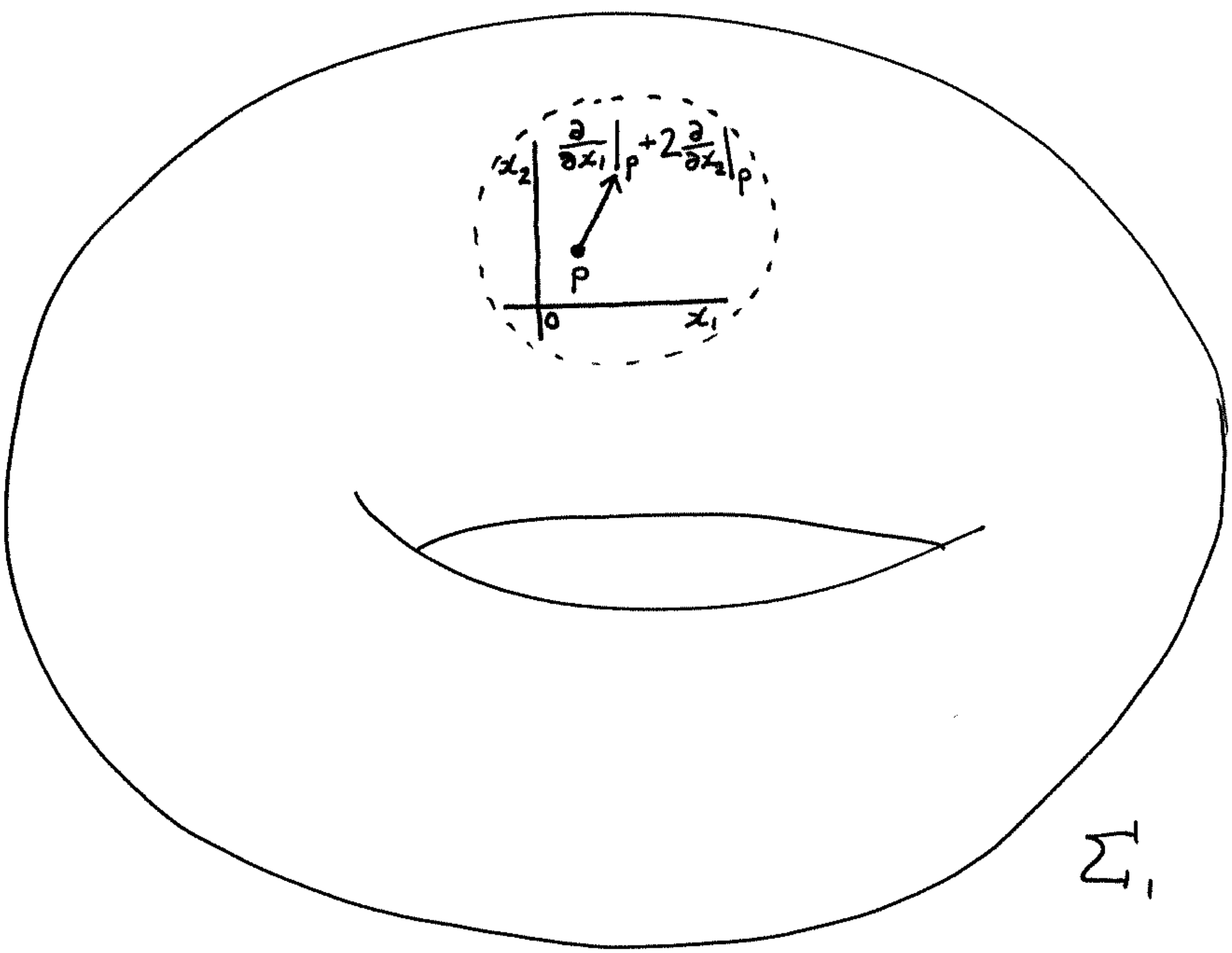
$$\begin{aligned} L_v([f]) &= D_p(f)(v) \\ &= u \\ &\neq 0 \end{aligned}$$



so that $L_v \neq 0$.

END

In light of the previous two theorems, if one is using local coordinates on a manifold M near a point $p \in M$, then one often writes $\sum_{i=1}^{\dim(M)} \alpha_i \frac{\partial}{\partial x_i}\Big|_p$ to denote a tangent vector in T_pM.



Also, we write $\sum_i \alpha_i \frac{\partial}{\partial x_i}\Big|_p$ interchangeably with $L_{\sum_i \alpha_i \frac{\partial}{\partial x_i}|_p}$.

Exercises:

① Prove $C_p^\infty(M) \cong C_0^\infty(\mathbb{R}^{\dim(M)})$

② If $\alpha \in \mathbb{R}$ and $\delta_1, \delta_2 \in D_p(M)$, prove that $\alpha\delta_1 + \delta_2$ satisfies the product formula. That is, that

$$(\alpha\delta_1+\delta_2)([fg]) = f(p)(\alpha\delta_1+\delta_2)([g]) + g(p)(\alpha\delta_1+\delta_2)([f])$$

for all $[f],[g] \in C_p^\infty(M)$.

③ Let $f \in C^\infty(\mathbb{R}^3)$ be $f(x,y,z) = x^3+y^2+2xyz+3zy+5$ and let $v \in T_{(2,3,1)}\mathbb{R}^3$ be $v = (\tfrac{1}{2}, 5, 1)$.

Find $L_v([f])$.

④ Let f be as in ③. Find

$$\left(\frac{1}{2} \frac{\partial}{\partial x}\Big|_{(2,3,1)} + 5 \frac{\partial}{\partial y}\Big|_{(2,3,1)} + \frac{\partial}{\partial z}\Big|_{(2,3,1)} \right) f.$$

⑤ $C_p^\infty(M) \longrightarrow \mathbb{R}$ where $[f] \longmapsto f(p)$ is a linear map. We let m_p be the kernel of this map, so that m_p is a maximal ideal in $C_p^\infty(M)$.

Let $m_p^2 = \left\{ \left[\sum_{i=1}^{k} f_i g_i \right] \;\middle|\; [f_i], [g_i] \in m_p \right\}$

Then m_p^2 is a vector subspace of m_p, so m_p/m_p^2 is a vector space. Let $\left(m_p/m_p^2\right)^*$ be the dual.

Given $\varphi \in \left(m_p/m_p^2\right)^*$, let $\delta_\varphi : C_p^\infty(M) \to \mathbb{R}$ be defined by $\delta_\varphi([f]) = \varphi\left([f - f(p)] + m_p^2\right)$.

Given $\delta \in D_p(M)$, let $\varphi_\delta : m_p/m_p^2 \to \mathbb{R}$ be defined by $\varphi_\delta\left([f] + m_p^2\right) = \delta([f])$.

(a) Prove φ_δ is well-defined and $\varphi_\delta \in \left(m_p/m_p^2\right)^*$

(b) Prove $\delta_\varphi \in D_p(M)$

(c) Prove $D_p(M) \longrightarrow \left(m_p/m_p^2\right)^*$ where $\delta \mapsto \varphi_\delta$ is linear.

(d) Prove $\varphi \mapsto \delta_\varphi$ is the inverse of $\delta \mapsto \varphi_\delta$ and conclude that $\left(m_p/m_p^2\right)^* \cong T_pM$.

§ Global Derivations

Let M be a smooth manifold.
Let $C^\infty(M)$ be the $\mathbb{R}$-algebra of smooth functions $f: M \to \mathbb{R}$.

A <u>derivation</u> on M is a linear function

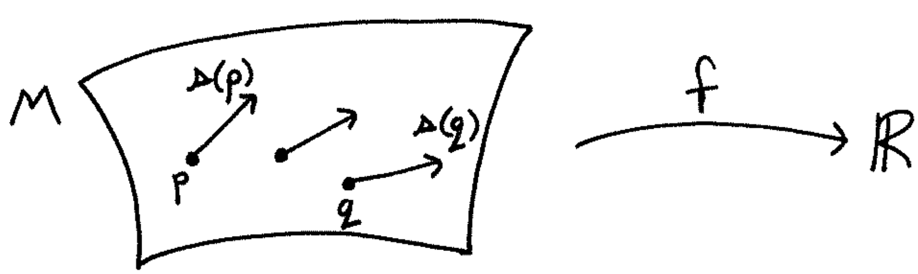
$$\delta : C^\infty(M) \longrightarrow C^\infty(M)$$

such that $\delta(fg) = f\,\delta(g) + g\,\delta(f)$ for any $f, g \in C^\infty(M)$.

We denote the set of all derivations on M as $D(M)$. Check that $D(M)$ is an $\mathbb{R}$-vector space.

Given a vector field $\Delta \in \Gamma(TM)$, we define

$$L_\Delta : C^\infty(M) \longrightarrow C^\infty(M)$$
$$L_\Delta(f)(p) = L_{\Delta(p)} f$$



Lemma: $L_\Delta \in D(M)$

proof:

I. L_Δ is linear:

Let $\alpha \in \mathbb{R}$ and $f, g \in C^\infty(M)$. Then for any $p \in M$,

$$\begin{aligned} L_\Delta(\alpha f + g)(p) &= L_{\Delta(p)}(\alpha f + g) \\ &= \alpha L_{\Delta(p)}(f) + L_{\Delta(p)}(g) \\ &= \alpha L_\Delta(f)(p) + L_\Delta(g)(p) \\ &= \big(\alpha L_\Delta(f) + L_\Delta(g)\big)(p) \end{aligned}$$

Thus, $L_\Delta(\alpha f + g) = \alpha L_\Delta(f) + L_\Delta(g)$.

II. Product rule:

Let $f, g \in C^\infty(M)$. Then for any $p \in M$,

$$\begin{aligned} L_\Delta(fg)(p) &= L_{\Delta(p)}(fg) \\ &= f(p) L_{\Delta(p)}(g) + g(p) L_{\Delta(p)}(f) \\ &= f(p) L_\Delta(g)(p) + g(p) L_\Delta(f)(p) \\ &= \big(f L_\Delta(g) + g L_\Delta(f)\big)(p) \end{aligned}$$

Thus, $L_\Delta(fg) = f L_\Delta(g) + g L_\Delta(f)$. $\boxed{\text{END}}$

Theorem: $\Gamma(TM) \cong D(M)$ as $\mathbb{R}$-vector spaces via $\Phi : \Gamma(TM) \longrightarrow D(M)$ where $\Phi(\Delta) = L_\Delta$.

proof:

I. Φ is linear:

Let $\alpha \in \mathbb{R}$, and let $\Delta_1, \Delta_2 \in \Gamma(TM)$. If $f \in C^\infty(M)$ and $p \in M$, then

$$\begin{aligned}
\Phi(\alpha\Delta_1 + \Delta_2)(f)(p) &= L_{\alpha\Delta_1+\Delta_2}(f)(p) \\
&= L_{(\alpha\Delta_1+\Delta_2)(p)} f \\
&= L_{\alpha\Delta_1(p)+\Delta_2(p)} f \\
&= \left(\alpha L_{\Delta_1(p)} + L_{\Delta_2(p)}\right)(f) \\
&= \alpha L_{\Delta_1(p)} f + L_{\Delta_2(p)} f \\
&= \alpha L_{\Delta_1}(f)(p) + L_{\Delta_2}(f)(p) \\
&= \alpha \Phi(\Delta_1)(f)(p) + \Phi(\Delta_2)(f)(p).
\end{aligned}$$

Thus, $\Phi(\alpha\Delta_1 + \Delta_2) = \alpha\Phi(\Delta_1) + \Phi(\Delta_2)$.

II. Φ is injective:

Let $\Delta_1, \Delta_2 \in \mathcal{P}(TM)$ with $\Delta_1 \neq \Delta_2$. Then there is some $p \in M$ such that $\Delta_1(p) \neq \Delta_2(p)$. Let $U \subseteq M$ be a neighborhood of p, and in local coordinates let $f : U \longrightarrow \mathbb{R}$ be a linear map with $f(\Delta_1(p)) \neq f(\Delta_2(p))$.

We extend f to $\bar{f} \in C^\infty(M)$ so that

$$\begin{aligned}\Phi(\Delta_i)(\bar{f})(p) &= L_{\Delta_i(p)} \bar{f} \\ &= L_{\Delta_i(p)} f \\ &= D_p f(\Delta_i(p)) \\ &= f(\Delta_i(p))\end{aligned}$$

Hence,

$$\Phi(\Delta_1)(\bar{f})(p) \neq \Phi(\Delta_2)(\bar{f})(p).$$

Thus, $\Phi(\Delta_1) \neq \Phi(\Delta_2)$.

III. Φ is surjective

Let $\delta \in D(M)$. Given $p \in M$, we define $\delta_p \in D_p M$ by $\delta_p([f]) = \delta(\bar{f})(p)$. Since $T_pM \cong D_p(M)$, there is some $\Delta_\delta(p) \in T_pM$ such that $L_{\Delta_\delta(p)} = \delta_p$. Thus we have defined a vector field $\Delta_\delta : M \to TM$, though we still have to check that Δ_δ is smooth.

To see that Δ_δ is smooth, we use local coordinates to write $\Delta_\delta(p) = \sum_i a_i(p) \frac{\partial}{\partial x_i}\Big|_p$. Note that Δ_δ will be smooth if each a_k is smooth, so we let x^k be projection onto the k^{th} coordinate and note that

$$a_k(p) = \left(\sum_i a_i(p) \frac{\partial}{\partial x_i}\Big|_p \right)(x^k) = L_{\Delta_\delta(p)} x^k$$

$$= \delta_p(x^k) = \delta(\overline{x^k})(p)$$

Since $\delta \in D(M)$, $a_k = \delta(\overline{x^k}) \in C^\infty(M)$, so we have that $\Delta_\delta \in \Gamma(TM)$.

Last note that

$$\Phi(\Delta_\delta)(f)(p) = L_{\Delta_\delta}(f)(p)$$

$$= L_{\Delta_\delta(p)} f$$

$$= \delta_p(f)$$

$$= \delta(f)(p)$$

Thus, $\Phi(\Delta_\delta) = \delta$.

$\boxed{\text{END}}$

As we saw in the proof of this theorem, smooth vector fields are often written in local coordinates as $\sum_i a_i(p) \frac{\partial}{\partial x_i}\Big|_p$ where each a_i is a smooth real-valued function.

We can, and do, write $\sum_i a_i(p) \frac{\partial}{\partial x_i}\Big|_p$ and $L_{\sum_i a_i(p) \frac{\partial}{\partial x_i}|_p}$ interchangeably. And when it's understood that the a_i are functions, we write $\sum_i a_i \frac{\partial}{\partial x_i}$.

Exercises:

① Prove if $\alpha \in \mathbb{R}$ and $\delta_1, \delta_2 \in D(M)$, then $\alpha\delta_1 + \delta_2$ satisfies the product rule, and therefore, $D(M)$ is an $\mathbb{R}$-vector space.

② Let $f \in C^\infty(\mathbb{R}^2)$ and $\Delta \in \Gamma(T\mathbb{R}^2)$ be given by $f(x,y) = 2x^3y^2 + 7$ and $\Delta(x,y) = (x+y)\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y}$. Find $L_\Delta(f)(x,y)$.

③ If M is a smooth manifold, $\Delta \in \Gamma(TM)$, and $f \in C^\infty(M)$, prove $L_\Delta f \in C^\infty(M)$.
(Hint: check this locally, and use #3 page 93.)

§ Lie Brackets

$D(M)$ is a vector space. The goal of this short chapter is to introduce a multiplication for $D(M)$, thus making $D(M)$ into an algebra.

If $\delta, \delta' \in D(M)$, then $\delta \circ \delta' : C^\infty(M) \to C^\infty(M)$ is linear, and one might hope that $\delta \circ \delta'$ also satisfies the product formula, because then $\delta \circ \delta'$ would be an element of $D(M)$. But $\delta \circ \delta'$ will usually not satisfy the product formula.

Example:

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in D(\mathbb{R}^2)$ and $\frac{\partial}{\partial x}\frac{\partial}{\partial y}$ does not satisfy the product formula since for $x, y \in C^\infty(\mathbb{R}^2)$ we see that

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}(xy) = \frac{\partial}{\partial x}(x) = 1$$

but

$$x\frac{\partial}{\partial x}\frac{\partial}{\partial y}(y) + y\frac{\partial}{\partial x}\frac{\partial}{\partial y}(x) = x\frac{\partial}{\partial x}(1) + y\frac{\partial}{\partial x}(0) = 0.$$

While $\delta \circ \delta'$ does not in general satisfy a product rule, $\delta \circ \delta' - \delta' \circ \delta$ does, so we can use the operation $(\delta, \delta') \longmapsto \delta \circ \delta' - \delta' \circ \delta$ to define an algebra structure on $D(M)$. This is the content of the following

<u>Lemma</u>: If M is a smooth manifold and $\delta, \delta' \in D(M)$, then $\delta \circ \delta' - \delta' \circ \delta \in D(M)$.

Now we can use the vector space isomorphism $D(M) \cong \Gamma(TM)$ to define an algebra structure on $\Gamma(TM)$ as follows:

If $X, Y \in \Gamma(TM)$, then $L_X, L_Y \in D(M)$, so $L_X \circ L_Y - L_Y \circ L_X \in D(M) \cong \Gamma(TM)$, thus there is a unique vector field on M – which we denote by $[X,Y]$ – such that $L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X$. We call $[X,Y]$ the <u>Lie bracket</u> of X and Y. The Lie bracket is the multiplication rule that makes $\Gamma(TM)$ an algebra.

Note that for $X, Y, Z \in \Gamma(TM)$,

(i) $[X, Y] = -[Y, X]$

(ii) $[X, X] = 0$

(iii) $\big[X, [Y, Z]\big] + \big[Z, [X, Y]\big] + \big[Y, [Z, X]\big] = 0$

The third equation is called the Jacobi Identity.

With these rules, $\Gamma(TM)$ is an example of a <u>Lie algebra</u>.

Exercises:

① If $\delta, \delta' \in D(M)$, prove that $\delta \circ \delta' - \delta' \circ \delta$ satisfies the product formula. Thus $D(M)$ is an algebra.

② For $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \in \Gamma(T\mathbb{R}^n)$, find $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]$.

③ Fix $X, Y \in \Gamma(TM)$. Prove that $[-, Y] : \Gamma(TM) \to \Gamma(TM)$ is linear. (Using that $[X,Y] = -[Y,X]$, we could then deduce that $[X, -] : \Gamma(TM) \to \Gamma(TM)$ is linear as well.)

④ If $f, g \in C^\infty(M)$ and $X, Y \in \Gamma(TM)$, prove

$$[fX, gY] = fg[X,Y] + fL_X(g)Y - gL_Y(f)X$$

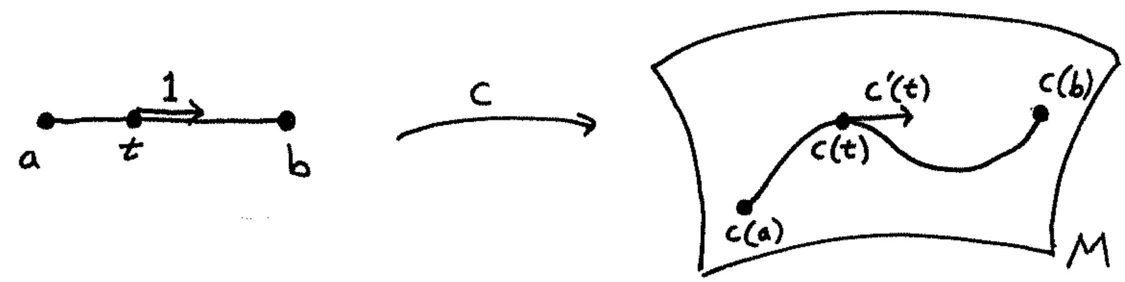
⑤ Use ② and ④ to show that if $b, a \in C^\infty(\mathbb{R}^n)$ then $\left[a\frac{\partial}{\partial x_i}, b\frac{\partial}{\partial x_j}\right] = a\frac{\partial b}{\partial x_i}\frac{\partial}{\partial x_j} - b\frac{\partial a}{\partial x_j}\frac{\partial}{\partial x_i}$

⑥ Use ③ and ⑤ to find $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right]$, where $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \in \Gamma(T\mathbb{R}^3)$.

⑦ Prove the Jacobi Identity.

§ Flows

Let $a, b \in \mathbb{R}$ with $a<b$ so that $[a,b] \subseteq \mathbb{R}$. Let M be a smooth manifold. Given a <u>smooth curve</u> $c:[a,b] \longrightarrow M$, let $c'(t) = D_t c(1) \in T_{c(t)}M$.



We have the following 2 propositions from the theory of ODEs:

Proposition:

If $p \in M$ and $X \in \Gamma(TM)$, then there is an interval $I_p \subseteq \mathbb{R}$ with $0 \in I_p$, and a unique smooth curve $c_p : I_p \to M$ such that

(i) $c_p(0) = p$

(ii) $c_p'(t) = X(c_p(t))$

Proposition:

If $p \in M$ and $X \in \Gamma(TM)$, then there are open neighborhoods $V \subseteq M$ of p and $I \subseteq \mathbb{R}$ of 0, such that for any $q \in V$, $I \subseteq I_q$ and $I \times V \to M$ defined by $(t,q) \mapsto c_q(t)$ is smooth.

Notation: $c_q(t)$ suggests that we are flowing a fixed $q \in M$ for a variable time t. If instead, we wish to suggest that we are flowing variable points in M for a fixed time t, then we may write $\theta_t^X(q)$ instead of $c_q(t)$. This notation also reminds us of the role of $X \in \Gamma(TM)$. Now the two rules from the previous page become: (i) $\theta_0^X(q) = q$, and (ii) $D_{t_0}\theta_t^X(q)(1) = X(\theta_{t_0}^X(q))$.

Proposition:

Given $X \in \Gamma(TM)$ and $p \in M$, let V and I be as above. If $t, t', t+t' \in I$, then $\theta_t^X : V \to \theta_t^X(V) \subseteq M$ is a diffeomorphism and $\theta_t^X \circ \theta_{t'}^X = \theta_{t+t'}^X$.

proof:

Flowing for t' and then for t is the same as flowing for $t+t'$. That is, $\theta_t^X \circ \theta_{t'}^X = \theta_{t+t'}^X$. Now note that since $\theta_0^X = id$, $\theta_{-t}^X : \theta_t(V) \to V$ is a smooth inverse of $\theta_t^X : V \to \theta_t^X(V)$.

END

Given $X \in \Gamma(TM)$ and $p \in M$, we let $p \in V \subseteq M$ and $0 \in I \subseteq \mathbb{R}$ as above. We call $\{\theta_t^X : V \to M\}_{t \in I}$ a <u>local one parameter group</u> for X near p.

If $V = M$ and $I = \mathbb{R}$, then $\{\theta_t^X\}_{t \in \mathbb{R}} \subseteq \mathrm{Diff}(M)$ is called the <u>one parameter group of diffeomorphisms</u> for X.

<u>Examples:</u>

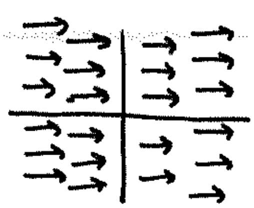
① For $t \in \mathbb{R}$, Let $f_t \in \mathrm{Diff}(\mathbb{R}^2)$ be $f_t(x,y) = (x+t, y)$. For $\frac{\partial}{\partial x} \in \Gamma(T\mathbb{R}^2)$, note that $f_0(x,y) = (x,y)$, and

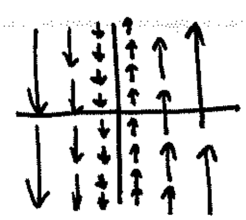
$$D_{t_0} f_t(x,y)(1) = (1,0)\Big|_{f_{t_0}(x,y)} = \frac{\partial}{\partial x}\Big|_{(x+t_0,y)}$$

Thus, $f_t = \theta_t^{\frac{\partial}{\partial x}}$. That is $\theta_t^{\frac{\partial}{\partial x}}(x,y) = (x+t, y)$.

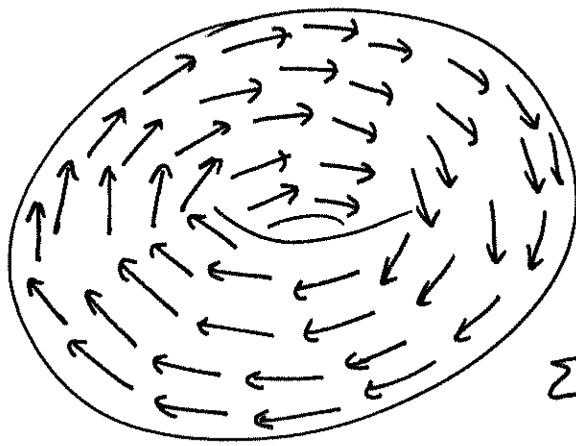
② For $x\frac{\partial}{\partial y} \in \Gamma(T\mathbb{R}^2)$, $\theta_t^{x\frac{\partial}{\partial y}}(x,y) = (x, y+tx)$ since

$$D_{t_0}\theta_t^{x\frac{\partial}{\partial y}}(x,y)(1) = (0,x)\Big|_{(x,y+t_0x)} = x\frac{\partial}{\partial y}\Big|_{\theta_{t_0}^{x\frac{\partial}{\partial y}}(x,y)}$$

$\frac{\partial}{\partial x}$ 

 $x\frac{\partial}{\partial y}$

③ For $X \in \Gamma(T\Sigma_1^1)$ as below



$\theta_t^X : \Sigma_1^1 \to \Sigma_1^1$ is a clockwise rotation of Σ_1^1.

Having a one parameter group for $X \in \Gamma(TM)$ is better than having a local one parameter group for X, and we'll see in the corallary to the following theorem that the key for having a full one parameter group for $X \in \Gamma(TM)$ is to have M be compact.

Theorem:

Suppose $X \in \Gamma(TM)$, that $K \subseteq M$ is compact, and that $X(p) = 0$ if $p \notin K$. Then, with our notation as above, $I = \mathbb{R}$ and $V = M$ so that X has a one parameter group of diffeomorphisms.

proof:

Cover K with finitely many open $V_i \subseteq M$ and let $I_i \subseteq \mathbb{R}$ be such that $\{\theta_t^{X,i} : V_i \to M\}_{t \in I_i}$ is a local one parameter group.

Let $I = \bigcap_i I_i$, and for $t \in I$ let

$$\theta_t^X(q) = \begin{cases} \theta_t^{X,i}(q) & \text{if } q \in V_i \\ q & \text{if } q \notin \bigcup_i V_i \end{cases}$$

By uniqueness of flows, $\theta_t^{X,i}(q) = \theta_t^{X,j}(q)$ if $q \in U_i \cap U_j$. Also, $\theta_t^{X,i}(q) = q$ if $q \in V_i - K$, so $\theta_t^X : M \to M$ is a diffeomorphism for all $t \in I$.

Since $\theta_t^{X,i} \circ \theta_{t'}^{X,i} = \theta_{t+t'}^{X,i}$ for all i, so $\theta_t^X \circ \theta_{t'}^X = \theta_{t+t'}^X$, and thus we can extend $\{\theta_t^X\}_{t \in I}$ to $\{\theta_t^X\}_{t \in \mathbb{R}}$ by $\theta_{kt}^X = (\theta_t^X)^k$ for all $k \in \mathbb{Z}$. $\boxed{\text{END}}$

Corollary:

If M is compact, any $X \in \Gamma(TM)$ has a one parameter group of diffeomorphisms of M.

proof:

Let K from the previous theorem by M.

END

Lemma:

Let $p \in M$, $f \in C^\infty(M)$, and $X \in \Gamma(TM)$.

Then

$$L_X f(p) = \lim_{h \to 0} \frac{f \circ \theta_h^X(p) - f(p)}{h}$$

proof:

Recall that $\theta_0^X(p) = p$ and that $D_0\left(\theta_t^X(p)\right)(1) = X\left(\theta_0^X(p)\right)$. Now we view $f \circ \theta_h^X(p) : \mathbb{R} \longrightarrow \mathbb{R}$ as a function of h, with p and X fixed, to see that

$$
\begin{aligned}
L_X f(p) &= L_{X(p)} f \\
&= D_p f\big(X(p)\big) \\
&= D_p f\Big(X\big(\theta_0^X(p)\big)\Big) \\
&= D_p f\Big(D_0\big(\theta_t^X(p)\big)(1)\Big) \\
&= D_0\big(f \circ \theta_t^X(p)\big)(1) \\
&= \frac{d}{dt}\Big|_0 \big(f \circ \theta_t^X(p)\big) \\
&= \lim_{h \to 0} \frac{f \circ \theta_h^X(p) - f \circ \theta_0^X(p)}{h} \\
&= \lim_{h \to 0} \frac{f \circ \theta_h^X(p) - f(p)}{h}
\end{aligned}
$$

$\boxed{\text{END}}$

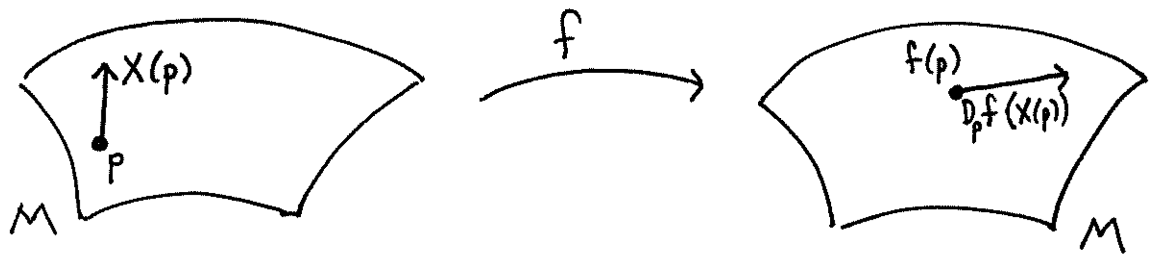
Exercises:

1. Define $X_0 \in \Gamma(TM)$ by $X_0(p) = 0$ for all $p \in M$. Show that $\theta_t^{X_0} = id$ for all $t \in \mathbb{R}$.
2. Find $\theta_t^{2\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}} \in \mathrm{Diff}(\mathbb{R}^2)$.

§ Push-forwards of vector fields

Let $f \in \mathrm{Diff}(M)$, where M is a smooth manifold, and let $X \in \Gamma(TM)$. The push-forward of X with respect to f is $f_* X \in \Gamma(TM)$ defined by

$$f_* X\,(f(p)) = D_p f\,(X(p))$$



Example:

For $\frac{\partial}{\partial x} \in \Gamma(T\mathbb{R}^2)$, we saw on page 120 that $\theta_h^{\partial/\partial x} \in \mathrm{Diff}(\mathbb{R}^2)$ is $\theta_h^{\frac{\partial}{\partial x}}(p,q) = (p+h, q)$. Thus, for $x\frac{\partial}{\partial y} \in \Gamma(T\mathbb{R}^2)$, we can find the push-forward of $x\frac{\partial}{\partial y}$ with respect to $\theta_h^{\frac{\partial}{\partial x}}$ as

$$\left(\theta_h^{\frac{\partial}{\partial x}}\right)_*\left(x\frac{\partial}{\partial y}\right)(p,q) = \left(\theta_h^{\frac{\partial}{\partial x}}\right)_*\left(x\frac{\partial}{\partial y}\right)\left(\theta_h^{\frac{\partial}{\partial x}}(p-h,q)\right)$$

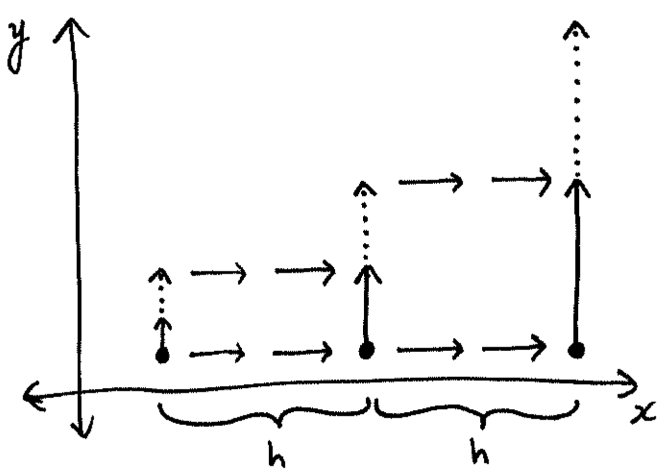
$$= \left(D_{(p-h,q)}\,\theta_h^{\frac{\partial}{\partial x}}\right)\left(x\frac{\partial}{\partial y}\Big|_{(p-h,q)}\right)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left((p-h)\frac{\partial}{\partial y}\Big|_{(p-h,q)}\right)$$

$$= (p-h)\frac{\partial}{\partial y}\Big|_{(p,q)}$$

That is, $\left(\theta_h^{\frac{\partial}{\partial x}}\right)_*\left(x\frac{\partial}{\partial y}\right) = (x-h)\frac{\partial}{\partial y}$.

In the drawing below, horizontal vectors are $\frac{\partial}{\partial x}$, vertical solid vectors are $\left(\theta_h^{\frac{\partial}{\partial x}}\right)_*\left(x\frac{\partial}{\partial y}\right)$, and vertical dotted vectors based at the same points are $x\frac{\partial}{\partial y}$.



So the horizontal vectors are pushing the dotted vectors forward to the vertical solid vectors.

Lemma:

If $f \in \mathrm{Diff}(M)$, $X \in \Gamma(TM)$, and $h \in C^\infty(M)$, then

$$L_X(h \circ f) = (L_{f_*X} h) \circ f$$

proof:

For $p \in M$ we have

$$\begin{aligned} L_X(h \circ f)(p) &= L_{X(p)}(h \circ f) \\ &= D_p(h \circ f)(X(p)) \\ &= D_{f(p)}h \circ D_p f\,(X(p)) \\ &= D_{f(p)}h\,(f_*X(f(p))) \\ &= L_{f_*X(f(p))}h \\ &= L_{f_*X}h\,(f(p)) \\ &= (L_{f_*X}h) \circ f\,(p). \end{aligned}$$

END

Proposition:

If $f \in \mathrm{Diff}(M)$, $X \in \Gamma(TM)$, $p \in M$, and if $\{\theta_t^X\}_{t\in I}$ is a local one parameter group near p, then $f_* X \in \Gamma(TM)$ has $\{f \circ \theta_t^X \circ f^{-1}\}_{t\in I}$ as a local one parameter group near $f(p)$. That is, we may take $\theta_t^{f_*X} = f \circ \theta_t^X \circ f^{-1}$.

proof:

Recall from page 119 that $\theta_t^{f_*X}$ is uniquely defined for $q \approx f^{-1}(p)$ by

(i) $\theta_0^{f_*X}(q) = q$, and

(ii) $D_{t_0} \theta_t^{f_*X}(q)(1) = f_*X(\theta_{t_0}^{f_*X}(q))$

so we just have to check that $f \circ \theta_t^X \circ f^{-1}$ satisfies these equations as well.

For (i), $f \circ \theta_0^X \circ f^{-1}(q) = f \circ f^{-1}(q) = q$, and for (ii) we have

$$D_{t_0}\left(f\circ\theta_t^X\circ f^{-1}(q)\right)(1) = D_{t_0}\left(f\circ\theta_t^X\left(f^{-1}(q)\right)\right)(1)$$

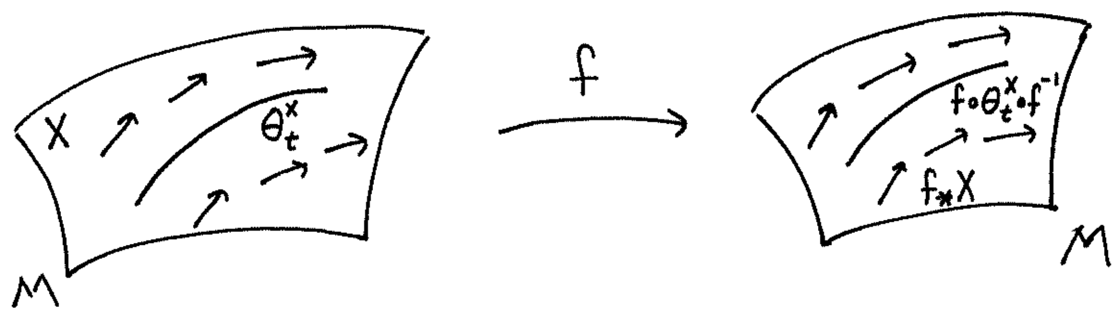
$$= \left(D_{\theta_{t_0}^X(f^{-1}(q))} f\right)\circ\left(D_{t_0}\left(\theta_t^X\left(f^{-1}(q)\right)\right)\right)(1)$$

$$= D_{\theta_{t_0}^X(f^{-1}(q))} f\left(X\left(\theta_{t_0}^X\left(f^{-1}(q)\right)\right)\right)$$

$$= f_*X\left(f\circ\theta_{t_0}^X\left(f^{-1}(q)\right)\right)$$

$$= f_*X\left(f\circ\theta_{t_0}^X\circ f^{-1}(q)\right).$$

$\boxed{\text{END}}$



Lemma:

Let $f:(-\varepsilon,\varepsilon)\times M \longrightarrow \mathbb{R}$ be smooth with $f(0,p)=0$ for all $p\in M$. Then there is a smooth $g:(-\varepsilon,\varepsilon)\times M \longrightarrow \mathbb{R}$ with $f(t,p)=t\,g(t,p)$ and $\frac{\partial f}{\partial t}(0,p)=g(0,p)$.

proof:

Let $g(t,p)=\int_0^1 \left(\frac{\partial}{\partial s}f\right)(st,p)\,ds$. Let

$h_{t,p}(s)=f(st,p)$ and let $m(s,t,p)=(st,p)$ so that $h_{t,p}(s)=f\circ m(s,t,p)$. Then

$h'_{t,p}(s)=f'(m(s,t,p))\,m'(s,t,p)=f'(st,p)\,t$.

Thus,

$$f(t,p)=f(t,p)-f(0,p)=h_{t,p}(1)-h_{t,p}(0)$$
$$=\int_0^1 h'_{t,p}(s)\,ds=\int_0^1 f'(st,p)\,t\,ds=t\,g(t,p).$$

Furthermore,

$$\frac{\partial f}{\partial t}(t,p)=t\frac{\partial g}{\partial t}(t,p)+g(t,p)$$

which implies that $\frac{\partial f}{\partial t}(0,p)=g(0,p)$. $\boxed{\text{END}}$

For $X, Y \in \Gamma(TM)$, we define the <u>Lie derivative</u> of Y in the direction of X to be the vector field $L_X Y \in \Gamma(TM)$ defined by

$$L_X Y(p) = \lim_{h \to 0} \frac{Y(p) - ((\theta_h^X)_* Y)(p)}{h}$$

<u>Example</u>:

For $\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} \in \Gamma(T\mathbb{R}^2)$, recall that $(\theta_h^{\frac{\partial}{\partial x}})_*(x\frac{\partial}{\partial y}) = (x-h)\frac{\partial}{\partial y}$. Thus, for $(p,q) \in \mathbb{R}^2$,

$$L_{\frac{\partial}{\partial x}}\left(x\frac{\partial}{\partial y}\right)(p,q) = \lim_{h \to 0} \frac{x\frac{\partial}{\partial y}\Big|_{(p,q)} - \left((\theta_h^{\frac{\partial}{\partial x}})_* x\frac{\partial}{\partial y}\right)(p,q)}{h}$$

$$= \lim_{h \to 0} \frac{p\frac{\partial}{\partial y}\Big|_{(p,q)} - (p-h)\frac{\partial}{\partial y}\Big|_{(p,q)}}{h}$$

$$= \lim_{h \to 0} \frac{h\frac{\partial}{\partial y}\Big|_{(p,q)}}{h}$$

$$= \frac{\partial}{\partial y}\Big|_{(p,q)}.$$

That is, $L_{\frac{\partial}{\partial x}}\left(x\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}$.

Note that $\left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right] = \frac{\partial x}{\partial x}\frac{\partial}{\partial y} - x\frac{\partial 1}{\partial y}\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$,

so we see in this example that

$$L_{\frac{\partial}{\partial x}}\left(x\frac{\partial}{\partial y}\right) = \left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right]$$

and the following proposition shows this relationship is true in general, thus providing us with another description of the Lie bracket.

Proposition:

If $X, Y \in \Gamma(TM)$, then $L_X Y = [X,Y]$.

proof:

Let $f \in C^\infty(M)$. Then $f \circ \theta_h^X \in C^\infty(M)$ as well, for any h. If $h=0$, then $(f \circ \theta_h^X - f)(h,p) = 0$. Thus, by the previous lemma, there are $g_h = g(h,-) \in C^\infty(M)$ such that

(i) $\left(f \circ \theta_h^X - f\right)(h,p) = h g_h(p)$, and hence

$$h g_h(p) + f(p) = f \circ \theta_h^X(p) \quad ; \text{ and}$$

(ii) $g_0(p) = \frac{\partial}{\partial h}\left(f \circ \theta_h^X - f\right)(0,p) = L_X f(p)$,

by the lemma on page 123.

Using the proposition on page 127, we know that $\left(L_{(\theta_h^X)_* Y} f\right) \circ \theta_h^X = L_Y\left(f \circ \theta_h^X\right)$, and therefore, for any $p \in M$ we have

$$\begin{aligned} L_{(\theta_h^X)_* Y} f(p) &= \left(L_{(\theta_h^X)_* Y} f\right) \circ \theta_h^X \circ \theta_{-h}^X(p) \\ &= L_Y\left(f \circ \theta_h^X\right) \circ \theta_{-h}^X(p) \\ &= L_{Y(\theta_{-h}^X(p))}\left(f \circ \theta_h^X\right) \\ &= L_{Y(\theta_{-h}^X(p))}\left(h g_h + f\right) \end{aligned}$$

Therefore,

$$\left(L_{L_X Y} f\right)(p) = L_{L_X Y(p)} f$$

$$= L_{\lim_{h\to 0} \frac{1}{h}\left[Y(p) - ((\theta_h^X)_* Y)(p)\right]} f$$

$$= \lim_{h\to 0} \frac{1}{h}\left(L_{[Y(p) - ((\theta_h^X)_* Y)(p)]} f\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left[L_Y f(p) - L_{(\theta_h^X)_* Y} f\ (p)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[L_Y f(p) - L_{Y(\theta_{-h}^X(p))}(h g_h + f)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[L_Y f(p) - L_{Y(\theta_{-h}^X(p))} f\right] - \lim_{h\to 0} \frac{1}{h}\left[L_{Y(\theta_{-h}^X(p))} h g_h\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[L_Y f(p) - L_Y f(\theta_{-h}^X(p))\right] - \lim_{h\to 0} \frac{1}{h}\left[L_Y h g_h (\theta_{-h}^X(p))\right]$$

$$= L_X(L_Y f)(p) - L_Y g_0(p)$$

$$= (L_X \circ L_Y)(f)(p) - (L_Y \circ L_X)(f)(p)$$

$$= L_{[X,Y]} f(p)$$

We conclude that $L_X Y = [X,Y]$. $\boxed{\text{END}}$

Exercise:

① Let M be a smooth manifold with $X, Y \in \Gamma(TM)$ and $f \in \mathrm{Diff}(M)$. Use the proposition on page 127 to prove that if $g \in C^\infty(M)$, then $\left(L_{[f_*X, f_*Y]}\, g\right) \circ f = \left(L_{f_*[X,Y]}\, g\right) \circ f$ and thus that

$$f_*[X,Y] = [f_*X, f_*Y]$$

Notice that in your proof of ①, and in the proof of the proposition on 127, f needn't be a diffeomorphism. Indeed, what is required of f is that it be a smooth map $f: M \to N$ that is surjective, and such that if $p, q \in M$ with $f(p) = f(q)$, then $D_pf(X(p)) = D_qf(X(q))$. If these conditions are met, then we can define for $X \in \Gamma(TM)$, the push-forward vector field $f_*X \in \Gamma(TN)$ by $f_*X(f(p)) = D_pf(X(p))$ for all $p \in M$. Then the proofs above still apply to show $f_*[X,Y] = [f_*X, f_*Y]$ for $X, Y \in \Gamma(TM)$.

§ Foliations

Let $n > k$, and suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a function of the form

$$F : \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$$

$$F(x,y) = \big(f(x,y), g(y)\big)$$

where $f : \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$.

Note that F maps k-dimensional planes of the form $\mathbb{R}^k \times \{y\}$ to k-dimensional planes of the form $\mathbb{R}^k \times \{g(y)\}$, and if F is invertible then $F(\mathbb{R}^k \times \{y\}) = \mathbb{R}^k \times \{g(y)\}$.

Note that if $j \leq k$ and $i \geq k+1$, then

$\dfrac{\partial \pi_i \circ F}{\partial x_j} = \dfrac{\partial \pi_i \circ g}{\partial x_j} = 0$ so that if $p \in \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$

then
$$D_p F = \left(\begin{array}{c|c} * & * \\ \hline 0_{n-k,k} & * \end{array}\right)$$

Conversely if $D_p F$ is of the above form for some $F : \mathbb{R}^n \to \mathbb{R}^n$, then $F(x,y) = \big(f(x,y), g(y)\big)$ as above.

Example:

If $F = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{R})$ then F maps $\mathbb{R} \times \{y\}$ onto $\mathbb{R} \times \{dy\}$.

Definition:

Let M be a smooth n-manifold. A k-dimensional foliation on M is a collection of charts $\{(U, \varphi)\}$ covering M such that all transition functions are smooth and of the form

$$\psi \circ \varphi^{-1} : \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$$

$$\psi \circ \varphi^{-1}(x, y) = (f(x, y), g(y))$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ and $g : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$. Or, alternatively, if for all $p \in \mathbb{R}^n$

$$D_p(\psi \circ \varphi^{-1}) = \left(\begin{array}{c|c} * & * \\ \hline 0_{n-k,k} & * \end{array} \right).$$

Definition:

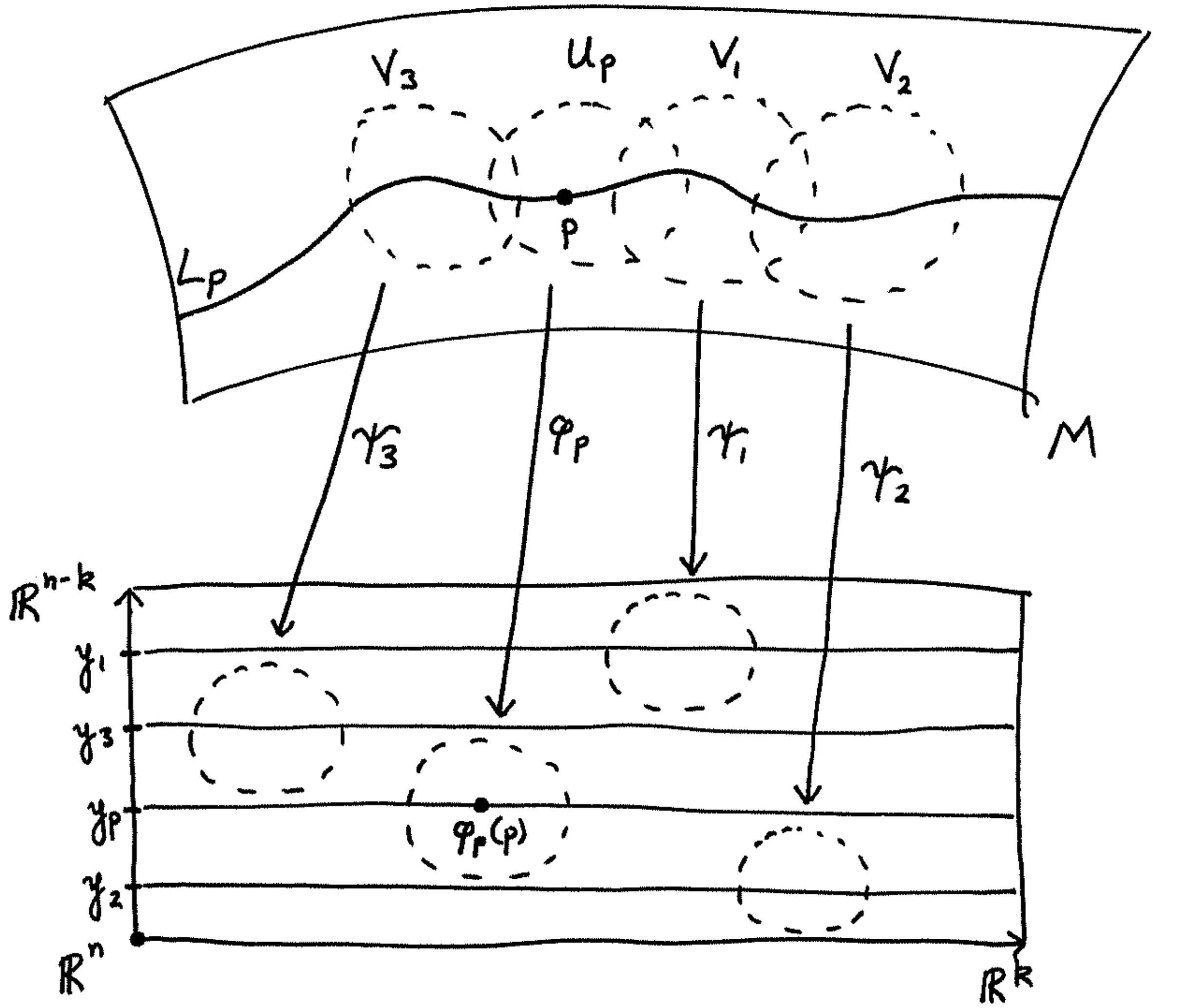
Suppose M is a smooth manifold with a k-dimensional foliation. For $p \in M$, choose a chart (U_p, φ_p) containing p, and suppose $y_p \in \mathbb{R}^{n-k}$ is such that $\varphi_p(p) \in \mathbb{R}^k \times \{y_p\}$.

If (V_1, ψ_1) is a chart with $U_p \cap V_1 \neq \emptyset$, then $\psi_1 \circ \varphi_p^{-1}(\mathbb{R}^k \times \{y_p\}) \subseteq \mathbb{R}^k \times \{y_1\}$ for some $y_1 \in \mathbb{R}^{n-k}$. Continue this process of extending $\varphi_p^{-1}(\mathbb{R}^k \times \{y_p\})$ by $\psi_i^{-1}(\mathbb{R}^k \times \{y_i\})$ and let

$$L_p = \varphi_p^{-1}(\mathbb{R}^k \times \{y_p\}) \cup \bigcup_i \psi_i^{-1}(\mathbb{R}^k \times \{y_i\})$$

L_p is the <u>leaf</u> of the foliation through p. It is a connected, immersed, k-dimensional submanifold of M.

If $p, q \in M$, then either $L_p = L_q$ or $L_p \cap L_q = \emptyset$. Also, $M = \bigcup_p L_p$.

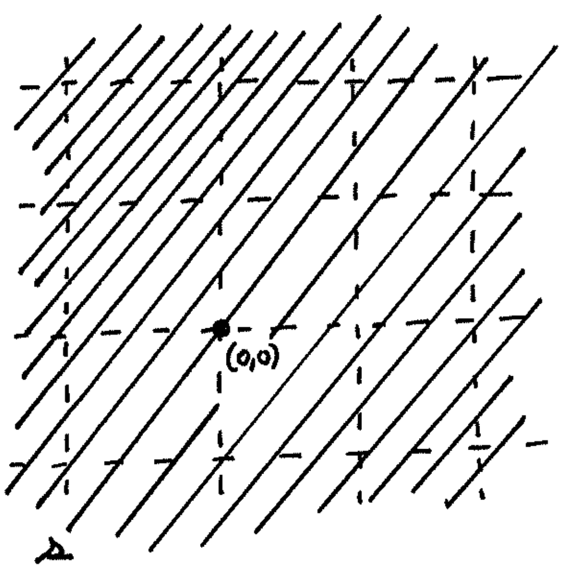


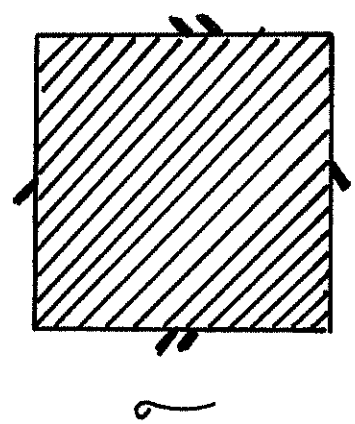
It is customary in describing foliations to describe the resulting leaves of the foliation rather than the system of charts. Several examples below follow this custom.

<u>Examples</u>

① $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ admits multiple foliations. If $\Delta \in \mathbb{P}^1(\mathbb{R})$, then choose a basis e_1, e_2 for $\mathbb{R}^2$ such that $e_1 \in \Delta$.

The transition functions for $\mathbb{R}^2/\mathbb{Z}^2$ are of the form $(x,y) \longmapsto (x+n, y+m)$ for $n,m \in \mathbb{Z}$ so the differential of any transition function is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with respect to any basis, and in particular with respect to the basis e_1, e_2. What this means for us in this example, is that transition functions for T^2 map lines parallel to Δ in $\mathbb{R}^2$ to themselves. Thus, $\Delta \in \mathbb{P}^1(\mathbb{R})$, and lines parallel to Δ in $\mathbb{R}^2$, determine a foliation on T^2 that we denote here as $\widetilde{\mathcal{F}_\Delta}$.

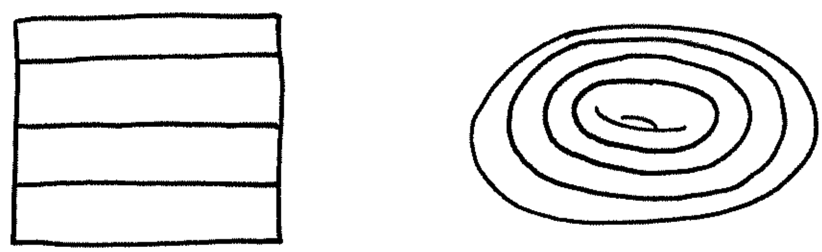




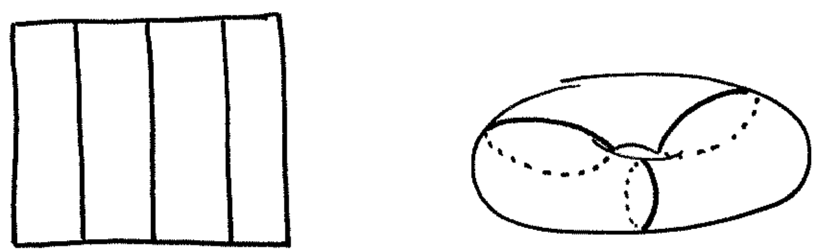
$\widetilde{\mathcal{F}_\Delta}$

If $s \in \mathbb{P}^1(\mathbb{Q})$, then each leaf of $\tilde{\mathcal{F}}_s$ is an embedded S^1. If $s \notin \mathbb{P}^1(\mathbb{Q})$, then each leaf of $\tilde{\mathcal{F}}_s$ is an immersed $\mathbb{R}$ with dense image in T^2.

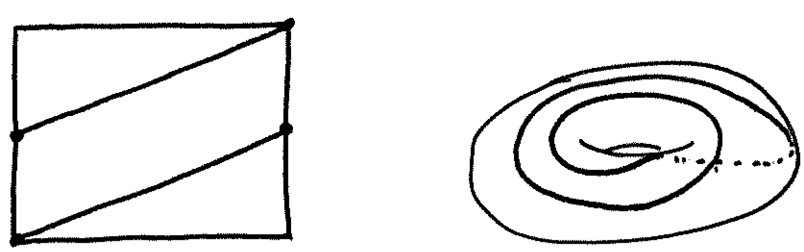
$\tilde{\mathcal{F}}_0$



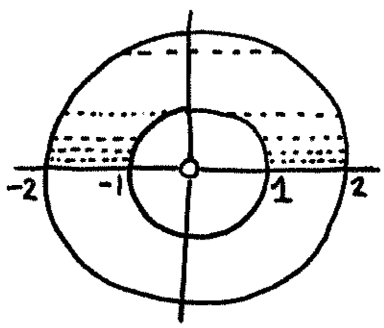
$\tilde{\mathcal{F}}_\infty$



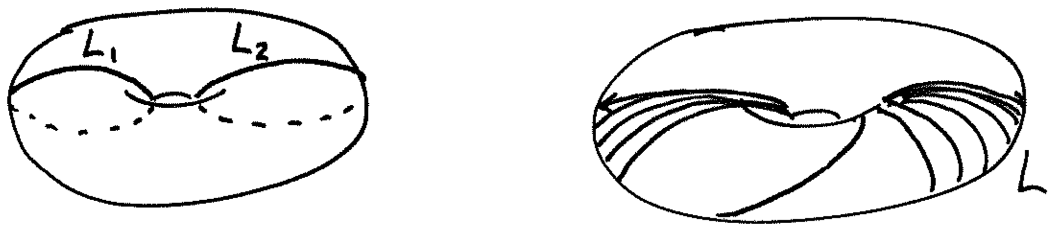
$\tilde{\mathcal{F}}_{1/2}$



② $T^2 = \langle\varphi\rangle \backslash \mathbb{R}^2 - \{0\}$ where $\varphi : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}$ is $\varphi(x,y) = (2x, 2y)$, and transition functions for this chart system on T^2 are powers of φ. Thus, transition functions map horizontal lines to horizontal lines in $\mathbb{R}^2 - \{0\}$, and horizontal lines map to leaves of the foliation under $\mathbb{R}^2 - \{0\} \longrightarrow T^2$.



The intervals $[-2,-1] \times \{0\}$ and $[1,2] \times \{0\}$ each map to S^1 leaves in T^2, which we denote as L_1 and L_2. Any other leaf $L \subseteq T^2$ is an immersed $\mathbb{R}$ with $\overline{L} = L \cup L_1 \cup L_2$.



③ Let $\varphi : \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R}^3 - \{0\}$ be $\varphi(x,y,z) = (2x, 2y, 2z)$.

Then for fixed z_0, 2-planes of the form $z = z_0$ map to leaves of a foliated $\langle\varphi\rangle \backslash \mathbb{R}^3 - \{0\} = S^2 \times S^1$. The leaf for $z_0 = 0$ is a Σ_1. All other leaves are immersed $\mathbb{R}^2$'s.

$S^2 \times S^1$ has a top half where $z \geq 0$, and a bottom half where $z \leq 0$. Each half is diffeomorphic to $D^2 \times S^1$ with $\partial(D^2 \times S^1)$ equal to the Σ_1 leaf. We could write this as $S^2 \times S^1 = D^2 \times S^1 \cup_{\Sigma_1} D^2 \times S^1$.

This foliation on $S^2 \times S^1$ is called the <u>Reeb foliation</u>.

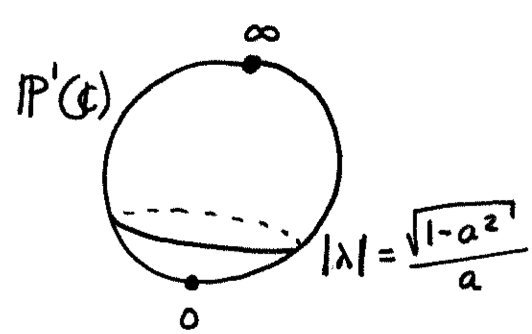
④ Let $S^3 = \{ (z, w) \in \mathbb{C}^2 \,/\, |z|^2 + |w|^2 = 1 \}$

Let $\pi : S^3 \rightarrow \mathbb{P}^1(\mathbb{C})$ be the function that maps a point to the complex line through 0 that contains it in $\mathbb{C}^2$. That is, $\pi(z, w) = w/z$.

For any $\lambda \in \mathbb{P}^1(\mathbb{C})$, $\pi^{-1}(\lambda) \subseteq S^3 \subseteq \mathbb{C}^2$ is the circle of unit vectors contained in the complex line $\lambda \subseteq \mathbb{C}^2$. Call this circle S^1_λ. Since any $(z,w) \in S^3$ is contained in $S^1_{w/z}$, we see that $\bigcup_{\lambda \in \mathbb{P}^1(\mathbb{C})} S^1_\lambda = S^3$ is a 1-dimensional foliation on S^3, called the <u>Hopf foliation</u> on S^3.

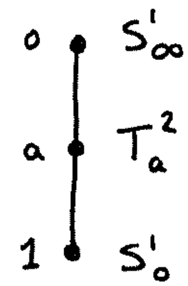
⑤ Continuing from the previous example, if $(z,w) \in S^3$ and $|z| = 0$, then $(z,w) \in S^1_\infty$. If $|z| = 1$, then $w = 0$ so $(z,w) \in S^1_0$.

If $|z| = a$ with $0 < a < 1$, then $|z|^2 + |w|^2 = 1$ implies $|w| = \sqrt{1 - |z|^2} = \sqrt{1 - a^2}$ so that $(z,w) \in S^1_\lambda$ with $|\lambda| = \frac{|w|}{|z|} = \frac{\sqrt{1-a^2}}{a}$.



For any $0<a<1$, the union of S^1_λ such that $|\lambda| = \frac{\sqrt{1-a^2}}{a}$ is a circle's worth of circles. It's diffeomorphic to a 2-torus, and we denote this set as T^2_a.

S^3 :



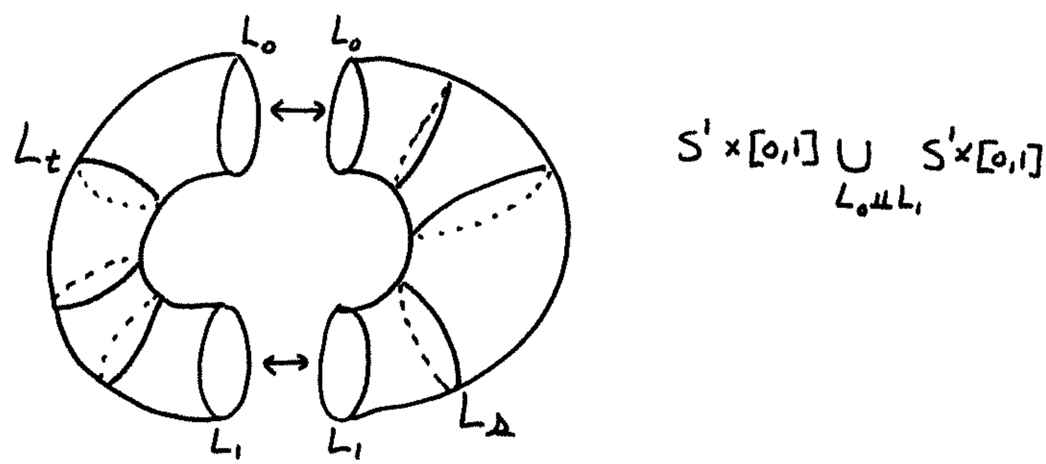
Let $\Sigma^1_1 = T^2_{\frac{1}{2}}$. Note that $S^1_\infty \cup \bigcup_{a\le 1/2} T^2_a$ is diffeomorphic to $D^2 \times S^1$. So is the bottom half of S^3, $S^1_0 \cup \bigcup_{a \ge 1/2} T^2_a$. And both halves of S^3 have boundary $T^2_{\frac{1}{2}} = \Sigma^1_1$. Thus, by endowing each half with the Reeb foliation, we have a 2-dimensional foliation on $S^3 = D^2 \times S^1 \cup_{\Sigma^1_1} D^2 \times S^1$.

⑥ The quotient of $M \times \mathbb{R}$ by the action of $(p,t) \longmapsto (f(p), t+1)$ where $f \in \mathrm{Diff}(M)$ has a 1-dimensional and $\dim(M)$-dimensional foliation.

⑦ $S^1 \times [0,1]$ is foliated by circles $S^1 \times \{t\}$ for $t \in [0,1]$. We denote $S^1 \times \{t\}$ by L_t.

Note that $\partial(S^1 \times [0,1]) = L_0 \amalg L_1$.

If we identify $L_0 \amalg L_1$ with itself using the identity map, then we can obtain a foliation on a torus.



If instead, we identify L_0 to L_0 using the identity map, and L_1 to L_1 using the antipodal map, then

$$S^1 \times [0,1] \underset{L_0 \amalg L_1}{\cup} S^1 \times [0,1]$$

produces a foliation of a Klein bottle.

⑧ Elaborating on example ①, any k-dimensional plane $P \subseteq \mathbb{R}^n$ produces a k-dimensional foliation of $\mathbb{R}^n$ by k-dimensional planes that are parallel to P.

This foliation induces a foliation on the quotient $\mathbb{R}^n / \mathbb{Z}^n = T^n$. It is a k-dimensional foliation on an n-dimensional manifold.

§ Frobenius' Theorem

Before coming to the main definition and theorem for this chapter, we'll begin with two lemmas that we'll need in the proof of the main theorem.

Lemma:

Suppose M is a smooth manifold, that $f \in \mathrm{Diff}(M)$ and $X \in \Gamma(TM)$. Then $f_* X = X$ if and only if $\theta_t^X \circ f = f \circ \theta_t^X$ for all relevant $t \in \mathbb{R}$.

proof:

To restate the lemma, $f_* X = X$ if and only if $\theta_t^X = \theta_t^{f_* X}$ by the proposition on page 128.

The sufficient condition is immediate. For the necessary, suppose $\theta_t^X = \theta_t^{f_* X}$. Then for any $p \in M$,

$$X(p) = D_0 \theta_t^X(1) = D_0 \theta_t^{f_* X}(1) = f_* X(p).$$ $\boxed{\text{END}}$

Lemma:

If M is a smooth manifold and $X, Y \in \Gamma(TM)$, then we have $[X,Y]=0$ if and only if $\theta_t^X \circ \theta_s^Y = \theta_s^Y \circ \theta_t^X$ for all relevant $s, t \in \mathbb{R}$.

proof:

If $\theta_t^X \circ \theta_s^Y = \theta_s^Y \circ \theta_t^X$, then by the previous lemma, $(\theta_t^X)_* Y = Y$. Thus,

$$[X,Y] = L_X Y = \lim_{h \to 0} \frac{Y - (\theta_h^X)_* Y}{h} = 0$$

For the converse, assume $[X,Y]=0$. Define $g: (-\varepsilon, \varepsilon) \longrightarrow T_pM$ by $g(t) = (\theta_t^X)_* Y(p)$ for $p \in M$. Then

$$D_t g(1) = \lim_{h \to 0} \frac{1}{h}\left[g(t+h) - g(t)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[(\theta_{t+h}^X)_* Y(p) - (\theta_t^X)_* Y(p)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[(\theta_t^X)_* (\theta_h^X)_* Y(p) - (\theta_t^X)_* Y(p)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[D_{\theta^X_{-t}(p)}\theta^X_t\left[(\theta^X_h)_* Y(\theta^X_{-t}(p)) - Y(\theta^X_{-t}(p))\right]\right]$$

$$= D_{\theta^X_{-t}(p)}\theta^X_t\left[\lim_{h\to 0}\frac{1}{h}\left[(\theta^X_h)_* Y(\theta^X_{-t}(p)) - Y(\theta^X_{-t}(p))\right]\right]$$

$$= D_{\theta^X_{-t}(p)}\theta^X_t\left[L_X(-Y)(\theta^X_{-t}(p))\right]$$

$$= D_{\theta^X_{-t}(p)}\theta^X_t\Big([X,-Y](\theta^X_{-t}(p))\Big)$$

$$= D_{\theta^X_{-t}(p)}\theta^X_t(0)$$

$$= 0$$

Therefore, $g(t)=g(0)$, so

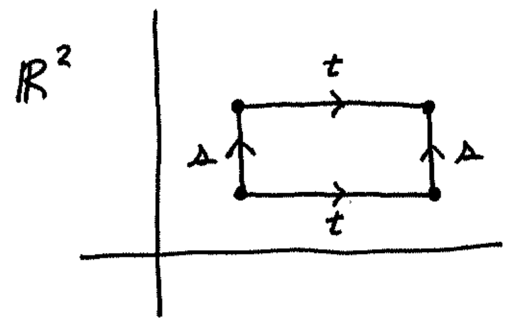
$$(\theta^X_t)_* Y(p) = (\theta^X_0)_* Y(p) = Y(p)$$

for $p\in M$, which is to say that $(\theta^X_t)_* Y = Y$. By the previous lemma, $\theta^Y_t \circ \theta^X_s = \theta^X_s \circ \theta^Y_t$.

$\boxed{\text{END}}$

Example:

For $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in \Gamma(T\mathbb{R}^2)$, $\theta_t^{\frac{\partial}{\partial x}}(x,y) = (x+t, y)$ and $\theta_s^{\frac{\partial}{\partial y}}(x,y) = (x, y+s)$. Thus, the two flows commute, and $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0$.



Definition:

If M is a smooth manifold, a <u>smooth k-plane field</u> on M is a map $p \mapsto \Delta_p \leq T_pM$ such that

(i) $\Delta_p \leq T_pM$ is a k-dimensional vector subspace.

(ii) If $p \in M$, then there is a neighborhood of p, $U \subseteq M$, and $X_1, \ldots, X_k \in \Gamma(TM)$ such that for any $q \in U$,

$$\operatorname{span}\{X_1(q), \ldots, X_k(q)\} = \Delta_q$$

Examples:

① Let $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \in \Gamma(T\mathbb{R}^3)$. For any $p \in \mathbb{R}^3$, $\left.\frac{\partial}{\partial x}\right|_p$ and $\left.\frac{\partial}{\partial y}\right|_p + x\left.\frac{\partial}{\partial z}\right|_p$ are linearly independent. Thus,

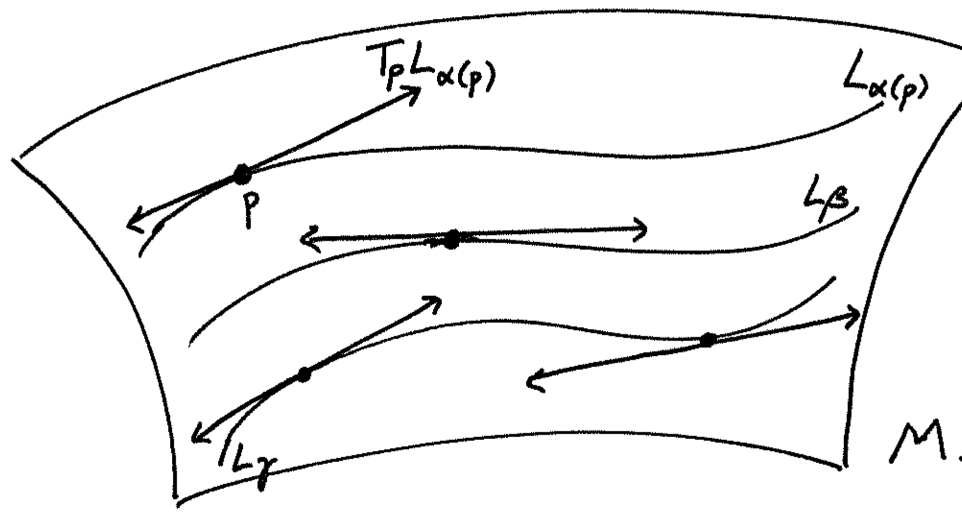
$$\Delta_p = \operatorname{span}\left\{ \left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p + x\left.\frac{\partial}{\partial z}\right|_p \right\}$$

is a smooth 2-plane field on $\mathbb{R}^3$.

② If M is a smooth manifold and $V \leq T_{p_0}M$ is a k-dimensional vector subspace for some p_0, and if $G \leq \operatorname{Diff}(M)$ acts simply-transitively on M, then $\Delta_{gp_0} = D_{p_0}g(V)$ for $g \in G$ is a smooth k-plane field on M.

③ Suppose a smooth manifold M has a k-dimensional foliation with leaves $L_\alpha \subseteq M$ for $\alpha \in A$.

If $p \in M$, let $\alpha(p) \in A$ be such that $p \in L_{\alpha(p)}$. Then $\Delta_p = T_p L_{\alpha(p)} \leq T_p M$ defines a k-dimensional plane field on M.



<u>Definition</u>:

Any plane field arising from a foliation as in the previous example is called <u>integrable</u>.

Frobenius' Theorem:

Suppose M is a smooth manifold with a smooth k-plane field $p \longmapsto \Delta_p$.

The plane field Δ_p is integrable if and only if for any $X, Y \in \Gamma(TM)$ with $X(p), Y(p) \in \Delta_p$ for all $p \in M$, we have $[X,Y](p) \in \Delta_p$ for all $p \in M$.

Example:

The plane field on $\mathbb{R}^3$ given by

$$\Delta_p = \operatorname{span}\left\{ \frac{\partial}{\partial x}\bigg|_p, \frac{\partial}{\partial y}\bigg|_p + x\frac{\partial}{\partial z}\bigg|_p \right\}$$

is not integrable. Indeed $\frac{\partial}{\partial x}\big|_p \in \Delta_p$ and $\frac{\partial}{\partial y}\big|_p + x\frac{\partial}{\partial z}\big|_p \in \Delta_p$ for all $p \in \mathbb{R}^3$, but $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right](p) = \frac{\partial}{\partial z}\big|_p \notin \Delta_p$, so we can apply Frobenius' Theorem.

<u>proof of theorem:</u>

Suppose Δ_p is integrable. That is, suppose M admits a foliation with resulting leaves N_α, $\alpha \in A$, such that if $p \in M$ and $\alpha(p) \in A$ with $p \in N_{\alpha(p)}$, then $T_p N_{\alpha(p)} = \Delta_p$.

Let $X, Y \in \Gamma(TM)$ be such that $X(p), Y(p) \in \Delta_p$ for all $p \in M$. We want to show that $[X,Y](p) \in \Delta_p$ for all $p \in M$.

Since $X(p), Y(p) \in \Delta_p = T_p N_{\alpha(p)}$, we have that $X|_{N_{\alpha(p)}}, Y|_{N_{\alpha(p)}} \in \Gamma(TN_{\alpha(p)})$. Therefore $[X|_{N_{\alpha(p)}}, Y|_{N_{\alpha(p)}}] \in \Gamma(TN_{\alpha(p)})$ so that

$$[X|_{N_{\alpha(p)}}, Y|_{N_{\alpha(p)}}](p) \in T_p N_{\alpha(p)} = \Delta_p.$$

Our claim, from which we conclude that $[X,Y](p) \in \Delta_p$, is that $[X,Y](p)$ equals

$$[X|_{N_{\alpha(p)}}, Y|_{N_{\alpha(p)}}](p).$$

To prove our claim, note that $\theta^X_h\big|_{N_{\alpha(p)}}$ satisfies the defining differential equation for $\theta_h^{X|_{N_{\alpha(p)}}}$, and thus $\theta^X_h\big|_{N_{\alpha(p)}} = \theta_h^{X|_{N_{\alpha(p)}}}$. Therefore,

$$\begin{aligned}
[X,Y](p) &= L_X Y(p) \\
&= \lim_{h\to 0} \frac{Y(p) - \left(\left(\theta^X_h\right)_* Y\right)(p)}{h} \\
&= \lim_{h\to 0} \frac{Y|_{N_{\alpha(p)}}(p) - \left(\left(\theta_h^{X|_{N_{\alpha(p)}}}\right)_* Y|_{N_{\alpha(p)}}\right)(p)}{h} \\
&= \left(L_{X|_{N_{\alpha(p)}}} Y|_{N_{\alpha(p)}}\right)(p) \\
&= \left[X|_{N_{\alpha(p)}}, Y|_{N_{\alpha(p)}}\right](p)
\end{aligned}$$

as was claimed. Thus, for any $X, Y \in \Gamma(TM)$ with $X(p), Y(p) \in \Delta_p$ for all $p \in M$, we have $[X,Y](p) \in \Delta_p$ for all $p \in M$. We have proved the sufficient implication of the theorem.

Now we'll prove the converse. Assume that if $X, Y \in \Gamma(TM)$ with $X(p), Y(p) \in \Delta_p$ for all $p \in M$ then $[X,Y](p) \in \Delta_p$ for all $p \in M$.

Let $n = \dim(M)$ and recall that $k = \dim(\Delta_p)$ for all $p \in M$. To show that the plane field is integrable, for any $z \in M$ we'll define a chart φ_z from a neighborhood of z to $\mathbb{R}^n$ such that if $w \in M$ then $D_a(\varphi_w^{-1} \circ \varphi_z) = \left(\begin{array}{c|c} * & * \\ \hline 0_{n-k,k} & * \end{array}\right)$ for all relevant $a \in \mathbb{R}^n$. Furthermore, we'll show that the resulting leaves of the k-dimensional foliation are tangent to Δ_p for all $p \in M$.

For $z \in M$, we fix local coordinates in M near z such that z corresponds to $0 \in \mathbb{R}^n$. Furthermore, choose the local coordinate system such that

$$\Delta_0 = \mathbb{R}^k \leq \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n = T_0\mathbb{R}^n.$$

We let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be projection so that for $p \approx 0$, $(D_p\pi)\big|_{\Delta_p} : \Delta_p \longrightarrow \mathbb{R}^k$ is bijective.

For $1 \leq i \leq k$ and $p \approx 0$, let

$X_i(p) = \left(\left(D_p\pi\right)\big|_{\Delta_p}\right)^{-1}\left(\frac{\partial}{\partial x_i}\Big|_{\pi(p)}\right)$ so that

$\Delta_p = \mathrm{span}\{X_1(p), \ldots, X_k(p)\}$ and $X_i(0) = \frac{\partial}{\partial x_i}\Big|_0$.

By hypothesis, $[X_i, X_j](p) \in \Delta_p$ for all i, j, and $p \approx 0$.

By the exercise on page 135,

$$\begin{aligned}\pi_*[X_i, X_j](\pi(p)) &= [\pi_* X_i, \pi_* X_j](\pi(p)) \\ &= \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right](\pi(p)) \\ &= 0.\end{aligned}$$

That is, $D_p\pi([X_i, X_j](p)) = 0$, so that

$[X_i, X_j](p) \in \operatorname{Ker}(D_p\pi)$. But $(D_p\pi)\big|_{\Delta_p}$ is injective, and $[X_i,X_j](p) \in \Delta_p$ so $[X_i,X_j](p)=0$, for any i,j and $p \approx 0$. That is, $[X_i,X_j]=0$, so we have by the lemma on page 149 that $\theta_s^{X_i} \circ \theta_t^{X_j} = \theta_t^{X_j} \circ \theta_s^{X_i}$ for any $s,t \in \mathbb{R}$ near 0.

Define $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$f(a_1,\dots,a_n) = \theta_{a_1}^{X_1} \circ \theta_{a_2}^{X_2} \circ \cdots \circ \theta_{a_k}^{X_k}(0,\dots,0,a_{k+1},\dots,a_n).$$

Note $f(0)=0$.

We claim for $1 \le i \le n$ that $D_0 f\left(\frac{\partial}{\partial x_i}\Big|_0\right) = \frac{\partial}{\partial x_i}\Big|_0$. Indeed, for $g \in C^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
L_{D_0 f\left(\frac{\partial}{\partial x_i}\big|_0\right)} g &= D_0 g\left(D_0 f\left(\frac{\partial}{\partial x_i}\Big|_0\right)\right) \\
&= D_0(g \circ f)\left(\frac{\partial}{\partial x_i}\Big|_0\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left[g\circ f(0,\dots,0,h,0,\dots,0) - g\circ f(0)\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[g\circ f(0,\dots,0,h,0,\dots,0) - g(0)\right]
\end{aligned}$$

If $i > k$, then

$$L_{D_0 f\left(\frac{\partial}{\partial x_i}\big|_0\right)} g = \lim_{h \to 0} \frac{1}{h}\left[g \circ f(0,\dots,0,h,0,\dots,0) - g(0)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[g(0,\dots,0,h,0,\dots,0) - g(0)\right]$$

$$= L_{\frac{\partial}{\partial x_i}\big|_0} g$$

If $i \leq k$, then

$$L_{D_0 f\left(\frac{\partial}{\partial x_i}\big|_0\right)} g = \lim_{h \to 0} \frac{1}{h}\left[g \circ f(0,\dots,0,h,0,\dots,0) - g(0)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[g \circ \theta_h^{X_i}(0) - g(0)\right]$$

$$= (L_{X_i} g)(0)$$

$$= L_{X_i(0)} g$$

$$= L_{\frac{\partial}{\partial x_i}\big|_0} g$$

Either way, $L_{D_0 f\left(\frac{\partial}{\partial x_i}\big|_0\right)} g = L_{\frac{\partial}{\partial x_i}\big|_0} g$ for all $g \in C^\infty(\mathbb{R}^n)$, so $D_0 f\left(\frac{\partial}{\partial x_i}\big|_0\right) = \frac{\partial}{\partial x_i}\big|_0$ as claimed.

The proved claim shows that $D_0 f$ maps a basis to a basis, and thus that $D_0 f$ is a linear isomorphism. By the Inverse Function Theorem, f is a diffeomorphism near 0.

We claim that for $a \in \mathbb{R}^n$ with $a \approx 0$, and for all $1 \leq i \leq k$,

$$D_a f\left(\frac{\partial}{\partial x_i}\Big|_a\right) = X_i(f(a)).$$

Indeed, let $a = (a_1, \ldots, a_n)$, and $g \in C^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned} L_{D_a f\left(\frac{\partial}{\partial x_i}\big|_a\right)} g &= D_b g\left(D_a f\left(\frac{\partial}{\partial x_i}\Big|_a\right)\right) \\ &= D_a(g \circ f)\left(\frac{\partial}{\partial x_i}\Big|_a\right) \end{aligned}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[g\circ f(a_1,\dots,a_{i-1},a_i+h,a_{i+1},\dots,a_n) - g\circ f(a)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[g\circ \theta^{X_1}_{a_1}\circ\cdots\circ\theta^{X_i}_{a_i+h}\circ\cdots\circ\theta^{X_k}_{a_k}(0,\dots,0,a_{k+1},\dots,a_n) - g\circ f(a)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[g\circ \theta^{X_i}_{h}\circ\theta^{X_1}_{a_1}\circ\cdots\circ\theta^{X_i}_{a_i}\circ\cdots\circ\theta^{X_k}_{a_k}(0,\dots,0,a_{k+1},\dots,a_n) - g\circ f(a)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[g\circ \theta^{X_i}_{h}\circ f(a) - g\circ f(a)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[g\circ \theta^{X_i}_{h}(f(a)) - g(f(a))\right].$$

$$= \left(L_{X_i}g\right)(f(a))$$

$$= L_{X_i(f(a))}g.$$

Since $L_{D_af\left(\frac{\partial}{\partial x_i}\big|_a\right)}g = L_{X_i(f(a))}g$ for all $g \in C^\infty(\mathbb{R}^n)$, we see that $D_af\left(\frac{\partial}{\partial x_i}\big|_a\right) = X_i(f(a))$ for all $a \approx 0$ and $1 \le i \le k$ as claimed.

It follows, that for $a \approx 0$

$$D_af\left(\mathbb{R}^k\times\{0\}\right) = \Delta_{f(a)}.$$

Now let $\varphi_z = f^{-1}$ be our chart on M about $z = 0 = f(0)$.

Note that if $p \in M$ is near z, then the last equation from the previous page states $D_p \varphi_z(\Delta_p) = \mathbb{R}^k \times \{0\} \leq T_{\varphi_z(p)} \mathbb{R}^n$. Hence, if $w \in M$ and φ_w is the similarly constructed chart about w, then

$$\begin{aligned} D_{\varphi_z(p)}(\varphi_w \circ \varphi_z^{-1})(\mathbb{R}^k \times \{0\}) &= D_p \varphi_w \left(D_{\varphi_z(p)} \varphi_z^{-1} \left(\mathbb{R}^k \times \{0\} \right) \right) \\ &= D_p \varphi_w (\Delta_p) \\ &= \mathbb{R}^k \times \{0\}. \end{aligned}$$

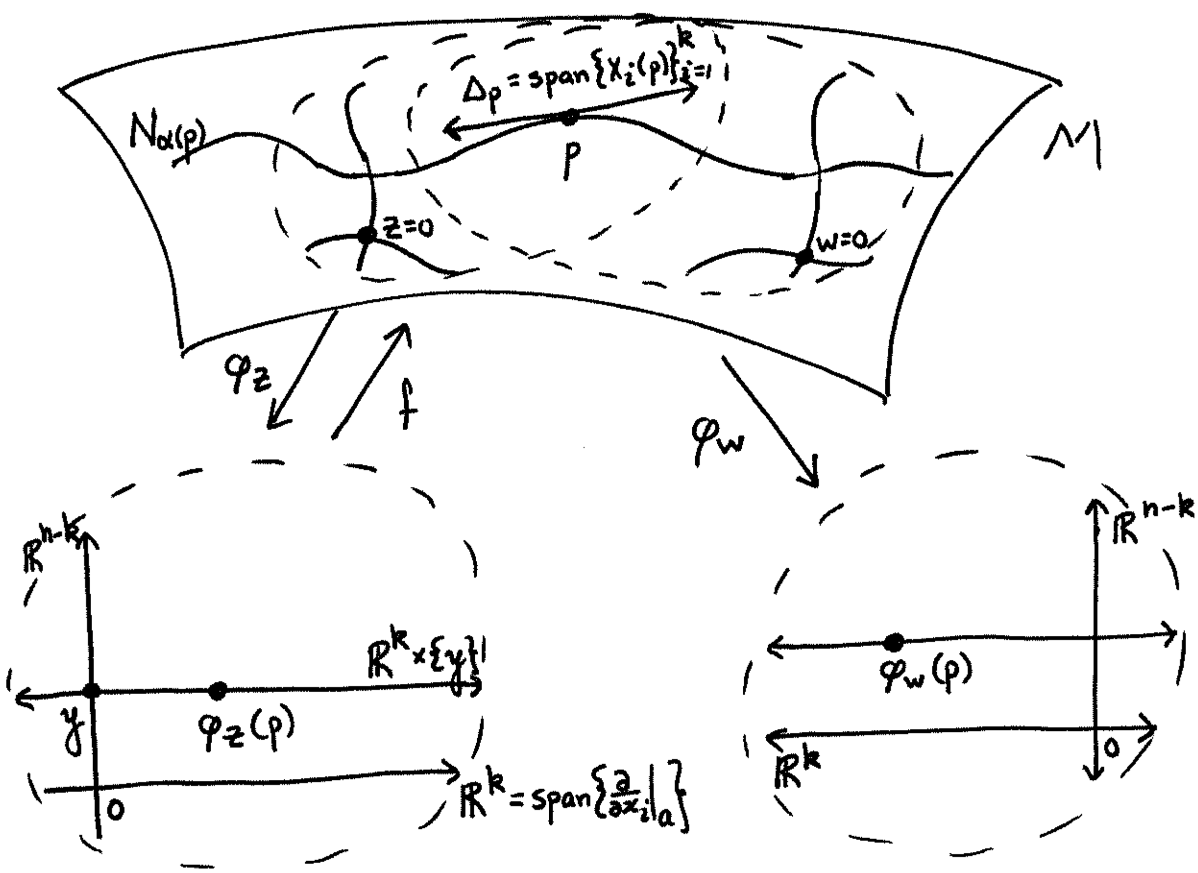
for all relevant p. That is,

$$D_{\varphi_z(p)}(\varphi_w \circ \varphi_z^{-1}) = \left(\begin{array}{c|c} * & * \\ \hline 0_{n-k,k} & * \end{array} \right)$$

so that the charts φ_z define a k-dimensional foliation on M. Also note, that if $y \in \mathbb{R}^{n-k}$ is such that $p \in \varphi_z^{-1}(\mathbb{R}^k \times \{y\})$, then $\varphi_z^{-1}(\mathbb{R}^k \times \{y\})$ is an open neighborhood of the leaf $N_{\alpha(p)}$ containing p. Hence,

$$\begin{aligned}\Delta_p &= \left(D_{\varphi(p)} \varphi_z^{-1}\right)\left(\mathbb{R}^k \times \{0\}\right)\\ &= T_p\left(\varphi_z^{-1}\left(\mathbb{R}^k \times \{y\}\right)\right)\\ &= T_p N_{\alpha(p)}.\end{aligned}$$

$\boxed{\text{END}}$



Exercises:

① We define a 2-plane field on $\mathbb{R}^3$ by

$$\Delta_p = \operatorname{span}\left\{ 2\frac{\partial}{\partial x}\Big|_p + y\frac{\partial}{\partial y}\Big|_p ,\ \frac{\partial}{\partial y}\Big|_p + y\frac{\partial}{\partial z}\Big|_p \right\}$$

for $p \in \mathbb{R}^3$. Is Δ_p integrable?

② For $p \in \mathbb{R}^2 - \{0\}$, let $\Delta_p \leq T_p(\mathbb{R}^2 - \{0\})$ be the line containing O, and orthogonal to the vector p.

For $(x,y) \in \mathbb{R}^2 - \{0\}$, let $\Delta_{(x,y)} \leq T_{(x,y)}(\mathbb{R}^2 - \{0\})$ be the line containing 0 with slop $-y/x \in \mathbb{R} \cup \{\infty\}$.

Are Δ_p and $\Delta_{(x,y)}$ integrable? If so, sketch the induced foliation on $\mathbb{R}^2 - \{0\}$.

§ Lie Groups

A Lie group is a smooth manifold G with a group operation such that

$$\begin{array}{ccc} G\times G \longrightarrow G & & G \longrightarrow G \\ (g,h) \longmapsto gh & \text{and} & g \longmapsto g^{-1} \end{array}$$

are smooth functions.

Examples

① $\mathbb{R}^n$

② $S^1 = \{ e^{i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R} \}$

③ If G and H are Lie groups, so is $G \times H$.

④ $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$ is a Lie group.

⑤ $GL_n(\mathbb{R})$

⑥ $SL_n(\mathbb{R})$

⑦ The Heisenberg group: $\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x,y,z \in \mathbb{R} \right\}$

⑧ Sol: $\left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x,y,t \in \mathbb{R} \right\}$

⑨ S^3: The quaternions are $\mathcal{H} = \left\{ x+iy+jz+kw \;\middle|\; \begin{matrix} x,y, \\ z,w \\ \text{are real} \end{matrix} \right\}$

so that $\mathcal{H}$ is diffeomorphic to $\mathbb{R}^4$.

$\mathcal{H}$ has a multiplication defined by

$i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.

We let S^3 be the unit sphere in $\mathcal{H}$ with this multiplication.

⑩ $S^0 \subseteq \mathbb{R}$, $S^1 \subseteq \mathbb{C}$, and $S^3 \subseteq \mathcal{H}$ are the only spheres that are Lie groups.

⑪ $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ are examples of complex Lie groups.

We will not prove the following important

Theorem:

If G is a Lie group, and if H is a closed subgroup of G (meaning that H is a subgroup and is a close subspace of G), then H is a Lie group.

(12) If $(x_i),(y_i) \in \mathbb{R}^n$, let $\langle (x_i),(y_i) \rangle = \sum_{i=1}^{n} x_i y_i$

Let $O(n) = \{ A \in GL_n(\mathbb{R}) \mid \langle Av, Aw \rangle = \langle v, w \rangle \text{ for all } v,w \in \mathbb{R}^n \}$.

Then $O(n)$ is a closed subgroup of $GL_n(\mathbb{R})$, so $O(n)$ is a Lie group. $O(n)$ is compact.

(13) $SO(n) = O(n) \cap SL_n(\mathbb{R})$.

(14) For $(x_i),(y_i) \in \mathbb{R}^{n+m}$, let $\Phi_{n,m}((x_i),(y_i)) = \sum_{i=1}^{n} x_i y_i - \sum_{i=n+1}^{n+m} x_i y_i$.

Let $O(n,m) = \{ A \in GL_{n+m}(\mathbb{R}) \mid \Phi_{n,m}(Av, Aw) = \Phi_{n,m}(v,w) \text{ for all } v,w \in \mathbb{R}^{n+m} \}$

$O(n,m)$ is a closed subgroup of $GL_{n+m}(\mathbb{R})$ so it's a Lie group. Note that $O(n,0)=O(n)$, but if $n,m>0$, then $O(n,m)$ is not compact.

(15) $SO(n,m) = O(n,m) \cap SL_{n+m}(\mathbb{R})$.

A Lie group H is <u>linear</u> if it is a subgroup of, and an immersed submanifold in, $GL_n(\mathbb{R})$ for some n.

The vast majority of Lie groups encountered are linear, including all examples above.

For instance, $\mathbb{R}^n$ is the group of $(n+1)\times(n+1)$ matrices of the form

$$\left(\begin{array}{c|c} \mathrm{Id}_{n,n} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \\ \hline 0 \cdots\cdots 0 & 1 \end{array}\right) \quad \text{where } x_i \in \mathbb{R}, \quad \text{and}$$

$$S^1 = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\} = SO(2).$$

Note that any real number is a sum of arbitrarily small numbers, and that any rotation in $SO(2)$ is a product of arbitrarily small rotations. These are two examples of the following

Proposition:

Let G be a connected Lie group. If $U \subseteq G$ is an open neighborhood of $1 \in G$, then

$$\bigcup_{n=1}^{\infty} U^n = G$$

where $U^n = \{ g \in G \mid g = u_1 u_2 \cdots u_n \text{ for } u_i \in U \}$.

proof:

Let $V = U \cap U^{-1}$ so that $V = V^{-1}$. Let $H = \bigcup_{n=1}^{\infty} V^n \subseteq \bigcup_{n=1}^{\infty} U^n$. We'll show that $H = G$.

First, to see that H is a group, if $\prod_{i=1}^{k} v_i\,,\ \prod_{i=1}^{m} v_i' \in H$, then $\left(\prod_{i=1}^{k} v_i\right)^{-1}\left(\prod_{i=1}^{m} v_i'\right) = \prod_{i=k}^{1} v_i^{-1} \prod_{i=1}^{m} v_i' \in H$.

Second, to see that H is open, note that
$$\begin{aligned} V &\longrightarrow hV \subseteq H \\ x &\longmapsto hx \end{aligned}$$
is a smooth map for all $h \in H$ and that $x \mapsto h^{-1}x$ is a smooth inverse. Thus, since V is open in G, so is hV. And since $h = h1 \in hV$, $H = \bigcup_{h\in H}\{h\} \subseteq \bigcup_{h\in H} hV \subseteq H$ so that $H = \bigcup_{h \in H} hV$ is open.

Last, to see that $H = G$, note that as above, since H is open, so too is gH for any $g \in G$. Hence, $\bigcup_{g \notin H} gH$ is open so $H = G - \left(\bigcup_{g\notin H} gH\right)$ is closed. Since H is open and closed and nonempty, and G is connected, $H = G$.

END

Exercises

Let G be a Lie group.

① For $g \in G$, let $L_g : G \longrightarrow G$ be $L_g(h) = gh$.
Prove that L_g is a diffeomorphism.

② For $g \in G$, let $C_g : G \longrightarrow G$ be $C_g(h) = ghg^{-1}$.
Prove that C_g is a diffeomorphism.

③ Let G^0 be the connected component of G that contains $1 \in G$. Prove that G^0 is a Lie group without using the Theorem on page 168. Prove that $G^0 \trianglelefteq G$.

Note: $R_g : G \longrightarrow G$ where $R_g(h) = hg$ is also a diffeomorphism.

§ Lie algebras of Lie groups

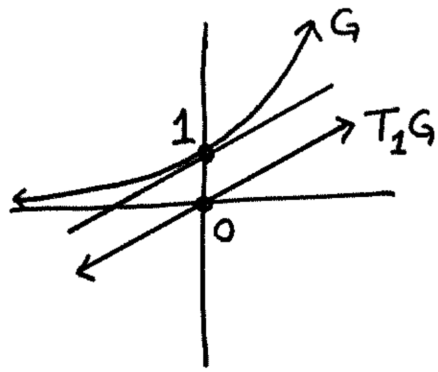
Let G be a Lie group, so that $1 \in G$ and $\dim(T_1 G) = \dim(G)$.

Examples:

① $T_0 \mathbb{R}^n = \mathbb{R}^n$

② $T_1 \mathbb{R}_{>0} = \mathbb{R}$

③ Let $G = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$. Since $\begin{pmatrix} e^0 & 0 \\ 0 & 1 \end{pmatrix} = 1 \in G$, and $\frac{d}{dt}\Big|_0 e^t = 1$ while $\frac{d}{dt}\Big|_0 1 = 0$, we have $T_1 G = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$.



④ If $H = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$, then

$T_1 H = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$ since $\left.\frac{d}{dt}\right|_0 e^{-t} = -1$.

⑤ Let $A = \left\{ \left(\begin{array}{ccc|c} 1 & & 0 & x_1 \\ & \ddots & & \vdots \\ & & & x_n \\ 0 & & & 1 \end{array} \right) \middle| \, x_i \in \mathbb{R} \right\}$ so that

$A \cong \mathbb{R}^n$. Let B be the matrices in A with $x_i = 0$ if $i > 1$, so that $B = \mathbb{R}$. Since $\left.\frac{d}{dx_1}\right|_0 x_1 = 1$, $\left.\frac{d}{dx_1}\right|_0 1 = 0$, and $\left.\frac{d}{dx_1}\right|_0 0 = 0$, we see that $T_1 B \leq T_1 A$ contains

$$\begin{pmatrix} & & \boxed{1} \\ & 0 & \\ & & \end{pmatrix}$$

By replacing x_1 with any x_i, we find a basis for $T_1 A$, so that

$$T_1 A = \left\{ \left(\begin{array}{c|c} & x_1 \\ 0 & \vdots \\ & x_n \\ \hline & 0 \end{array} \right) \middle| \, x_i \in \mathbb{R} \right\}.$$

⑥ Let $H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$. Then

$$T_1 H = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}.$$

⑦ Let $S = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, t \in \mathbb{R} \right\}$. Then

$$T_1 S = \left\{ \begin{pmatrix} t & 0 & x \\ 0 & -t & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \, x, y, t \in \mathbb{R} \right\}.$$

⑧ We denote $T_1 SL_n(\mathbb{R})$ as $\mathfrak{sl}_n \mathbb{R}$.

Note that $\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}$, $\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}$ are in $SL_3(\mathbb{R})$, so $\mathfrak{sl}_3 \mathbb{R}$ contains the following 8 linearly independent vectors: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Note that each of these 8 vectors has trace 0.

Furthermore, matrices with trace 0 are a vector subspace of $M_{3\times3}(\mathbb{R})$, a subspace of dimension 8 since $\text{trace}: M_{3\times3}(\mathbb{R}) \longrightarrow \mathbb{R}$ and $3\times3 - 1 = 8$.

Since $\dim(\mathfrak{sl}_3\mathbb{R}) = \dim(SL_3(\mathbb{R})) = 9-1 = 8$, we conclude that $\mathfrak{sl}_3\mathbb{R}$ is the space of matrices in $M_{3\times3}(\mathbb{R})$ whose trace is 0.

Similarly, $\mathfrak{sl}_n\mathbb{R} = \{A \in M_{n\times n}(\mathbb{R}) \mid \text{trace}A = 0\}$.

(9) We denote $T_1 GL_n(\mathbb{R})$ by $\mathfrak{gl}_n\mathbb{R}$.

Because $GL_n(\mathbb{R})$ is an open set in $M_{n\times n}(\mathbb{R})$ containing 1, $\mathfrak{gl}_n\mathbb{R} = M_{n\times n}(\mathbb{R})$.

(10) $\mathfrak{sl}_n\mathbb{C} = T_1 SL_n(\mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) \mid \text{trace}A = 0\}$, and $\mathfrak{gl}_n\mathbb{C} = T_1 GL_n(\mathbb{C}) = M_{n\times n}(\mathbb{C})$.

⑪ We let $\mathfrak{so}_2 = T_1 SO(2)$. Recall that

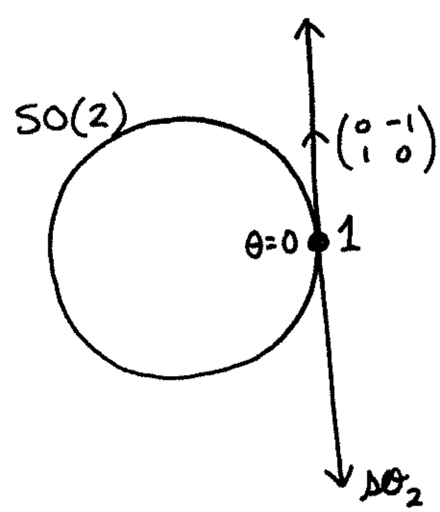
$$SO(2) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \middle| \, \theta \in \mathbb{R} \right\} \cong S^1.$$ Since

$$\begin{pmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix} = 1,$$ we note that

$\frac{d}{d\theta}\Big|_0 \cos(\theta) = 0$ and $\frac{d}{d\theta}\Big|_0 \sin(\theta) = 1$ to

conclude that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}_2$, and thus

that $\mathfrak{so}_2 = \left\{ \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$.



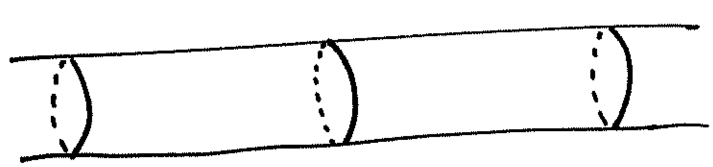
(12) Let $G = \mathbb{R} \times S^1$ be given as matrices of the form

$$\begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Where $x, \theta \in \mathbb{R}$. Then $T_1 G$ is the vector space of matrices of the form

$$\begin{pmatrix} 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z \\ 0 & 0 & z & 0 \end{pmatrix}$$

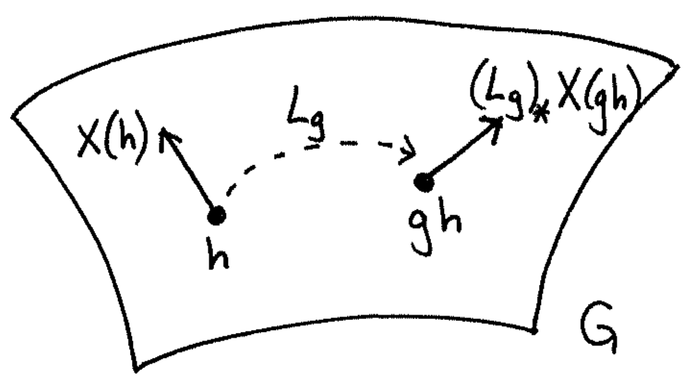
where $y, z \in \mathbb{R}$.



We say $X \in \Gamma(TG)$ is <u>left-invariant</u> if for all $g \in G$, $(L_g)_* X = X$, where, recall, $(L_g)_* X \in \Gamma(TG)$ is defined by

$$(L_g)_* X(gh) = D_h L_g(X(h))$$

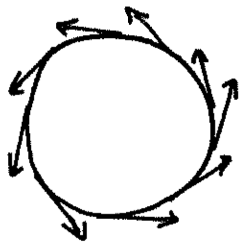
for all $h \in G$.



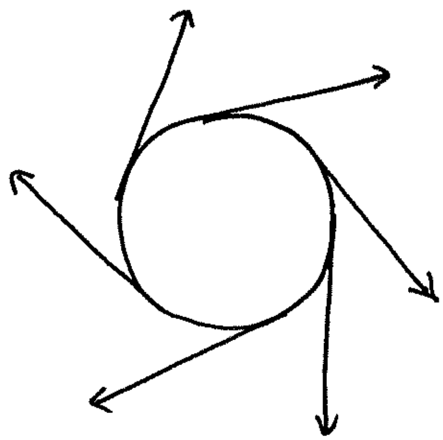
$$(L_g)_* : \Gamma(TG) \longrightarrow \Gamma(TG) \text{ is linear,}$$

since $D_h L_g$ is linear, so $\{X \in \Gamma(TG) \mid (L_g)_* X = X\}$ is a vector subspace of $\Gamma(TG)$ for all $g \in G$. The intersection of such subspaces for all $g \in G$ is thus a vector space, the space of left-invariant vector fields, denoted $\Gamma(TG)^L$.

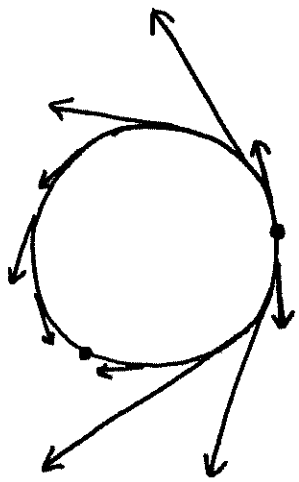
Examples:



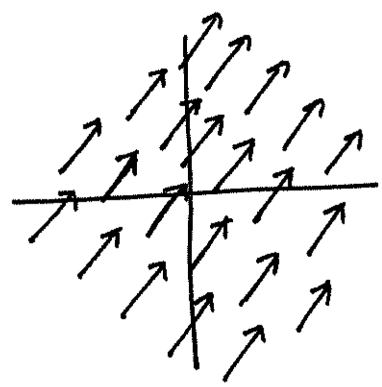
$\in \Gamma(TS^1)^L$



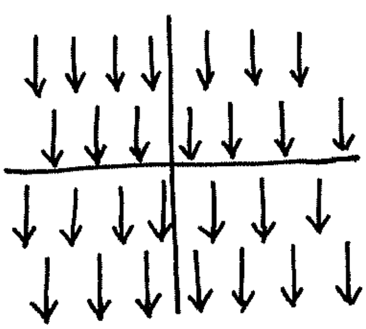
$\in \Gamma(TS^1)^L$



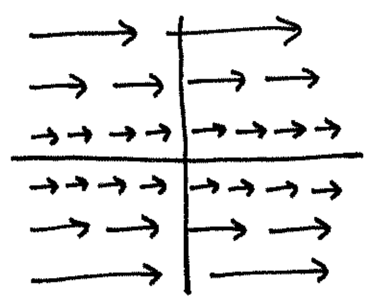
$\notin \Gamma(TS^1)^L$



$\in \Gamma(T\mathbb{R}^2)^L$



$\in \Gamma(T\mathbb{R}^2)^L$



$\notin \Gamma(T\mathbb{R}^2)^L$

Example:

Let $a_1, \dots, a_n \in \mathbb{R}$. Then $\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \in \Gamma(T\mathbb{R}^n)^L$.

Indeed, for $v, w \in \mathbb{R}^n$, $L_v(x) = v + x$ so $D_w L_v = Id$ and therefore,

$$(L_v)_*\left(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\right)(v+w) = D_w L_v\left(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\Big|_w\right)$$

$$= \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\Big|_{v+w}$$

so that $(L_v)_*\left(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\right) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$ for all $v \in \mathbb{R}^n$.

Note that there is clear assignment $T_0\mathbb{R}^n \longrightarrow \Gamma(T\mathbb{R}^n)^L$ given by

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\Big|_0 \longmapsto \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$

We'll see below that this is a linear isomorphism.

Definition:

If $v \in T_1G$, we define $X_v \in \Gamma(TG)$ by

$$X_v(g) = D_1 L_g(v)$$

for all $g \in G$. Note that since $D_1 L_g$ is linear, $X_{v+w} = X_v + X_w$ for all $v, w \in T_1 G$. Thus, $T_1 G \longrightarrow \Gamma(TG)$ where $v \longmapsto X_v$ is linear.

Lemma:

If $v \in T_1 G$, then $X_v \in \Gamma(TG)^L$, so $v \longmapsto X_v$ is a linear map $T_1 G \longrightarrow \Gamma(TG)^L$.

proof:

If $g \in G$, then for any $h \in G$

$$(L_g)_* X_v(gh) = D_h L_g(X_v(h)) = D_h L_g(D_1 L_h(v))$$
$$= D_1(L_g \circ L_h)(v) = D_1 L_{gh}(v) = X_v(gh). \quad \boxed{\text{END}}$$

The map from the previous lemma has a linear inverse $\Gamma(TG)^L \longrightarrow T_1G$ given by $X \longmapsto X(1)$. This is the content of the following

<u>Proposition:</u>

$\Gamma(TG)^L \cong T_1G$ as $\mathbb{R}$-vector spaces. In particular, $\Gamma(TG)^L$ is finite-dimensional.

<u>proof:</u>

Let $X \in \Gamma(TG)^L$, and let $v = X(1)$. We want to show $X_v = X$. Indeed, if $g \in G$ then

$$X_v(g) = D_1L_g(v) = D_1L_g(X(1)) = (L_g)_* X(g) = X(g).$$

We also need to check that for any $v \in T_1G$, we have $X_v(1) = v$. Indeed, $L_1 = id$, so $D_1L_1 = id$. Thus, $X_v(1) = D_1L_1(v) = v$.

$\boxed{\text{END}}$

If $\Gamma(TG)^L$ has an algebra structure, then we could use the identification $\Gamma(TG)^L \cong T_1G$ to define an algebra structure on T_1G. And $\Gamma(TG)^L$ does have an algebra structure, where multiplication is given by the Lie bracket. This is the content of the following

Lemma:

If $X, Y \in \Gamma(TG)^L$, then $[X,Y] \in \Gamma(TG)^L$.

proof:

By the exercise on page 135, if $g \in G$, and $X, Y \in \Gamma(TG)^L$, then

$$(L_g)_*[X,Y] = [(L_g)_*X, (L_g)_*Y] = [X,Y].$$

$\boxed{\text{END}}$

The previous lemma shows that $\Gamma(TG)^L$ is an algebra, so we can define an algebra structure on T_1G as

$$[v,w] = [X_v, X_w](1)$$

for any $v,w \in T_1G$. By the nature of this definition, we immediately have the following

Theorem:

$T_1G \cong \Gamma(TG)^L$ as $\mathbb{R}$-algebras.

Definition:

T_1G (or $\Gamma(TG)^L$) is called the Lie algebra of G. We'll often denote it as $\mathfrak{g}$.

Example:

If $a_1,\dots,a_n \in \mathbb{R}$ and $b_1,\dots,b_n \in \mathbb{R}$, then $\left[\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}\right] = 0$. That is, the Lie bracket of any two elements of $\Gamma(T\mathbb{R}^n)^L \cong T_0\mathbb{R}^n$ is 0.

Proposition:

If G is a linear Lie group, then for $A,B \in \mathfrak{g} \subseteq M_{n\times n}(\mathbb{R})$ we have

$$[A,B] = AB - BA$$

proof:

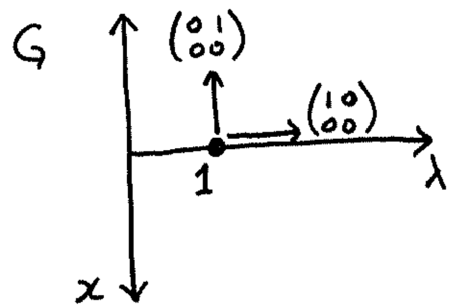
See exercises on pages 190 and 196.

END

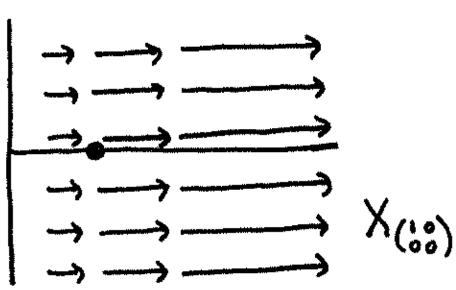
To rectify the previous example with the proposition, realize $\mathbb{R}^n$ as a linear Lie group and check that for $A,B \in T_0\mathbb{R}^n$, $AB-BA=0$.

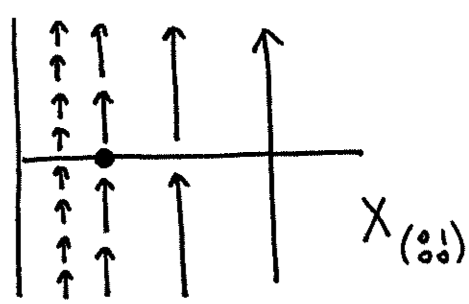
Example:

Let $G = \left\{ \begin{pmatrix} \lambda & x \\ 0 & 1 \end{pmatrix} \,\middle|\, \lambda > 0, x \in \mathbb{R} \right\}$. Note that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in T_1 G$ is tangent to the line $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ for $\lambda > 0$, and that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in T_1 G$ is tangent to the line $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$



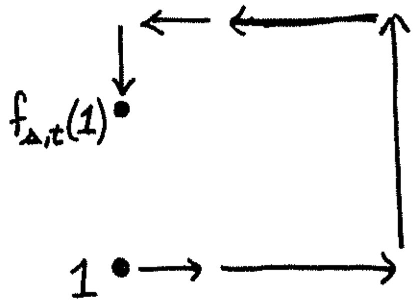
Identifying $\begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix} \in G$ with $(p,q) \in \mathbb{R}^2$, note that $(1,x)(p,q) = (p, q+x)$ and $(\lambda, 0)(p,q) = (\lambda p, \lambda q)$. So given $v \in T_1 G$, X_v is obtained by translating v up and down, and scaling left and right.





Let $f_{s,t}(1) = \theta_{-s}^{X_{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}} \circ \theta_{-t}^{X_{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}} \circ \theta_{s}^{X_{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}} \circ \theta_{t}^{X_{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}}(1)$

for $s, t \in \mathbb{R}$. Notice that $f_{s,t}(1) \neq 1$. That is, the flows for $X_{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}$ and $X_{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}$ do not commute.



By the lemma on page 149, we can be assured that $\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right] \neq \begin{pmatrix}0&0\\0&0\end{pmatrix}$. In fact, $f_{s,t}(1)$ is in the direction of $\begin{pmatrix}0&1\\0&0\end{pmatrix}$ from 1, and

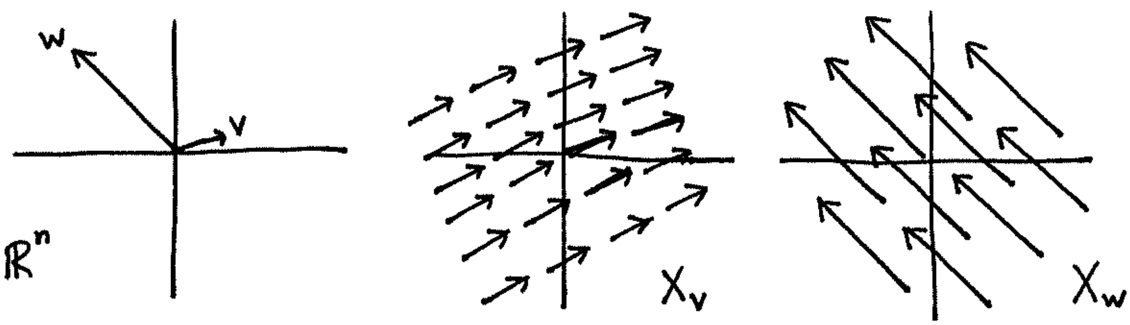
$$\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right] = \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} - \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$= \begin{pmatrix}0&1\\0&0\end{pmatrix} - \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

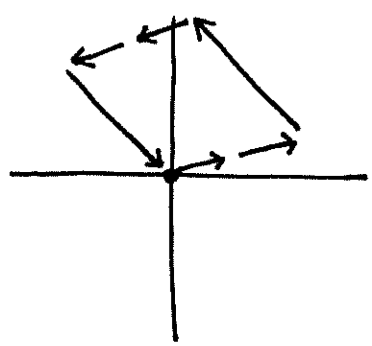
$$= \begin{pmatrix}0&1\\0&0\end{pmatrix}$$

Example:

For $v, w \in T_0\mathbb{R}^n$, $X_v, X_w \in \Gamma(T\mathbb{R}^n)^L$ are "constant" vector fields as discussed on page 181.



Thus, the flows θ^{X_v} and θ^{X_w} commute. That is, $\theta^{X_w}_{-s}\theta^{X_v}_{-t}\theta^{X_w}_{s}\theta^{X_v}_{t} = id$ for all $s, t \in \mathbb{R}$



Therefore, $[v, w] = 0$.

Exercises:

① Let $A \in gl_n\mathbb{R}$ so that $X_A \in \Gamma(TGL_n(\mathbb{R}))^L$.
For $g \in GL_n(\mathbb{R})$, show that $X_A(g) = gA$.

② Let $A = \begin{pmatrix} 3 & 5 & 7 \\ 2 & 8 & 1 \\ 3 & 6 & 4 \end{pmatrix} \in gl_3\mathbb{R}$ and $g = \begin{pmatrix} 2 & 5 & 1 \\ 0 & 1 & 0 \\ 1 & 7 & 1 \end{pmatrix} \in GL_3(\mathbb{R})$.

Find $X_A(g)$.

For ③-⑦, let $A, B \in gl_n\mathbb{R}$, $g \in GL_n(\mathbb{R})$, and let $x^{ij} : GL_n(\mathbb{R}) \to \mathbb{R}$ be projection to the ij-coordinate.

③ Show $L_{X_A}(x^{ij})(g) = \sum_{l=1}^{n} g_{il} A_{lj}$.

④ Prove $L_{X_B}(L_{X_A}(x^{ij}))(g) = \sum_{l=1}^{n} A_{lj}\left(\sum_{k=1}^{n} g_{ik} B_{kl}\right)$.

⑤ Prove $L_{[X_A, X_B]}(x^{ij})(g) = L_{X_{(AB-BA)}}(x^{ij})(g)$.

⑥ Prove $L_{[A,B]}x^{ij} = L_{AB-BA}\, x^{ij}$.

⑦ Prove $[A, B] = AB - BA$.

§ Homomorphisms of Lie groups

Let G and H be Lie groups. A Lie group homomorphism $\varphi: G \to H$ is a smooth function that is a homomorphism of groups.

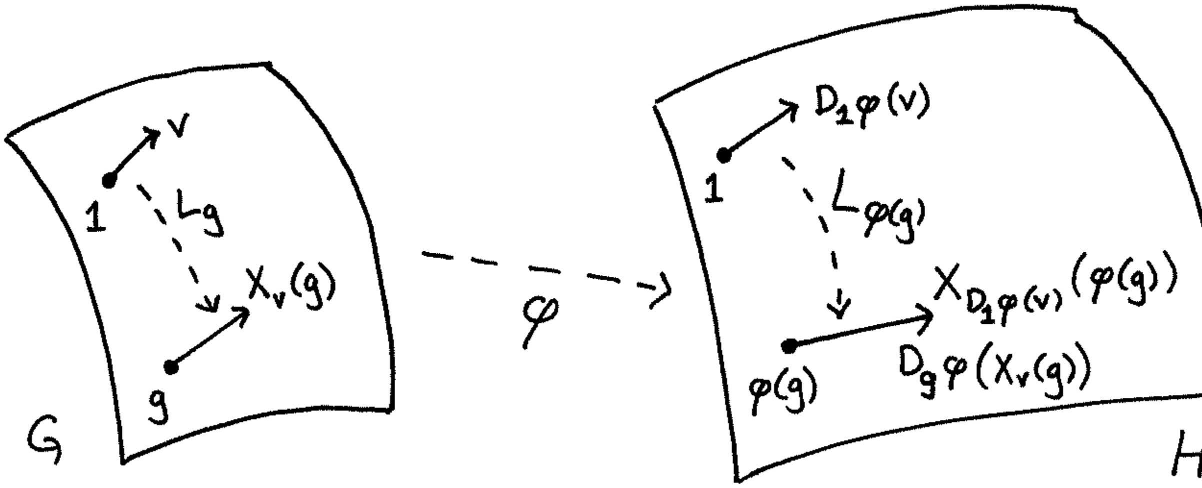
Example:

$f: \mathbb{R} \to S^1$ where $f(\theta) = e^{i\theta}$ is a Lie group homomorphism.

Lemma:

Suppose $\varphi: G \to H$ is a Lie group homomorphism and that $v \in \mathfrak{g}$. Then, for any $g \in G$,

$$D_g\varphi\big(X_v(g)\big) = X_{D_1\varphi(v)}\big(\varphi(g)\big)$$



proof:

For $p \in G$,

$$L_{\varphi(g)} \circ \varphi(p) = L_{\varphi(g)}(\varphi(p)) = \varphi(g)\varphi(p) = \varphi(gp) = \varphi \circ L_g(p),$$

which is to say $L_{\varphi(g)} \circ \varphi = \varphi \circ L_g$.

Therefore,

$$\begin{aligned} D_g\varphi(X_v(g)) &= D_g\varphi \circ D_1 L_g(v) \\ &= D_1(\varphi \circ L_g)(v) \\ &= D_1(L_{\varphi(g)} \circ \varphi)(v) \\ &= D_1 L_{\varphi(g)}(D_1\varphi(v)) \\ &= X_{D_1\varphi(v)}(\varphi(g)). \end{aligned}$$

$\boxed{\text{END}}$

Below, $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively.

Proposition:

If $\varphi: G \longrightarrow H$ is a homomorphism of Lie groups, then $D_1\varphi: \mathfrak{g} \longrightarrow \mathfrak{h}$ is an algebra homomorphism. That is, if $v, w \in \mathfrak{g}$ then $D_1\varphi([v,w]) = [D_1\varphi(v), D_1\varphi(w)]$.

proof:

Let $f \in C^\infty(H)$ and $g \in G$. Then, by the previous lemma

$$\left(L_{X_{D_1\varphi(v)}} f\right) \circ \varphi(g) = L_{X_{D_1\varphi(v)}(\varphi(g))} f$$

$$= L_{D_g\varphi(X_v(g))} f$$

$$= D_{\varphi(g)} f\left(D_g\varphi\left(X_v(g)\right)\right)$$

$$= D_g(f \circ \varphi)\left(X_v(g)\right)$$

$$= L_{X_v(g)}(f \circ \varphi)$$

$$= L_{X_v}(f \circ \varphi)(g)$$

Using the previous equation for $v, w \in \mathfrak{g}$ and $f, L_{X_{D,\varphi(v)}} f, L_{X_{D,\varphi(w)}} f \in C^\infty(H)$, we have

$$L_{[X_{D,\varphi(v)}, X_{D,\varphi(w)}](\varphi(g))} f$$

$$= \left(L_{[X_{D,\varphi(v)}, X_{D,\varphi(w)}]} f\right) \circ \varphi(g)$$

$$= \left(L_{X_{D,\varphi(v)}} \circ L_{X_{D,\varphi(w)}} f\right) \circ \varphi(g) - \left(L_{X_{D,\varphi(w)}} \circ L_{X_{D,\varphi(v)}} f\right) \circ \varphi(g)$$

$$= L_{X_v}\left(\left(L_{X_{D,\varphi(w)}} f\right) \circ \varphi\right)(g) - L_{X_w}\left(\left(L_{X_{D,\varphi(v)}} f\right) \circ \varphi\right)(g)$$

$$= L_{X_v}\left(L_{X_w}(f \circ \varphi)\right)(g) - L_{X_w}\left(L_{X_v}(f \circ \varphi)\right)(g)$$

$$= L_{[X_v, X_w]}(f \circ \varphi)(g)$$

$$= D_g(f \circ \varphi)\left([X_v, X_w](g)\right)$$

$$= D_{\varphi(g)} f\left(D_g \varphi\left([X_v, X_w](g)\right)\right)$$

$$= L_{D_g \varphi([X_v, X_w](g))} f$$

Therefore, for all $g \in G$, and $v, w \in \mathfrak{g}$,

$$\left[X_{D_1\varphi(v)}, X_{D_1\varphi(w)}\right](\varphi(g)) = D_g\varphi\left([X_v, X_w](g)\right)$$

Setting $g = 1$, we see

$$\left[D_1\varphi(v), D_1\varphi(w)\right] = \left[X_{D_1\varphi(v)}, X_{D_1\varphi(w)}\right](1)$$

$$= D_1\varphi\left([X_v, X_w](1)\right)$$

$$= D_1\varphi\left([v, w]\right).$$

$\boxed{\text{END}}$

<u>Corollary:</u>

If a Lie group homomorphism $\varphi : G \to H$ is an immersion at $1 \in G$, then $\mathfrak{g}$ is a subalgebra of $\mathfrak{h}$.

G is a <u>Lie subgroup</u> of a Lie group H if G is a Lie group, G is an immersed submanifold of H, and the inclusion $G \hookrightarrow H$ is a homomorphism.

<u>Corollary</u>:

If G is a Lie subgroup of a Lie group H, then $\mathfrak{g}$ is a subalgebra of $\mathfrak{h}$.

<u>proof</u>:

Apply the previous corollary to the inclusion $G \hookrightarrow H$.

$\boxed{\text{END}}$

Examples:

Let

$$U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$

$$V = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$

$$A = \left\{ \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \;\middle|\; y \in \mathbb{R}^{\times} \right\}.$$

Then $SO(2) \cong S^1$, $U \cong \mathbb{R}$, $V \cong \mathbb{R}$, and $A \cong \mathbb{R}^{\times}$ are Lie subgroups of $SL_2(\mathbb{R})$. Their respective Lie subalgebras of $\mathfrak{sl}_2\mathbb{R}$ are

$$\mathfrak{so}_2 = \left\{ \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\},$$

$$T_1 U = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\},$$

$$T_1 V = \left\{ \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}, \quad \text{and}$$

$$T_1 A = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}.$$

Exercises:

① Let G be a Lie subgroup of $GL_n(\mathbb{R})$. Prove that if $A, B \in \mathfrak{g} \leq \mathfrak{gl}_n\mathbb{R}$, then $[A,B] = AB - BA$.

② Using ①, prove that any Lie group homomorphism $\varphi : SL_2(\mathbb{R}) \longrightarrow \mathbb{R}$ has $D_1\varphi = 0$.

§ Exponential map

Lemma:

Let G be a Lie group with $X \in \Gamma(TG)^L$. Then there is a one parameter group of diffeomorphisms $\mathbb{R} \cong \{\theta_t^X\}_{t \in \mathbb{R}} \leq \mathrm{Diff}(G)$

proof:

For $1 \in G$, let $\{\theta_t^{X,1}\}_{t \in I_1}$ be the local flow of X near 1. By the proposition on page 128, if $g \in G$ then the flow near $L_g(1) = g$ for $(L_g)_* X = X$ is $\{L_g \circ \theta_t^{X,1} \circ L_g^{-1}\}_{t \in I_1}$. Because flows for X are uniquely determined by X, the flows for $g \in G$ agree in the intersections of their domains. Therefore, we define $\theta_t^X \in \mathrm{Diff}(G)$ for $t \in I_1$ by $\theta_t^X(h) = L_g \circ \theta_t^{X,1} \circ L_g^{-1}(h)$ if $h \approx g$. If $k \in \mathbb{Z}$, let $\theta_{kt}^X = (\theta_t^X)^k$.

$\boxed{\text{END}}$

If G is a Lie group with Lie algebra $\mathfrak{g}$, we define the exponential map

$$\exp : \mathfrak{g} \longrightarrow G$$

by $\exp(v) = \Theta_1^{X_v}(1)$.

Examples:

① Let $y \in GL_1(\mathbb{R}) = \mathbb{R}^\times$, $t \in \mathfrak{gl}_1\mathbb{R}$, and $s \in \mathbb{R}$.

Recall from exercise ① on page 190 that $X_t(y) = yt$.

We verify that $\Theta_s^{X_t}(y) = e^{st}y$ by checking the flow equations from page 119:

(i) $e^{0\cdot t}y = y$

(ii) $D_{s_0}(e^{st}y)\left(\frac{d}{ds}\Big|_{s_0}\right) = e^{s_0 t}yt = X_t(e^{s_0 t}y)$

Thus, $\exp(t) = \Theta_1^{X_t}(1) = e^{1\cdot t}\cdot 1 = e^t$.

② For $G=\mathbb{R}^n$, we have $\mathfrak{g}=\mathbb{R}^n$. Given $a=(a_1,\dots,a_n)\in\mathbb{R}^n$, recall from page 181 that $X_{\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}|_0} = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, and thus

$$\theta_t^{X_{\sum_i a_i \frac{\partial}{\partial x_i}|_0}}(x) = \theta_t^{\sum_i a_i \frac{\partial}{\partial x_i}}(x) = x + ta.$$

Hence, $\exp\left(\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\Big|_0\right) = \theta_1^{X_{\sum_i a_i \frac{\partial}{\partial x_i}|_0}}(0) = a$.

Here, $a=(a_1,\dots,a_n)$ is usually identified with $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\Big|_0$ so that $\exp = id_{\mathbb{R}^n}$.

Lemma:

Let $v\in\mathfrak{g}$ and $s,t\in\mathbb{R}$. Then $\theta_{st}^{X_v} = \theta_t^{X_{sv}}$.

proof:

If $g\in G$ then

$$sX_v(g) = sD_1L_g(v) = D_1L_g(sv) = X_{sv}(g).$$

By the chain rule,

$$D_{t_0}\left(\theta^{X_v}_{\lambda t}(g)\right)\left(\frac{d}{dt}\Big|_{t_0}\right) = \lambda\, D_{\lambda t_0}\left(\theta^{X_v}_{\lambda t}(g)\right)\left(\frac{d}{dt}\Big|_{\lambda t_0}\right)$$
$$= \lambda\, X_v\left(\theta^{X_v}_{\lambda t_0}(g)\right)$$
$$= X_{\lambda v}\left(\theta^{X_v}_{\lambda t_0}(g)\right).$$

Therefore, $\theta^{X_v}_{\lambda t} = \theta^{X_{\lambda v}}_{t}$.

$\boxed{\text{END}}$

Lemma:

Let $v \in \mathfrak{g}$, and $g, h \in G$ and $t \in \mathbb{R}$.
Then $\theta^{X_v}_t(gh) = g\,\theta^{X_v}_t(h)$.

proof:

Since $(L_g)_* X_v = X_v$, we have by the proposition on page 128 that $\theta^{X_v}_t \circ L_g = L_g \circ \theta^{X_v}_t$. Thus, $\theta^{X_v}_t \circ L_g(h) = L_g \circ \theta^{X_v}_t(h)$.

$\boxed{\text{END}}$

Proposition:

If $v \in \mathfrak{g}$, then $\mathbb{R}v$ is a subgroup of $\mathfrak{g}$. The restriction

$$\exp\big|_{\mathbb{R}v} : \mathbb{R}v \longrightarrow G$$

is a homomorphism.

proof:

For $s, t \in \mathbb{R}$,

$$\begin{aligned}
\exp(sv)\exp(tv) &= \theta_1^{X_{sv}}(1)\,\theta_1^{X_{tv}}(1) \\
&= \theta_s^{X_v}(1)\,\theta_t^{X_v}(1) \\
&= \theta_t^{X_v}\left(\theta_s^{X_v}(1)\cdot 1\right) \\
&= \theta_t^{X_v}\left(\theta_s^{X_v}(1)\right) \\
&= \theta_{t+s}^{X_v}(1) \\
&= \theta_1^{X_{(t+s)v}}(1) \\
&= \exp\left((t+s)v\right).
\end{aligned}$$

$\boxed{\text{END}}$

Examples:

Let $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ so that $T_1(T^2) = T_0(\mathbb{R}^2)$.

Let $v \in T_1(T^2)$. If v is nonzero and has rational slope, then $\exp(\mathbb{R}v) = S^1 \leq T^2$.

If $\mathbb{R}v$ has irrational slope, then $\exp(\mathbb{R}v) = \mathbb{R} \leq T^2$.

Definition:

For a Lie group G, if $v \in \mathfrak{g}$ and $v \neq 0$, we call $\exp(\mathbb{R}v) \leq G$ the <u>one parameter subgroup</u> of G corresponding to v.

Proposition:

$\exp : \mathfrak{g} \to G$ is smooth.

proof:

For $(v,g) \in \mathfrak{g} \times G$, $T_{(v,g)}(\mathfrak{g} \times G) = \mathfrak{g} \times T_g G$, so we define $Y \in \Gamma(T(\mathfrak{g} \times G))$ by $Y(v,g) = (0, X_v(g))$. Then $\theta_t^Y \in \mathrm{Diff}(\mathfrak{g} \times G)$ is $\theta_t^Y(v,g) = (v, \theta_t^{X_v}(g))$. If $\pi : \mathfrak{g} \times G \to G$ is projection, then

$$\pi \circ \theta_1^Y(v,1) = \pi\left(v, \theta_1^{X_v}(1)\right) = \exp(v).$$

$\boxed{\text{END}}$

Proposition:

$D_0 \exp : \mathfrak{g} \to \mathfrak{g}$ is the identity.

proof:

For any $v \in \mathfrak{g}$, we let $c : \mathbb{R} \to \mathfrak{g}$ be $c(t) = tv$. Hence,

$$\begin{aligned} D_0 \exp(v) &= D_0 \exp\left(D_0 c\left(\tfrac{d}{dt}\big|_0\right)\right) \\ &= D_0(\exp \circ c)\left(\tfrac{d}{dt}\big|_0\right) = D_0(\exp(tv))\left(\tfrac{d}{dt}\big|_0\right) \\ &= D_0\left(\theta_1^{X_{tv}}(1)\right)\left(\tfrac{d}{dt}\big|_0\right) = D_0\left(\theta_t^{X_v}(1)\right)\left(\tfrac{d}{dt}\big|_0\right) \\ &= X_v\left(\theta_0^{X_v}(1)\right) = X_v(1) = v. \end{aligned}$$

$\boxed{\text{END}}$

Corollary:

There are open neighborhoods $\mathcal{U}_0 \subseteq \mathfrak{g}$ and $\mathcal{U}_1 \subseteq G$, of 0 and 1 respectively, such that $\exp|_{\mathcal{U}_0} : \mathcal{U}_0 \to \mathcal{U}_1$ is a diffeomorphism.

proof:

Apply the inverse function theorem using that $D_0 \exp$ is a linear isomorphism.

$\boxed{\text{END}}$

Proposition:

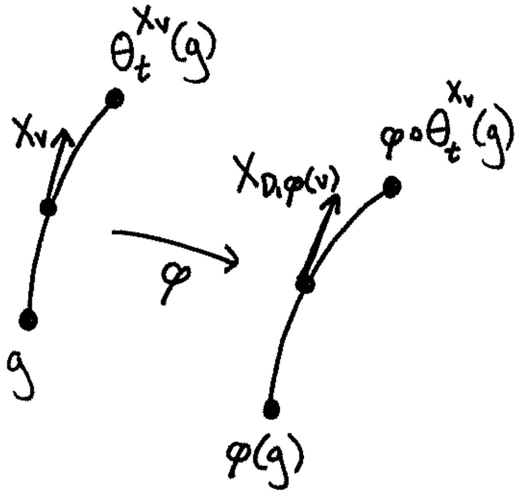
If $\varphi : G \longrightarrow H$ is a homomorphism of Lie groups whose respective Lie algebras are $\mathfrak{g}$ and $\mathfrak{h}$, then the following commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "D_1\varphi"] \arrow[d, "\exp_G"'] & \mathfrak{h} \arrow[d, "\exp_H"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

proof:

Let $g, g_0 \in G$ and $v \in \mathfrak{g}$. First note that $\varphi \circ L_g = L_{\varphi(g)} \circ \varphi$ since $\varphi \circ L_g(g_0) = \varphi(g g_0) = \varphi(g)\varphi(g_0)$ $= L_{\varphi(g)} \circ \varphi(g_0)$. Therefore, $\varphi \circ \theta_t^{X_v}$ satisfies the flow equation for $X_{D_1\varphi(v)} \in \Gamma(TH)^L$ restricted to $\varphi(G)$. Indeed,

$$D_{t_0}\left(\varphi \circ \theta_t^{X_v}(g)\right)\left(\frac{d}{dt}\Big|_{t_0}\right) = \left(D_{\theta_{t_0}^{X_v}(g)}\varphi\right) \circ \left(D_{t_0}\theta_t^{X_v}(g)\right)\left(\frac{d}{dt}\Big|_{t_0}\right)$$



$$= D_{\theta_{t_0}^{X_v}(g)}\varphi\left(X_v\left(\theta_{t_0}^{X_v}(g)\right)\right)$$

$$= D_{\theta_{t_0}^{X_v}(g)}\varphi\left(D_1 L_{\theta_{t_0}^{X_v}(g)}(v)\right)$$

$$= D_1\left(\varphi \circ L_{\theta_{t_0}^{X_v}(g)}\right)(v)$$

$$= D_1\left(L_{\varphi\circ\theta_{t_0}^{X_v}(g)} \circ \varphi\right)(v)$$

$$= \left(D_1 L_{\varphi\circ\theta_{t_0}^{X_v}(g)}\right)\left(D_1\varphi\right)(v)$$

$$= X_{D_1\varphi(v)}\left(\varphi\circ\theta_{t_0}^{X_v}(g)\right).$$

So $\varphi \circ \theta_t^{X_v}(g) = \theta_t^{X_{D_1\varphi(v)}}(\varphi(g))$.

Therefore, since $\varphi(1)=1$, let $g=1$ and $t=1$:

$$\exp_H \circ D_1\varphi(v) = \Theta_1^{X_{D_1\varphi(v)}}(1) = \varphi\circ\Theta_1^{X_v}(1) = \varphi\circ\exp_G(v).$$

END

Proposition:

$$\exp : \mathfrak{gl}_n\mathbb{R} \longrightarrow GL_n(\mathbb{R}) \text{ is } \exp(A) = \sum_{k=0}^{\infty}\frac{A^k}{k!}$$

proof:

If $A\in\mathfrak{gl}_n\mathbb{R}$, then $\sum_{k=0}^{\infty}\frac{A^k}{k!}\in\mathfrak{gl}_n\mathbb{R}$. To see that $\sum_{k=0}^{\infty}\frac{A^k}{k!}\in GL_n(\mathbb{R})$, let $B\in GL_n(\mathbb{C})$ be such that $BAB^{-1} = \begin{pmatrix}\lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots & \lambda_n\end{pmatrix}$. Then

$$BA^kB^{-1} = (BAB^{-1})^k = \begin{pmatrix}\lambda_1^k & & * \\ & \lambda_2^k & \\ 0 & & \ddots & \lambda_n^k\end{pmatrix}, \text{ so}$$

$$B\left(\sum_{k=0}^{\infty}\frac{A^k}{k!}\right)B^{-1} = \sum_{k=0}^{\infty}\frac{(BAB^{-1})^k}{k!} = \begin{pmatrix}e^{\lambda_1} & & * \\ & e^{\lambda_2} & \\ 0 & & \ddots & e^{\lambda_n}\end{pmatrix}\in GL_n(\mathbb{C})$$

since $\prod_{i=1}^{n}e^{\lambda_i}\neq 0$. Hence, $\sum_{k=0}^{\infty}\frac{A^k}{k!}$ is contained in

$$\mathfrak{gl}_n\mathbb{R}\cap B^{-1}GL_n(\mathbb{C})B = \mathfrak{gl}_n\mathbb{R}\cap GL_n(\mathbb{C}) = GL_n(\mathbb{R}).$$

We want to show that if $t \in \mathbb{R}$, then $\theta_t^{X_A}(1) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$. Indeed, first note that $\sum_{k=0}^{\infty} \frac{(0 \cdot A)^k}{k!} = 1$. Second, note that

$$
\begin{aligned}
D_{t_0}\left(\sum_{k=0}^{\infty} \frac{(tA)^k}{k!}\right)\left(\frac{d}{dt}\Big|_{t_0}\right) &= \sum_{k=0}^{\infty} D_{t_0}\left(\frac{t^k A^k}{k!}\right)\left(\frac{d}{dt}\Big|_{t_0}\right) \\
&= \sum_{k=0}^{\infty} \frac{k t_0^{k-1} A^k}{k!} \\
&= \sum_{k=1}^{\infty} \frac{t_0^{k-1} A^k}{(k-1)!} \\
&= \sum_{k=0}^{\infty} \frac{t_0^k A^{k+1}}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(t_0 A)^k}{k!} A \\
&= D_1 L_{\sum_{k=0}^{\infty} \frac{(t_0 A)^k}{k!}} A \\
&= X_A\left(\sum_{k=0}^{\infty} \frac{(t_0 A)^k}{k!}\right)
\end{aligned}
$$

Thus, $\theta_t^{X_A}(1) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$, so letting $t = 1$ yields $\exp(A) = \theta_1^{X_A}(1) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

$\boxed{\text{END}}$

Corollary:

If G is a Lie subgroup of the Lie group H, then $\exp_H\big|_{\mathfrak{g}} = \exp_G$.

proof:

Let $\varphi: G \to H$ be the inclusion so that $D_1\varphi: \mathfrak{g} \to \mathfrak{h}$ is the inclusion. Then

$$\exp_H\big|_{\mathfrak{g}} = \exp_H \circ D_1\varphi = \varphi \circ \exp_G = \exp_G.$$

$\boxed{\text{END}}$

Corollary:

If G is a Lie subgroup of $GL_n(\mathbb{R})$ and $A \in \mathfrak{g} \leq \mathfrak{gl}_n\mathbb{R}$, then $\exp_G(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

proof:

$$\exp_G(A) = \exp_{GL_n(\mathbb{R})}\big|_{\mathfrak{g}}(A) = \exp_{GL_n(\mathbb{R})}(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

$\boxed{\text{END}}$

Examples:

① For $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2\mathbb{R}$, $\exp\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^1 & 0 \\ 0 & e^{-1} \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 1/e \end{pmatrix}$.

② For $\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$ in the Lie algebra of

$\mathbb{R}^2 = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$, we have

$\exp\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ since $\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

if $k \geq 2$.

③ For $\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \in \mathfrak{so}_2$, $\exp\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} = \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix} \in SO(2)$.

Exercises:

① Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $I \in \mathrm{Diff}(G)$ be $I(g) = g^{-1}$. Prove $D_1 I(v) = -v$ for all $v \in \mathfrak{g}$.

② Prove a one parameter subgroup of a Lie group is a Lie subgroup.

③ Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Prove G is abelian if and only if $[v,w]=0$ for all $v,w \in \mathfrak{g}$. (See pages 149, 170, 182, 184, 185, 200, 201, 202, and 206. Such Lie algebras are called abelian.)

④ Use ② on page 198 to prove that if $\varphi : SL_2(\mathbb{R}) \longrightarrow \mathbb{R}$ is a Lie group homomorphism, then $\varphi(g) = 0$ for all $g \in SL_2(\mathbb{R})^\circ$.

⑤ Let H be a Lie group with connected Lie subgroups $G_1, G_2 \leq H$ whose Lie algebras are $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Prove that $\mathfrak{g}_1 \subseteq \mathfrak{g}_2$ implies $G_1 \subseteq G_2$ and that $\mathfrak{g}_1 = \mathfrak{g}_2$ implies $G_1 = G_2$. (Use the proposition on page 170.)

⑥ Let $U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$. Find all connected Lie subgroups of $SL_2(\mathbb{R})$ containing U.

§ Correspondence of subgroups and subalgebras

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We saw on page 196 that if $H \leq G$ is a Lie subgroup, then $T_1 H \leq \mathfrak{g}$ is a subalgebra. The following theorem provides a converse.

Theorem:

Suppose $\mathfrak{h} \leq \mathfrak{g}$ is a subalgebra. Then there exists a unique connected Lie subgroup $H \leq G$ such that $T_1 H = \mathfrak{h}$.

proof:

We will prove the existence of H. Uniqueness follows from exercise 5 on page 212.

Given $g \in G$, we define a plane field on G by $\Delta_g = D_1 L_g(\mathfrak{h}) \subseteq T_g G$.

Fix a basis $\{v_i\}_{i=1}^n$ for $\mathfrak{h}$, so that $\{X_{v_i}(g)\}_{i=1}^n = \{D_1 L_g(v_i)\}_{i=1}^n$ is a basis for Δ_g. Since $\mathfrak{h}$ is an algebra, $[v_i, v_j] \in \mathfrak{h}$, so
$[X_{v_i}, X_{v_j}](g) = X_{[v_i,v_j]}(g) = D_1 L_g([v_i,v_j]) \in D_1 L_g(\mathfrak{h}) = \Delta_g$
for all $g \in G$.

To show that Δ_g is integrable, assume $Y, Z \in \Gamma(TG)$ are such that $Y(g), Z(g)$ are in Δ_g for all g. Then $Y(g) = \sum_{i=1}^n a_i(g) X_{v_i}(g)$ and $Z(g) = \sum_{i=1}^n b_i(g) X_{v_i}(g)$ for some $a_i, b_i \in C^\infty(G)$.

Thus, $X_{v_i}(g), X_{v_j}(g), [X_{v_i}, X_{v_j}](g) \in \Delta_g$ implies that

$$[Y,Z](g) = \Big[\sum_i a_i X_{v_i}, \sum_j b_j X_{v_j}\Big](g)$$

$$= \sum_{i,j} [a_i X_{v_i}, b_j X_{v_j}](g)$$

$$= \sum_{i,j} \left(a_i b_j [X_{v_i}, X_{v_j}] + a_i L_{X_{v_i}}(b_j) X_{v_j} - b_j L_{X_{v_j}}(a_i) X_{v_i} \right)(g)$$

$$\in \operatorname{span}\left\{ [X_{v_i}, X_{v_j}](g),\ X_{v_j}(g),\ X_{v_i}(g) \right\}_{i,j}$$

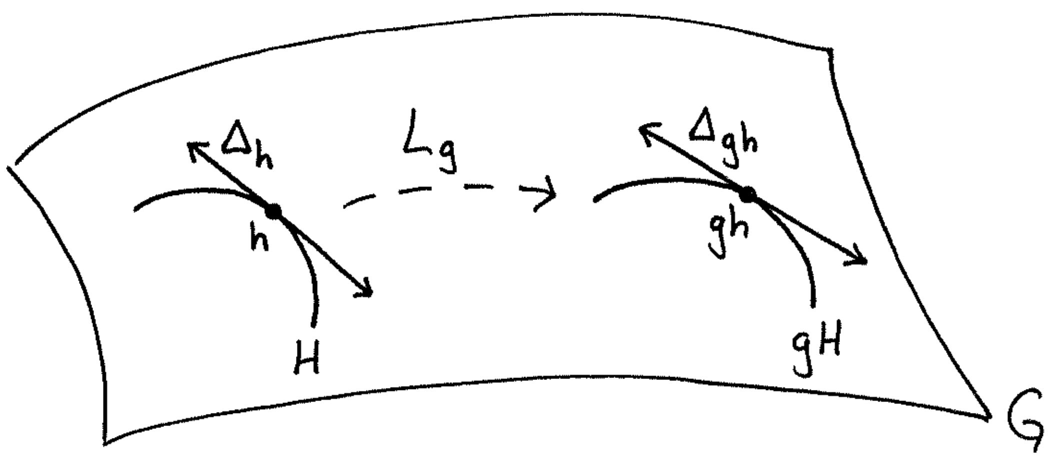
$$\leq \Delta_g .$$

where the first line above is the formula from ④ on page 117. Therefore, Δ_g satisfies the hypothesis of Frobenius' Theorem from page 154, so Δ_g is integrable. Let H be the connected leaf of the resulting foliation that contains $1 \in G$. Then H is an immersed submanifold of G, and $T_1 H = \Delta_1 = D_1 L_1(\mathfrak{h}) = \mathfrak{h}$. We only need to show that H is a subgroup of G.

For $g \in G$, since $L_g \in \mathrm{Diff}(G)$, $L_g(H) = gH$ is also a connected, immersed submanifold of G. And if $h\in H$, so that $gh\in gH$, then H being a leaf of the foliation means $T_h H = \Delta_h = D_1 L_h(\mathfrak{h})$ and thus

$$T_{gh}(gH) = D_h L_g(T_h H) = D_h L_g(D_1 L_h(\mathfrak{h}))$$

$$= D_1(L_g \circ L_h)(\mathfrak{h}) = D_1 L_{gh}(\mathfrak{h}) = \Delta_{gh}.$$



Therefore, the manifolds gH are exactly the leaves of the foliation.

Let $h,k\in H$. H is a subgroup of G if $h^{-1}k\in H$.

As above, $h^{-1}H$ is a leaf and it contains $h^{-1}h=1$. H is also the leaf that contains 1, so $h^{-1}H=H$. Hence, $h^{-1}k\in h^{-1}H=H$.

$\boxed{\text{END}}$

Exercise:

① Let $H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$ and let $Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; z \in \mathbb{R} \right\}$ so that Z is a Lie subgroup of the nilpotent Lie group H. (Z is the center of H.)

Find all the connected 2-dimensional Lie subgroups of H that contain Z. Describe them as both submanifolds of H and as groups.

§ Homogeneous spaces

Let G be a Lie group. H is a <u>closed Lie subgroup</u> of G if H is a Lie subgroup of G and $H \subseteq G$ is a closed subset.

If $H \leq G$ is a closed Lie subgroup, and if $\varphi : H \to G$ is inclusion, then for any compact set $K \subseteq G$, we have that $\varphi^{-1}(K) = H \cap K$ is compact in H. That is, φ is proper, so that H is an embedded submanifold of G by ② on page 48.

The quotient topology for the space of left-H-cosets G/H is defined by having $\{g_\alpha H\}_{\alpha \in A} \subseteq G/H$ be open if and only if $\bigcup_{\alpha \in A} g_\alpha H \subseteq G$ is open.

Note that the natural action of G on G/H is by homeomorphisms, and that the quotient map $\pi : G \to G/H$ is open and continuous.

Lemma:

If G is a Lie group and $H \leq G$ is a closed Lie subgroup, then G/H is Hausdorff.

proof:

Suppose $g_1 H \neq g_2 H$. Let $g = g_1^{-1} g_2$, so that $g \notin H$. Because H is closed, H^c is open. Because right-multiplication is a diffeomorphism, $H^c g^{-1}$ is open. Note too that $1 = g g^{-1} \in H^c g^{-1}$.

Because $1 \cdot 1 = 1 \in H^c g^{-1}$, the inverse image of $H^c g^{-1}$ under the multiplication map $G \times G \longrightarrow G$ is an open set containing $(1,1) \in G \times G$. Let $U \subseteq G$ be an open neighborhood of 1 such that $U \times U \longrightarrow H^c g^{-1}$, that is, $U^2 \subseteq H^c g^{-1}$. Replacing U with $U \cap U^{-1}$, we may assume that $U = U^{-1}$.

If $h \in H$, then Ugh and Uh are open in G. Thus, $UgH = \bigcup_{h \in H} Ugh$ and $UH = \bigcup_{h \in H} Uh$ are open in G. By definition of the quotient topology, UgH and UH are open subsets of G/H. Since $1 \in U$, UgH and UH are neighborhoods of gH and H, respectively. We wish to show they are disjoint neighborhoods.

If not, then $u_1 g h_1 = u_2 h_2$ for some $u_1, u_2 \in U$ and $h_1, h_2 \in H$. Then we would have the following contradiction:

$$h_2 h_1^{-1} = u_2^{-1} u_1 g \in U^{-1} U g = U^2 g \subseteq H^c g^{-1} g = H^c$$

Therefore, UgH and UH are disjoint open neighborhoods of gH and H. Hence, since left-multiplication of $g_1 \in G$ on G/H is a homeomorphism, we have that $g_1(UgH)$ and $g_1(UH)$ are disjoint open neighborhoods of $g_1 g H = g_2 H$ and $g_1 H$. $\boxed{\text{END}}$

Theorem:

If G is a Lie group with a closed Lie subgroup $H \leq G$, then the space G/H is a smooth manifold of dimension $\dim(G) - \dim(H)$.

proof:

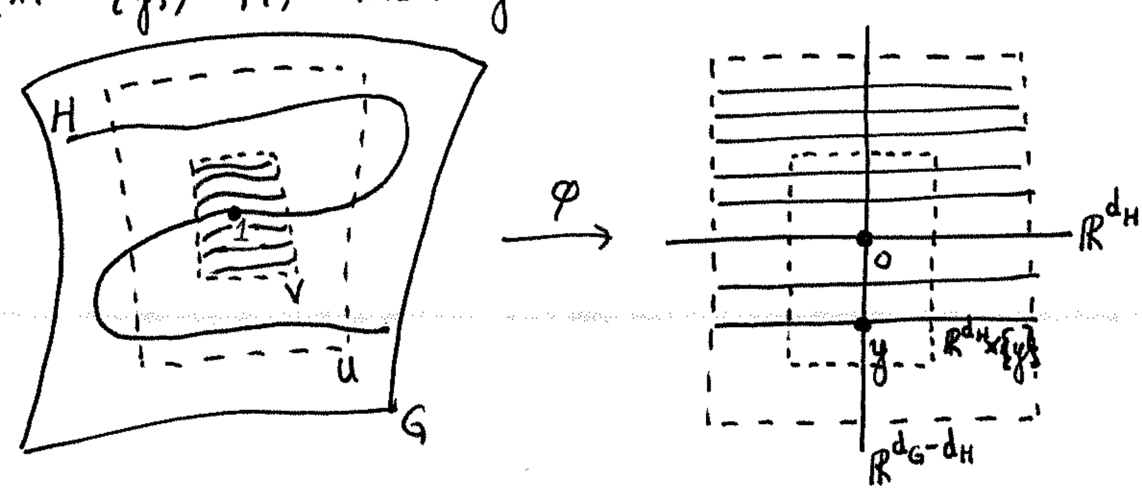
Let $d_G = \dim(G)$, $d_H = \dim(H)$, and $\pi : G \to G/H$ be the quotient map.

If $G = \bigcup_{i=1}^{\infty} K_i$ for compact $K_i \subseteq G$, then $\pi(K_i) \subseteq G/H$ is compact, and $G/H = \bigcup_{i=1}^{\infty} \pi(K_i)$ since π is surjective.

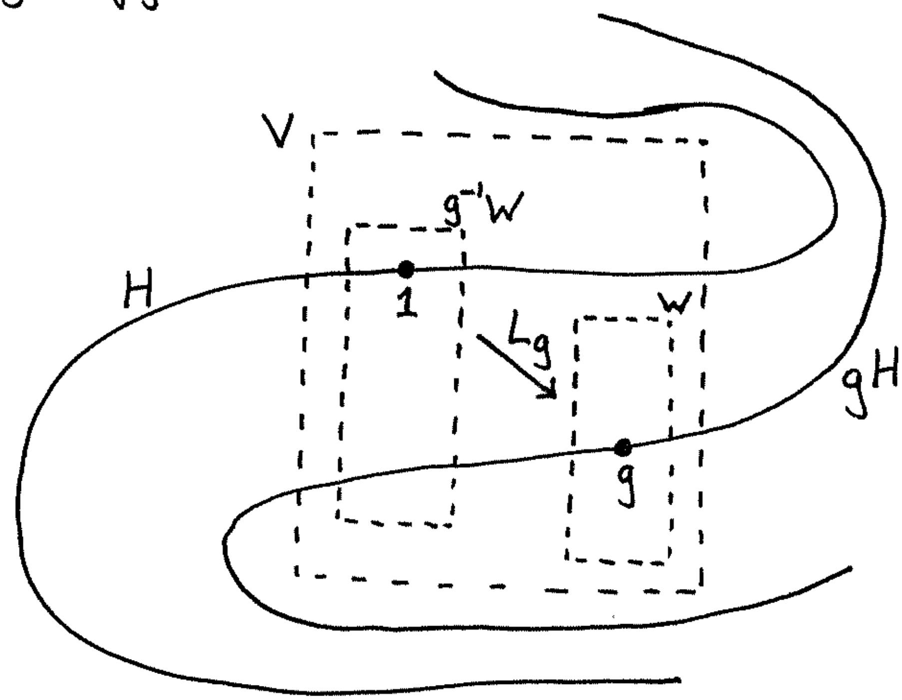
Recall from the proof of the previous chapter that the cosets $gH \subseteq G$ are the leaves of a foliation on G. We let $\varphi : \mathcal{U} \to \mathbb{R}^{d_G}$ be a chart for this foliation such that $1 \in \mathcal{U}$. We assume $\varphi(1) = 0$, so that $\varphi^{-1}(0) \in H$.

For $y \in \mathbb{R}^{d_G - d_H}$, $\varphi^{-1}(\mathbb{R}^{d_H} \times \{y\}) \subseteq g_y H$ for some $g_y \in G$, since φ is a chart for the foliation whose leaves are cosets.

Let $Y_H = \{ y \in \mathbb{R}^{d_G - d_H} \mid \varphi^{-1}(\mathbb{R}^{d_H} \times \{y\}) \subseteq H \}$.
In H, each $\varphi^{-1}(\mathbb{R}^{d_H} \times \{y\}) \subseteq H$, for $y \in Y_H$, is open, and this collection of sets is pairwise disjoint. Thus, $\varphi^{-1}(Y_H) \subseteq H$ is discrete. But $H \subseteq G$ is closed, hence embedded, so $\varphi^{-1}(Y_H)$ is discrete in G. Thus, $Y_H \subseteq \mathbb{R}^{d_G - d_H}$ is discrete. Therefore, we may replace $\mathcal{U}$ with an open $V \subseteq G$ containing $1 \in G$ such that for the chart $\varphi : V \to \mathbb{R}^{d_G}$, if $\varphi^{-1}(\mathbb{R}^{d_H} \times \{y\}) \subseteq H$, then $y = 0$.



Choose $W \subseteq V$ such that if $g \approx 1$, then $g^{-1}W \subseteq V$. Then $H \cap g^{-1}W \subseteq H \cap V = \varphi^{-1}(\mathbb{R}^{d_H} \times \{0\})$. Therefore, $gH \cap W \subseteq \varphi^{-1}(\mathbb{R}^{d_H} \times \{y_g\})$ for a single $y_g \in \mathbb{R}^{d_G - d_H}$.

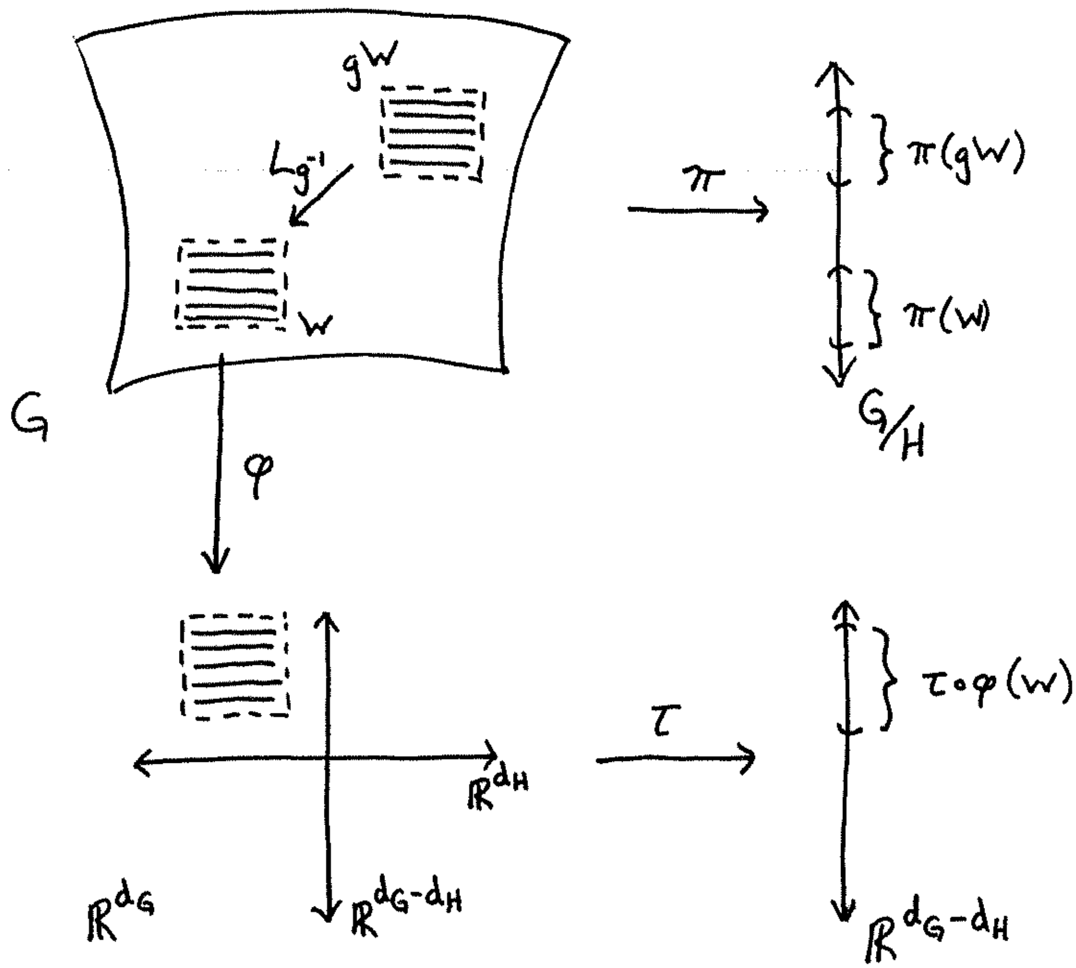


If we let $\tau : \mathbb{R}^{d_G} \longrightarrow \mathbb{R}^{d_G - d_H}$, then we have seen that

$$\tau \circ \varphi \circ \pi^{-1} : \pi(W) \longrightarrow \mathbb{R}^{d_G - d_H}$$

is well defined. More generally, if $g \in G$ then $\tau \circ \varphi \circ L_{g^{-1}} \circ \pi^{-1} : \pi(gW) \longrightarrow \mathbb{R}^{d_G - d_H}$

is well-defined.



Each $gW \subseteq G$ is open, and $\bigcup_{g\in G} gW = G$, so each $\pi(gW)$ is open and $\bigcup_{g\in G} \pi(gW) = G/H$. Thus, we take the functions

$$\tau\circ\varphi\circ L_{g^{-1}}\circ\pi^{-1} : \pi(gW) \longrightarrow \mathbb{R}^{d_G-d_H}$$

for each $g\in G$ as our charts for G/H.

Last, note that if $\tau^{-1}: \mathbb{R}^{d_G - d_H} \to \mathbb{R}^{d_G}$ denotes the inclusion, then the transition functions

$$\left(\tau \circ \varphi \circ L_{g^{-1}} \circ \pi^{-1}\right) \circ \left(\tau \circ \varphi \circ L_{k^{-1}} \circ \pi^{-1}\right)^{-1} = \tau \circ \varphi \circ L_{g^{-1}k} \circ \varphi^{-1} \circ \tau^{-1}$$

are smooth.

$\boxed{\text{END}}$

<u>Lemma:</u>

Suppose a group G acts transitively on a set X. Let $x \in X$, and let $G_x = \{ g \in G \mid gx = x \}$. Then there is a bijection

$$f: G/G_x \longrightarrow X$$

<u>proof:</u>

Let $f(gG_x) = gx$. Surjectivity of f follows from transitivity, and if $gG_x \neq hG_x$ then $h^{-1}g \notin G_x$ so $h^{-1}gx \neq x$ which implies that $f(gG_x) = gx \neq hx = f(hG_x)$.

$\boxed{\text{END}}$

Examples:

① $\mathbb{Z}^n$ is a closed 0-dimensional Lie subgroup of $\mathbb{R}^n$. Therefore, $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is an $n-0=n$ - dimensional smooth manifold.

② Similar to the above, $M_{n\times n}(\mathbb{Z}) = \mathbb{Z}^{n^2}$ is discrete and hence closed in $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$. Passing to the subspace $SL_n(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$, we see that $SL_n(\mathbb{Z}) = SL_n(\mathbb{R}) \cap M_{n\times n}(\mathbb{Z})$ is discrete and hence closed in $SL_n(\mathbb{R})$. Thus, $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ is a smooth manifold of dimension $n^2-1 = (n^2-1)-(0)$.

③ Let $\mathbb{P}^{n-1}(\mathbb{R})$ be the set of lines in $\mathbb{R}^n$ passing through 0. The Lie group $GL_n(\mathbb{R})$ acts transitively on this set, and if $x \in \mathbb{P}^{n-1}(\mathbb{R})$ is the line

$x = \{(\lambda, 0, 0, \dots, 0) \mid \lambda \in \mathbb{R}\}$, then

$$GL_n(\mathbb{R})_x = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & \\ 0 & * \\ \vdots & \\ 0 & \end{array} \right) \in GL_n(\mathbb{R}) \right\}$$

$GL_n(\mathbb{R})_x$ is a Lie group, as it's an open subset of $\mathbb{R}^{n^2-(n-1)}$, and it's closed in $GL_n(\mathbb{R})$. Therefore,

$$\mathbb{P}^{n-1}(\mathbb{R}) = GL_n(\mathbb{R}) / GL_n(\mathbb{R})_x$$

is a smooth manifold of dimension $n^2 - \left(n^2 - (n-1)\right) = n-1$.

④ Let $Gr_{\mathbb{R}}(m,n)$ be the set of m-dimensional $\mathbb{R}$-vector subspaces of $\mathbb{R}^n$. (Here $m \leq n$) $GL_n(\mathbb{R})$ acts transitively on $Gr_{\mathbb{R}}(m,n)$, and if $V_m \in Gr_{\mathbb{R}}(m,n)$ is the vector subspace

$$V_m = \{(\lambda_1, \lambda_2, \dots, \lambda_m, 0, \dots, 0) \mid \lambda_i \in \mathbb{R}\}$$

then $GL_n(\mathbb{R})_{V_m} = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0_{n-m,m} & * \end{array} \right) \in GL_n(\mathbb{R}) \right\}$

is a closed Lie subgroup of $GL_n(\mathbb{R})$ of dimension $n^2 - (n-m)m$. Therefore,

$$Gr_{\mathbb{R}}(m,n) = GL_n(\mathbb{R}) \big/ GL_n(\mathbb{R})_{V_m}$$

is a smooth manifold of dimension $(n-m)m$.

$Gr_{\mathbb{R}}(m,n)$ is called a <u>Grassmanian</u>.

$Gr_{\mathbb{R}}(1,n) = \mathbb{P}^{n-1}(\mathbb{R})$.

A manifold of the form G/H where G is a Lie group and $H \leq G$ is a closed subgroup is called a <u>homogeneous space</u>.

Exercise:

① A complete flag in $\mathbb{R}^n$ is an inclusion of $\mathbb{R}$-vector subspaces of the form

$$\{0\} \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{n-1} \subsetneq \mathbb{R}^n$$

so that $\dim(W_i) = i$.

Show that the set of complete flags is a homogeneous space. What's its dimension?

This homogeneous space is called a flag manifold.

§ Exterior Algebras

A *p-tensor* on $\mathbb{R}^n$ is a function

$$\varphi : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{p\text{-times}} \longrightarrow \mathbb{R}$$

such that for $v_i, w_i \in \mathbb{R}^n$ and $a \in \mathbb{R}$ we have

$$\varphi(v_1, \ldots, v_k, v_{k+1} + w_{k+1}, v_{k+2}, \ldots, v_p)$$
$$=$$
$$\varphi(v_1, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_p) + \varphi(v_1, \ldots, v_k, w_{k+1}, v_{k+2}, \ldots, v_p)$$

and

$$\varphi(v_1, \ldots, v_k, a v_{k+1}, v_{k+2}, \ldots, v_p) = a\varphi(v_1, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_p)$$

We let $T^p(\mathbb{R}^n)$ be all the p-tensors on $\mathbb{R}^n$. With pointwise addition and scalar multiplication for real valued functions, $T^p(\mathbb{R}^n)$ is a vector space.

Example:

For $1 \le i \le n$, let $dx_i : \mathbb{R}^n \to \mathbb{R}$ be projection onto the i^{th} coordinate, so that

$$dx_i \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_i$$

Note that if $\{e_j\}_{j=1}^n$ is the standard ordered basis for $\mathbb{R}^n$, then

$$dx_i(e_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Tensor product

If $\varphi \in T^k(\mathbb{R}^n)$ and $\psi \in T^\ell(\mathbb{R}^n)$, we let $\varphi \otimes \psi \in T^{k+\ell}(\mathbb{R}^n)$ be defined by

$$\varphi \otimes \psi (v_1, \dots, v_{k+\ell}) = \varphi(v_1, \dots, v_k)\, \psi(v_{k+1}, \dots, v_{k+\ell})$$

where $v_i \in \mathbb{R}^n$.

Lemma:

$\{ dx_{i_1} \otimes \cdots \otimes dx_{i_k} \mid (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \}$ is a basis for $T^k(\mathbb{R}^n)$.

proof:

Note that

$$dx_{i_1} \otimes \cdots \otimes dx_{i_k}(e_{j_1}, \ldots, e_{j_k}) = dx_{i_1}(e_{j_1}) \cdots dx_{i_k}(e_{j_k})$$
$$= \begin{cases} 1 & \text{if } (i_1, \ldots, i_k) = (j_1, \ldots, j_k) \\ 0 & \text{otherwise} \end{cases}$$

Thus, if $a_{i_1, \ldots, i_k} \in \mathbb{R}$ and

$$\sum_{(i_1, \ldots, i_k)} a_{i_1, \ldots, i_k} dx_{i_1} \otimes \cdots \otimes dx_{i_k} = 0$$

then

$$a_{j_1, \ldots, j_k} = \Big(\sum_{(i_1, \ldots, i_k)} a_{i_1, \ldots, i_k} dx_{i_1} \otimes \cdots \otimes dx_{i_k} \Big)(e_{j_1}, \ldots, e_{j_k})$$
$$= 0(e_{j_1}, \ldots, e_{j_k})$$
$$= 0$$

for all $a_{j_1, \ldots, j_k} \in \mathbb{R}$, which is to say that our proposed basis is linearly independent.

If $\varphi \in T^k(\mathbb{R}^n)$, then for any $(i_1, \ldots, i_k)$ $\varphi(e_{i_1}, \ldots, e_{i_k}) \in \mathbb{R}$ so $\psi = \sum_{(i_1,\ldots,i_k)} \varphi(e_{i_1}, \ldots, e_{i_k}) dx_{i_1} \otimes \cdots \otimes dx_{i_k}$ is in the span of our proposed basis.

Note that for any $(j_1, \ldots, j_k)$,

$$\psi(e_{j_1}, \ldots, e_{j_k}) = \varphi(e_{j_1}, \ldots, e_{j_k}).$$

Therefore,

$$\begin{aligned}\psi\left(\sum_{i=1}^n b_i^1 e_i, \ldots, \sum_{i=1}^n b_i^k e_i\right) &= \sum_{(i_1,\ldots,i_k)} b_{i_1}^1 \cdots b_{i_k}^k \psi(e_{i_1}, \ldots, e_{i_k}) \\ &= \sum_{(i_1,\ldots,i_k)} b_{i_1}^1 \cdots b_{i_k}^k \varphi(e_{i_1}, \ldots, e_{i_k}) \\ &= \varphi\left(\sum_{i=1}^n b_i^1 e_i, \ldots, \sum_{i=1}^n b_i^k e_i\right)\end{aligned}$$

so that $\psi = \varphi$.

$\boxed{\text{END}}$

The symmetric group acts linearly on $T^k(\mathbb{R}^n)$. If $\sigma \in S_k$ and $\varphi \in T^k(\mathbb{R}^n)$ we let $\varphi^\sigma \in T^k(\mathbb{R}^n)$ be $\varphi^\sigma(v_1, \ldots, v_k) = \varphi(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(k)})$.

A tensor $\varphi \in T^k(\mathbb{R}^n)$ is <u>alternating</u> if $\varphi^\sigma = -\varphi$ for any odd $\sigma \in S_k$.

<u>Example:</u>

Regarding $v_1, \dots, v_n \in \mathbb{R}^n$ as column vectors so that $(v_1, \dots, v_n) \in M_{n \times n}(\mathbb{R})$, $\det(v_1, \dots, v_n)$ defines an alternating n-tensor $\det \in T^n(\mathbb{R}^n)$.

<u>Definitions:</u>

We let $\Lambda^k(\mathbb{R}^n)$ be the alternating k-tensors on $\mathbb{R}^n$ so that $\Lambda^k(\mathbb{R}^n)$ is a vector subspace of $T^k(\mathbb{R}^n)$.

Given $\varphi \in \Lambda^k(\mathbb{R}^n)$ and $\psi \in \Lambda^\ell(\mathbb{R}^n)$, we define the <u>wedge product</u> $\varphi \wedge \psi \in \Lambda^{k+\ell}(\mathbb{R}^n)$ as

$$\varphi \wedge \psi = \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)(\varphi \otimes \psi)^\sigma$$

where $\operatorname{sgn}(\sigma) = 1$ if $\sigma \in A_{k+\ell}$ and $\operatorname{sgn}(\sigma) = -1$ otherwise.

Lemma:

If $\varphi \in \Lambda^k(\mathbb{R}^n)$ and $\psi \in \Lambda^\ell(\mathbb{R}^n)$, then $\varphi \wedge \psi \in \Lambda^{k+\ell}(\mathbb{R}^n)$.

proof:

First note $\varphi \wedge \psi \in T^{k+\ell}(\mathbb{R}^n)$. Second, if $\tau \in S_k$ is odd, then

$$(\varphi \wedge \psi)^\tau = \left(\frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\,(\varphi \otimes \psi)^\sigma\right)^\tau$$

$$= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\,(\varphi \otimes \psi)^{\sigma\tau}$$

$$= \frac{1}{k!\,\ell!} \sum_{\sigma\tau^{-1} \in S_{k+\ell}} \operatorname{sgn}(\sigma\tau^{-1})\,(\varphi \otimes \psi)^{\sigma\tau^{-1}\tau}$$

$$= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau^{-1})\,(\varphi \otimes \psi)^{\sigma}$$

$$= \operatorname{sgn}(\tau^{-1})\,(\varphi \wedge \psi)$$

$$= -\,\varphi \wedge \psi.$$

$\boxed{\text{END}}$

Lemma:

If $\varphi \in \Lambda^k(\mathbb{R}^n)$ and $\psi \in \Lambda^\ell(\mathbb{R}^n)$, then $\varphi \wedge \psi = (-1)^{k\ell}\psi \wedge \varphi$. In particular, if k is odd, then $\varphi \wedge \varphi = 0$.

proof:

Let $\mu \in S_{k+\ell}$ be the permutation

$$\begin{array}{cc} \underbrace{1 \cdots k} & \underbrace{k+1 \cdots k+\ell} \\ \underbrace{1 \cdots \ell} & \underbrace{\ell+1 \cdots \ell+k} \end{array}$$

so that $\mathrm{sgn}(\mu) = (-1)^{k\ell}$ and $\varphi \otimes \psi = (\psi \otimes \varphi)^{\mu^{-1}}$.

Then

$$\begin{aligned}
\varphi \wedge \psi &= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma)(\varphi \otimes \psi)^\sigma \\
&= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma)(\psi \otimes \varphi)^{\mu^{-1}\sigma} \\
&= \frac{1}{k!\,\ell!} \sum_{\mu\sigma \in S_{k+\ell}} \mathrm{sgn}(\mu\sigma)(\psi \otimes \varphi)^{\mu^{-1}\mu\sigma} \\
&= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\mu)\,\mathrm{sgn}(\sigma)(\psi \otimes \varphi)^{\sigma} \\
&= \mathrm{sgn}(\mu) \frac{1}{\ell!\,k!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma)(\psi \otimes \varphi)^{\sigma} \\
&= (-1)^{k\ell}\psi \wedge \varphi.
\end{aligned}$$

$\boxed{\text{END}}$

Lemma:

If $\varphi \in \Lambda^k(\mathbb{R}^n)$, $\psi \in \Lambda^\ell(\mathbb{R}^n)$, and $\xi \in \Lambda^m(\mathbb{R}^n)$, then $\varphi \wedge (\psi \wedge \xi) = (\varphi \wedge \psi) \wedge \xi$.

proof:

$$\varphi \wedge (\psi \wedge \xi) = \varphi \wedge \left(\frac{1}{\ell!\,m!} \sum_{\sigma \in S_{\ell+m}} \mathrm{sgn}(\sigma) (\psi \otimes \xi)^\sigma \right)$$

$$= \frac{1}{k!\,(\ell+m)!} \sum_{\varepsilon \in S_{k+\ell+m}} \mathrm{sgn}(\varepsilon) \left(\varphi \otimes \left(\frac{1}{\ell!\,m!} \sum_{\sigma \in S_{\ell+m}} \mathrm{sgn}(\sigma) (\psi \otimes \xi)^\sigma \right) \right)^\varepsilon$$

$$= \frac{1}{k!\,(\ell+m)!\,\ell!\,m!} \sum_{\varepsilon \in S_{k+\ell+m}} \mathrm{sgn}(\varepsilon) \left(\sum_{\sigma \in S_{\ell+m}} \mathrm{sgn}(\sigma)\, \varphi \otimes (\psi \otimes \xi)^\sigma \right)^\varepsilon$$

$$= \frac{1}{k!\,\ell!\,m!\,(\ell+m)!} \sum_{\sigma \in S_{\ell+m}} \sum_{\varepsilon \in S_{k+\ell+m}} \mathrm{sgn}(\varepsilon\sigma) \left(\varphi \otimes (\psi \otimes \xi)^\sigma \right)^\varepsilon$$

$$= \frac{1}{k!\,\ell!\,m!\,(\ell+m)!} \sum_{\sigma \in S_{\ell+m}} \sum_{\sigma^{-1}\varepsilon \in S_{k+\ell+m}} \mathrm{sgn}(\sigma^{-1}\varepsilon\sigma) \left(\varphi \otimes (\psi \otimes \xi)^\sigma \right)^{\sigma^{-1}\varepsilon}$$

$$= \frac{1}{k!\,\ell!\,m!\,(\ell+m)!} \sum_{\sigma \in S_{\ell+m}} \sum_{\sigma^{-1}\varepsilon \in S_{k+\ell+m}} \mathrm{sgn}(\varepsilon) (\varphi \otimes \psi \otimes \xi)^\varepsilon$$

$$= \frac{1}{k!\,\ell!\,m!\,(\ell+m)!} (\ell+m)! \sum_{\varepsilon \in S_{k+\ell+m}} \mathrm{sgn}(\varepsilon) (\varphi \otimes \psi \otimes \xi)^\varepsilon$$

$$= \frac{1}{k!\,\ell!\,m!} \sum_{\varepsilon \in S_{k+\ell+m}} \mathrm{sgn}(\varepsilon) (\varphi \otimes \psi \otimes \xi)^\varepsilon.$$

Similarly, the above sum equals $(\varphi \wedge \psi) \wedge \xi$. $\boxed{\text{END}}$

Lemma:

If $\varphi \in \Lambda^k(\mathbb{R}^n)$, $\psi \in \Lambda^\ell(\mathbb{R}^n)$, $\xi \in \Lambda^\ell(\mathbb{R}^n)$, and $\lambda \in \mathbb{R}$, then $\varphi \wedge (\lambda\psi + \xi) = \lambda(\varphi \wedge \psi) + (\varphi \wedge \xi)$ and $(\lambda\psi + \xi) \wedge \varphi = \lambda(\psi \wedge \varphi) + (\xi \wedge \varphi)$.

proof:

We'll check the first equation. The second is similar, or follows from the lemma on page 236.

The first equation is linearity of left wedging with φ. But the wedge product is defined using sums, scalars, tensor product, and the $S_{k+\ell}$-action. All of these are linear, so $\varphi \wedge -$ must be linear as well.

$\boxed{\text{END}}$

We define $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$, and if $\lambda \in \mathbb{R}$ we let $\lambda \wedge \varphi = \lambda\varphi$ for $\varphi \in \Lambda^k(\mathbb{R}^n)$. With these operations, the lemmas above tell us that $\bigoplus_{k=0}^{\infty} \Lambda^k(\mathbb{R}^n)$ is a graded algebra, the exterior algebra of $\mathbb{R}^n$

For any $(i_1, i_2, \dots, i_k) \in \{1, \dots, n\}^k$, $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \in \Lambda^k(\mathbb{R}^n)$.
If $i_l = i_m$ for $l \neq m$, then

$$\begin{aligned} dx_{i_1} \wedge \cdots \wedge dx_{i_k} &= \pm\, dx_{i_l} \wedge dx_{i_m} \wedge \cdots \wedge dx_{i_k} \\ &= \pm\, 0 \wedge \cdots \wedge dx_{i_k} \\ &= 0. \end{aligned}$$

So if $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is nonzero, then the i_j are distinct, $k \leq n$, and up to scaling by -1, we may assume $i_1 < i_2 < \cdots < i_k$. For such a k-tuple, first note as in the proof of associativity of the wedge product, that

$$\begin{aligned} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} &= \frac{1}{1!1!\cdots 1!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\left(dx_{i_1} \otimes \cdots \otimes dx_{i_k}\right)^{\sigma} \\ &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\left(dx_{i_1} \otimes \cdots \otimes dx_{i_k}\right)^{\sigma} \end{aligned}$$

Second, if $A \in M_{n \times k}(\mathbb{R})$, then viewing the k-columns of A as vectors in $\mathbb{R}^n$, we have

$$
\begin{aligned}
dx_{i_1}\wedge\cdots\wedge dx_{i_k}(A) &= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\left(dx_{i_1}\otimes\cdots\otimes dx_{i_k}\right)^{\sigma}(A)\\
&= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\left(\prod_{i_1}^{i_k} A_{i_j,\sigma^{-1}(i_j)}\right)\\
&= \sum_{\sigma^{-1}\in S_k}\operatorname{sgn}(\sigma^{-1})\left(\prod_{i_1}^{i_k} A_{i_j,\sigma(i_j)}\right)\\
&= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\left(\prod_{i_1}^{i_k} A_{i_j,\sigma(i_j)}\right)\\
&= \det\left(A^{(i_1,\dots,i_k)}\right)
\end{aligned}
$$

where $A^{(i_1,\dots,i_k)}\in M_{k\times k}(\mathbb{R})$ is the matrix obtained from A by deleting the m-th rows from A if $m\notin\{i_1,\dots,i_k\}$.

In particular, $dx_1\wedge dx_2\wedge\cdots\wedge dx_n\in\Lambda^n(\mathbb{R}^n)$ is the determinant function

$$\det : M_{n\times n}(\mathbb{R})\longrightarrow\mathbb{R}.$$

Lemma:

$\{dx_{i_1}\wedge\cdots\wedge dx_{i_k} \mid i_1<\cdots<i_k\}$ is a basis for $\Lambda^k(\mathbb{R}^n)$.

proof:

Suppose $j_1<\cdots<j_k$. Then for $(e_{j_1},\ldots,e_{j_k})\in M_{n\times k}(\mathbb{R})$ and $i_1<\cdots<i_k$ we have that

$$dx_{i_1}\wedge\cdots\wedge dx_{i_k}(e_{j_1},\ldots,e_{j_k}) = \det\left((e_{j_1},\ldots,e_{j_k})^{(i_1,\ldots,i_k)}\right)$$
$$= \begin{cases} 1 & \text{if } (i_1,\ldots,i_k)=(j_1,\ldots,j_k) \\ 0 & \text{otherwise} \end{cases}$$

Thus, if for $a_{i_1,\ldots,i_k}\in\mathbb{R}$ we have

$$\sum_{i_1<\cdots<i_k} a_{i_1,\ldots,i_k}\, dx_{i_1}\wedge\cdots\wedge dx_{i_k} = 0$$

then

$$a_{j_1,\ldots,j_k} = \left(\sum_{i_1<\cdots<i_k} a_{i_1,\ldots,i_k}\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}\right)(e_{j_1},\ldots,e_{j_k})$$
$$= 0\,(e_{j_1},\ldots,e_{j_k})$$
$$= 0$$

for all $a_{j_1,\ldots,j_k}\in\mathbb{R}$, which is to say that our proposed basis is linearly independent.

If $\varphi \in \Lambda^k(\mathbb{R}^n)$, then $\varphi \in T^k(\mathbb{R}^n)$ so that $\varphi = \sum_{(i_1,\dots,i_k)} a_{i_1,\dots,i_k}\, dx_{i_1} \otimes \cdots \otimes dx_{i_k}$. Since $\varphi \in \Lambda^k(\mathbb{R}^n)$,

$$\begin{aligned}
\varphi &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2 \varphi \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \varphi^\sigma \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \left(\sum_{(i_1,\dots,i_k)} a_{i_1,\dots,i_k}\, dx_{i_1} \otimes \cdots \otimes dx_{i_k} \right)^\sigma \\
&= \frac{1}{k!} \sum_{(i_1,\dots,i_k)} a_{i_1,\dots,i_k} \left(\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \left(dx_{i_1} \otimes \cdots \otimes dx_{i_k} \right)^\sigma \right) \\
&= \frac{1}{k!} \sum_{(i_1,\dots,i_k)} a_{i_1,\dots,i_k} \left(dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) \\
&= \sum_{i_1 < \cdots < i_k} b_{i_1,\dots,i_k} \left(dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right)
\end{aligned}$$

for some $b_{i_1,\dots,i_k} \in \mathbb{R}$. Therefore, our proposed basis spans.

$\boxed{\text{END}}$

Corollary:

$$\dim\left(\Lambda^k(\mathbb{R}^n)\right) = \binom{n}{k}, \quad \Lambda^n(\mathbb{R}^n) = \mathbb{R}\cdot\det,$$

$\Lambda^\ell(\mathbb{R}^n) = 0$ if $\ell > n$, and if we denote the exterior algebra of $\mathbb{R}^n$ as $\Lambda(\mathbb{R}^n)$, then $\Lambda(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \Lambda^k(\mathbb{R}^n)$.

Definition:

For a linear function $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, let $A^* : \Lambda^1(\mathbb{R}^m) \longrightarrow \Lambda^1(\mathbb{R}^n)$ be $A^*(\varphi) = \varphi \circ A$ and $A^* : \Lambda^0(\mathbb{R}^m) \longrightarrow \Lambda^0(\mathbb{R}^n)$ be $A^*(\lambda) = \lambda$.

Having defined A^* for the generators dx_i of the algebra $\Lambda(\mathbb{R}^m)$, there is a unique algebra homomorphism $A^* : \Lambda(\mathbb{R}^m) \longrightarrow \Lambda(\mathbb{R}^n)$ extending A^* as in the previous paragraph so that, for example,

$$A^*\left(\sum_{i_1 < \cdots < i_k} a_{i_1,\ldots,i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) = \sum_{i_1 < \cdots < i_k} a_{i_1,\ldots,i_k} (dx_{i_1} \circ A) \wedge \cdots \wedge (dx_{i_k} \circ A)$$

A^* is a graded homomorphism, meaning that $A^*(\Lambda^k(\mathbb{R}^m)) \subseteq \Lambda^k(\mathbb{R}^n)$.

Lemma:

If $\varphi \in \Lambda^k(\mathbb{R}^m)$, $A: \mathbb{R}^n \to \mathbb{R}^m$ is linear, and $v_1, \dots, v_k \in \mathbb{R}^n$, then $A^*\varphi(v_1,\dots,v_k) = \varphi(Av_1,\dots,Av_k)$

proof:

It suffices to check for $\varphi = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where

$$A^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k})(v_1,\dots,v_k) = \left((dx_{i_1} \circ A) \wedge \cdots \wedge (dx_{i_k} \circ A)\right)(v_1,\dots,v_k)$$

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \left((dx_{i_1} \circ A) \otimes \cdots \otimes (dx_{i_k} \circ A)\right)^{\sigma}(v_1,\dots,v_k)$$

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \left((dx_{i_1} \circ A) \otimes \cdots \otimes (dx_{i_k} \circ A)\right)(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})$$

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) (dx_{i_1} \otimes \cdots \otimes dx_{i_k})(Av_{\sigma^{-1}(1)},\dots,Av_{\sigma^{-1}(k)})$$

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) (dx_{i_1} \otimes \cdots \otimes dx_{i_k})^{\sigma}(Av_1,\dots,Av_k)$$

$$= (dx_{i_1} \wedge \cdots \wedge dx_{i_k})(Av_1,\dots,Av_k).$$

$\boxed{\text{END}}$

<u>Lemma:</u>

If $A:\mathbb{R}^n \to \mathbb{R}^m$ and $B:\mathbb{R}^m \to \mathbb{R}^\ell$ are linear, then $(BA)^* = A^*B^*$.

<u>proof:</u>

It suffices to check for the algebra generators $dx_i \in \Lambda(\mathbb{R}^\ell)$ where $(BA)^* dx_i = dx_i \circ BA = A^*(dx_i \circ B) = A^* B^* dx_i$.

$\boxed{\text{END}}$

<u>Lemma:</u>

If $A:\mathbb{R}^n \to \mathbb{R}^n$ is linear and $\varphi \in \Lambda^n(\mathbb{R}^n)$, then $A^*\varphi = \det(A)\varphi$.

<u>proof:</u>

$\Lambda^n(\mathbb{R}^n)$ is one-dimensional with basis $dx_1 \wedge \cdots \wedge dx_n$, so it suffices to find $\lambda_A \in \mathbb{R}$ such that $A^*(dx_1 \wedge \cdots \wedge dx_n) = \lambda_A dx_1 \wedge \cdots \wedge dx_n$. Note

$$\begin{aligned}
\lambda_A &= \lambda_A \, dx_1 \wedge \cdots \wedge dx_n \, (e_1, \dots, e_n) \\
&= A^*(dx_1 \wedge \cdots \wedge dx_n)(e_1, \dots, e_n) \\
&= dx_1 \wedge \cdots \wedge dx_n \, (Ae_1, \dots, Ae_n) \\
&= \det(A).
\end{aligned}$$

$\boxed{\text{END}}$

Corollary:

If $\det : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$ is the determinant, then $\det(AB) = \det(A)\det(B)$.

proof:

For $\det \in \Lambda^n(\mathbb{R}^n)$ we have

$$\begin{aligned}\det(AB)\det &= (AB)^*\det \\ &= B^*A^*\det \\ &= B^*\det(A)\det \\ &= \det(B)\det(A)\det.\end{aligned}$$

So $\det(AB) = \det(B)\det(A)$.

$\boxed{\text{END}}$

Corollary:

If $A \in GL_n(\mathbb{R})$ and $B \in M_{n\times n}(\mathbb{R})$, then $\det(A^{-1}) = \det(A)^{-1}$ and $\det(ABA^{-1}) = \det(B)$.

Exercises:

① Write

$$\left(3dx_1\wedge dx_2+4dx_3\wedge dx_2+2dx_3\wedge dx_3\right)\wedge\left(2dx_1+3dx_2+5dx_3\right)$$

as a scalar multiple of $\det : M_{3\times 3}(\mathbb{R})\longrightarrow\mathbb{R}$.

② Let $A(x_1,x_2)=(y_1,y_2,y_3)$ be the linear function $A:\mathbb{R}^2\longrightarrow\mathbb{R}^3$ defined by $A=\begin{pmatrix}2&4\\0&3\\1&5\end{pmatrix}$. Write

$$A^*\left(2dy_1\wedge dy_2+dy_2\wedge dy_3-3dy_1\wedge dy_3\right)$$

as a scalar multiple of $dx_1\wedge dx_2$.

③ Let $I_n:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be the identity. Prove that $I_n^*:\Lambda(\mathbb{R}^n)\longrightarrow\Lambda(\mathbb{R}^n)$ is the identity.

④ If $g\in GL_n(\mathbb{R})$, show that $g^*\big|_{\Lambda^k(\mathbb{R}^n)}:\Lambda^k(\mathbb{R}^n)\to\Lambda^k(\mathbb{R}^n)$ is invertible for any k.

⑤ Suppose $\mathcal{U}\subseteq\mathbb{R}^m$, $\mathcal{U}^m$ is open and that $g:\mathcal{U}\to GL_n(\mathbb{R})$ is a smooth function. Show that $p\longmapsto(g(p))^*\big|_{\Lambda^k(\mathbb{R}^n)}$ is a smooth function $\mathcal{U}\longrightarrow GL_{\binom{n}{k}}(\mathbb{R})$.

§ Differential Forms

In the remainder, we will denote an arbitrary $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$ with $i_1 < \cdots < i_k$ as I_k, and we'll write $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ as dx_{I_k}. Thus an arbitrary element of $\Lambda^k(\mathbb{R}^n)$ is $\sum_{I_k} a_{I_k} dx_{I_k}$ where $a_{I_k} \in \mathbb{R}$ for all I_k.

Until we conclude our proof of Stokes' Theorem, we assume M is a smooth manifold with boundary.

Proposition:

Given a vector bundle $\pi_E : E \longrightarrow M$, there's a vector bundle $\pi_\Lambda : \Lambda^k(E) \to M$ for any $k \in \mathbb{Z}_{\geq 0}$ such that for all $p \in M$, $\pi_\Lambda^{-1}(p) \cong \Lambda^k(\pi_E^{-1}(p))$.

proof:

Let $h_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$ be the local trivializations for E, where $U_i \subseteq M$ and $h_j \circ h_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \longrightarrow (U_i \cap U_j) \times \mathbb{R}^n$ is given by $h_j \circ h_i^{-1}(p, v) = (p, g_{ji}(p)(v))$ for smooth $g_{ji} : U_i \cap U_j \longrightarrow GL_n(\mathbb{R})$.

Let $V_i = U_i \times \Lambda^k(\mathbb{R}^n)$ and $\Lambda^k(E) = \bigcup_i V_i \big/ \sim$

where $\left(p, \sum_{I_k}' a_{I_k} dx_{I_k}\right) \sim \left(q, \sum_{I_k}' b_{I_k} dx_{I_k}\right)$ for

$\left(p, \sum_{I_k}' a_{I_k} dx_{I_k}\right) \in V_i$ and $\left(q, \sum_{I_k}' b_{I_k} dx_{I_k}\right) \in V_j$

if $p = q$ and $\left(g_{ji}(p)\right)^* \left(\sum_{I_k}' b_{I_k} dx_{I_k}\right) = \left(\sum_{I_k}' a_{I_k} dx_{I_k}\right)$.

Define $\pi_\Lambda : \Lambda^k(E) \to M$ by

$\pi_\Lambda \left(p, \sum_{I_k}' a_{I_k} dx_{I_k}\right) = p$.

$\boxed{\text{END}}$

A <u>differential k-form</u> on M is a smooth section of $\Lambda^k(TM)$. The collection of all differential k-forms is denoted $\Omega^k(M)$, so that $\Omega^k(M) = \Gamma(\Lambda^k(TM))$ is a vector space with pointwise addition and scalor multiplication.

The $\omega \in \Omega^k(M)$ are expressed in local coordinates as $\omega(p) = \sum_{I_k}' a_{I_k}(p) dx_{I_k}$ for smooth real valued functions a_{I_k}.

Examples:

① Recall that $\Lambda^0(T_pM) = \mathbb{R}$ for all $p \in M$, so that $\Omega^0(M) = C^\infty(M)$.

② Suppose $f \in C^\infty(M) = \Omega^0(M)$. Recall that $\Lambda^1(T_pM)$ is the space of linear functionals $T_pM \longrightarrow \mathbb{R}$, for any $p \in M$. Hence, we let $df \in \Omega^1(M)$ be defined for $p \in M$ as

$$df(p) = D_pf : T_pM \longrightarrow T_{f(p)}\mathbb{R} = \mathbb{R}.$$

In local coordinates, for $\{e_1, \ldots, e_n\}$ the standard ordered basis for $\mathbb{R}^n$, and for $a_i \in \mathbb{R}$, we have

$$\begin{aligned} df(p)\left(\sum_{i=1}^n a_i e_i\right) &= D_pf\left(\sum_{i=1}^n a_i e_i\right) \\ &= \sum_{i=1}^n a_i D_pf(e_i) \\ &= \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}\Big|_p \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_p dx_i\left(\sum_{i=1}^n a_i e_i\right) \end{aligned}$$

Thus $df(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_p dx_i$ so we see that $df \in \Omega^1(M)$ is indeed smooth.

Note also that $d : \Omega^0(M) \longrightarrow \Omega^1(M)$ is linear.

③ The most important examples of differential forms with regards to integration are forms $\omega \in \Omega^n(M)$ where $n = \dim(M)$.

Recall that for any $p \in M$, $\Lambda^n(T_pM) \cong \mathbb{R}$ corresponds to scalar multiples of a determinant on T_pM. Thus, $\omega \in \Omega^n(M)$ corresponds to a smooth choice of a determinant on each tangent space of M.

Exercises:

① Let $\omega \in \Omega^2(\mathbb{R}^3)$ be defined by
$$\omega(x,y,z) = x^2\, dx \wedge dy + 3xyz\, dy \wedge dz.$$
For $\begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}0\\2\\4\end{pmatrix} \in T_{(-1,1,2)}\mathbb{R}^3$, find $\omega(-1,1,2)\left(\begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}0\\2\\4\end{pmatrix}\right)$.

② Let $\pi : E \to M$ be a smooth vector bundle. Let $M_0 \subseteq E$ be those points in E that represent the zero vector in some fiber over a point in M. We can verify using local trivializations of E that M_0 is an embedded submanifold of E, diffeomorphic to M.

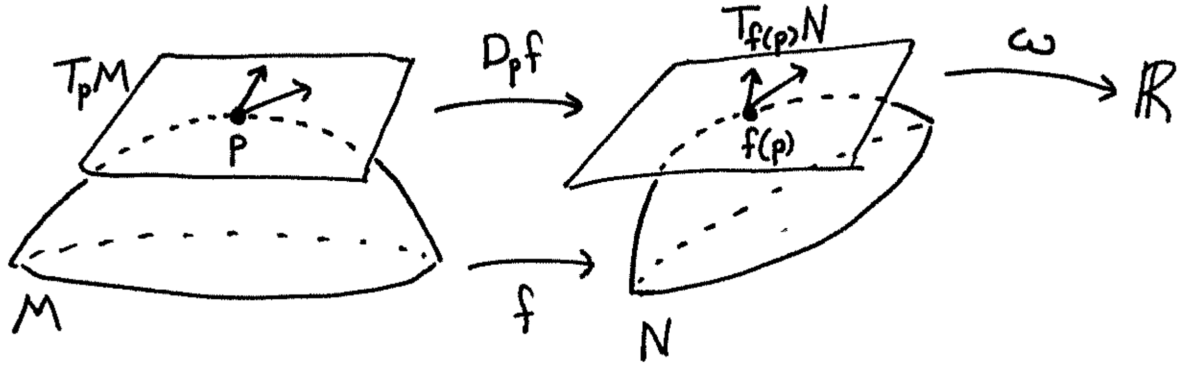
If M is a connected, smooth manifold of dimension n, prove that $\Lambda^n(TM) - M_0$ has at most 2 connected components.

§Pullbacks

Let $f: M \to N$ be a smooth map between smooth manifolds. If $\omega \in \Omega^k(N)$, we define $f^*\omega \in \Omega^k(M)$ by

$$f^*\omega(p) = (D_p f)^* \omega(f(p))$$

for $p \in M$. We call $f^*\omega$ the <u>pullback</u> of ω with respect to f.



<u>Lemma:</u>

$f^*: \Omega^k(N) \longrightarrow \Omega^k(M)$ is linear, and $f^*(\omega) \wedge f^*(\theta) = f^*(\omega \wedge \theta)$ for all $\omega \in \Omega^n(N)$ and $\theta \in \Omega^m(N)$.

proof:

Sums, scalar multiples, and wedge products of forms are defined pointwise. The lemma follows since at each point $p \in M$, $(D_p f)^* : \Lambda(T_{f(p)}N) \longrightarrow \Lambda(T_p M)$ is an algebra homomorphism.

END

Lemma:

If $f : M \longrightarrow N$ and $g : N \longrightarrow Q$ are smooth and $\omega \in \Omega^k(Q)$, then $(g \circ f)^* \omega = f^* g^* \omega$.

proof:

As above, this needs to be checked at each $p \in M$, where

$$\left(D_p (g \circ f) \right)^* = \left(D_{f(p)} g \circ D_p f \right)^* = D_p f^* \, D_{f(p)} g^*$$

by a lemma on page 245.

END

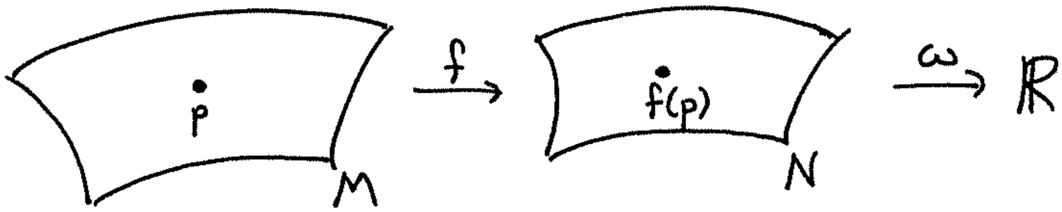
Examples:

① Let $f : M \longrightarrow N$ be smooth. Recall that $\Omega^0(N) = C^\infty(N)$ and $\Omega^0(M) = C^\infty(M)$. Let $p \in M$.

Because $(D_p f)^* : \Lambda^0(T_{f(p)}N) = \mathbb{R} \longrightarrow \Lambda^0(T_pM) = \mathbb{R}$ was defined as the identity, we have for $\omega \in \Omega^0(N)$ that

$$f^*\omega(p) = (D_pf)^*\omega(f(p)) = \omega(f(p))$$

so $f^*\omega = \omega \circ f$.



② If $U, V \subseteq \mathbb{R}^n, \mathbb{U}^n$ are open and $f : U \longrightarrow V$ is smooth, then for $p \in U$

$$f^*(dx_1 \wedge \cdots \wedge dx_n)(p) = (D_pf)^*(dx_1 \wedge \cdots \wedge dx_n)(f(p))$$
$$= \det(D_pf)(dx_1 \wedge \cdots \wedge dx_n)(p)$$

by lemma on page 245. Hence,

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = \det(D_pf)(dx_1 \wedge \cdots \wedge dx_n).$$

Furthermore, if $a \in C^\infty(V) = \Omega^0(V)$, then

$$f^*(a\, dx_1 \wedge \cdots \wedge dx_n) = f^*(a \wedge dx_1 \wedge \cdots \wedge dx_n)$$

$$= f^*(a) \wedge f^*(dx_1 \wedge \cdots \wedge dx_n)$$

$$= (a \circ f) \wedge \det(D_p f)(dx_1 \wedge \cdots \wedge dx_n)$$

$$= (a \circ f) \det(D_p f)(dx_1 \wedge \cdots \wedge dx_n).$$

Exercises:

① Let $f:\mathbb{R}^2 \to \mathbb{R}^3$ be $f(x,y) = (x, x^2, xy)$ and let $\omega \in \Omega^2(\mathbb{R}^3)$ be

$\omega(x,y,z) = 2y\,dx \wedge dz + 3xz\,dy \wedge dz.$

Find $f^*\omega \in \Omega^2(\mathbb{R}^2)$.

② Let $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be smooth. Show that if $\omega \in \Omega^k(\mathbb{R}^m)$, then $f^*\omega$ is smooth, that is that $f^*\omega \in \Omega^k(\mathbb{R}^n)$.

Since smoothness is a local condition, it follows that if $f:M \to N$ is smooth and $\omega \in \Omega^k(N)$, then $f^*\omega \in \Omega^k(M)$.

§ Exterior differentiation

Let $U \subseteq \mathbb{R}^n, \mathbb{H}^n$ be open, and let $\omega \in \Omega^k(U)$ so that $\omega = \sum_{I_k}' a_{I_k} dx_{I_k}$ for $a_{I_k} \in C^\infty(U) = \Omega^0(U)$.

Recall that we defined $d(a_{I_k}) \in \Omega^1(U)$ by $d(a_{I_k})(p) = D_p a_{I_k}$ for all $p \in U$.

Now we define $d\omega \in \Omega^{k+1}(U)$ by

$$d\omega = \sum_{I_k}' d(a_{I_k}) \wedge dx_{I_k}.$$

Proposition:

1. $d : \Omega^k(U) \longrightarrow \Omega^{k+1}(U)$ is linear for all k.
2. If $\omega \in \Omega^k(U)$ and $\theta \in \Omega^\ell(U)$, then
$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$$
3. $d \circ d : \Omega^k(U) \longrightarrow \Omega^{k+2}(U)$ is 0.

proof:

① $$d\left(\sum_{I_k}' a_{I_k}\, dx_{I_k} + \lambda \sum_{I_k}' b_{I_k}\, dx_{I_k}\right)$$

$$= d\left(\sum_{I_k}' (a_{I_k} + \lambda b_{I_k})\, dx_{I_k}\right)$$

$$= \sum_{I_k}' d(a_{I_k} + \lambda b_{I_k}) \wedge dx_{I_k}$$

$$= \sum_{I_k}' (da_{I_k} + \lambda\, db_{I_k}) \wedge dx_{I_k}$$

$$= \sum_{I_k}' da_{I_k} \wedge dx_{I_k} + \lambda \sum_{I_k}' db_{I_k} \wedge dx_{I_k}$$

$$= d\left(\sum_{I_k}' a_{I_k}\, dx_{I_k}\right) + \lambda\, d\left(\sum_{I_k}' b_{I_k}\, dx_{I_k}\right).$$

② $$d\left(\left(\sum_{I_k}' a_{I_k}\, dx_{I_k}\right) \wedge \left(\sum_{I_\ell}' b_{I_\ell}\, dx_{I_\ell}\right)\right)$$

$$= d\left(\sum_{I_k}' \sum_{I_\ell}' a_{I_k} b_{I_\ell}\, dx_{I_k} \wedge dx_{I_\ell}\right)$$

$$= \sum_{I_k}' \sum_{I_\ell}' d(a_{I_k} b_{I_\ell}) \wedge dx_{I_k} \wedge dx_{I_\ell}$$

$$= \sum_{I_k}' \sum_{I_\ell}' (a_{I_k}\, db_{I_\ell} + b_{I_\ell}\, da_{I_k}) \wedge dx_{I_k} \wedge dx_{I_\ell}$$

$$= \sum_{I_k}' \sum_{I_\ell}' a_{I_k} db_{I_\ell} \wedge dx_{I_k} \wedge dx_{I_\ell} + \sum_{I_k}' \sum_{I_\ell}' b_{I_\ell} da_{I_k} \wedge dx_{I_k} \wedge dx_{I_\ell}$$

$$= (-1)^k \sum_{I_k}' \sum_{I_\ell}' a_{I_k} dx_{I_k} \wedge db_{I_\ell} \wedge dx_{I_\ell} + \sum_{I_k}' \sum_{I_\ell}' da_{I_k} \wedge dx_{I_k} \wedge b_{I_\ell} dx_{I_\ell}$$

$$= (-1)^k \left(\sum_{I_k}' a_{I_k} dx_{I_k}\right) \wedge \left(\sum_{I_\ell}' db_{I_\ell} \wedge dx_{I_\ell}\right) + \left(\sum_{I_k}' da_{I_k} \wedge dx_{I_k}\right) \wedge \left(\sum_{I_\ell}' b_{I_\ell} dx_{I_\ell}\right)$$

$$= (-1)^k \left(\sum_{I_k}' a_{I_k} dx_{I_k}\right) \wedge d\left(\sum_{I_\ell}' b_{I_\ell} dx_{I_\ell}\right) + d\left(\sum_{I_k}' a_{I_k} dx_{I_k}\right) \wedge \left(\sum_{I_\ell}' b_{I_\ell} dx_{I_\ell}\right)$$

③ $$d\left(d\left(\sum_{I_k}' a_{I_k} dx_{I_k}\right)\right) = d\left(\sum_{I_k}' da_{I_k} \wedge dx_{I_k}\right)$$

$$= d\left(\sum_{I_k}' \left(\sum_{i=1}^{n} \frac{\partial a_{I_k}}{\partial x_i} dx_i\right) \wedge dx_{I_k}\right)$$

$$= d\left(\sum_{I_k}' \sum_{i=1}^{n} \frac{\partial a_{I_k}}{\partial x_i} dx_i \wedge dx_{I_k}\right)$$

$$= \sum_{I_k}' \sum_{i=1}^{n} d\left(\frac{\partial a_{I_k}}{\partial x_i}\right) \wedge dx_i \wedge dx_{I_k}$$

$$= \sum_{I_k}' \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 a_{I_k}}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_{I_k}$$

$$= \sum_{I_k}' \left(\sum_{i=1}^{n} \frac{\partial^2 a_{I_k}}{\partial x_i \partial x_i} dx_i \wedge dx_i + \sum_{i<j} \left(\frac{\partial^2 a_{I_k}}{\partial x_j \partial x_i} - \frac{\partial^2 a_{I_k}}{\partial x_j \partial x_i}\right) dx_j \wedge dx_i\right) \wedge dx_{I_k}$$

$$= \sum_{I_k}' 0 \wedge dx_{I_k}$$

$$= 0.$$

$\boxed{\text{END}}$

Lemma:

Let $U, V \subseteq \mathbb{R}^n, \cancel{U}^n$ be open and suppose $g: U \to V$ is smooth. Then $d \circ g^* = g^* \circ d$.

proof:

If $a \in \Omega^0(V)$ and $p \in U$ then

$$d \circ g^*(a)(p) = d(a \circ g)(p) = D_p(a \circ g) = D_{g(p)} a \circ D_p g$$
$$= (D_p g)^* D_{g(p)} a = (D_p g)^* da(g(p)) = g^* \circ d(a)(p)$$

so $d \circ g^*(a) = g^* \circ d(a)$. Also, $dg^* dx_{I_k} = 0$ by exercise #2.

Thus, for $\sum_{I_k}' a_{I_k} dx_{I_k} \in \Omega^k(V)$

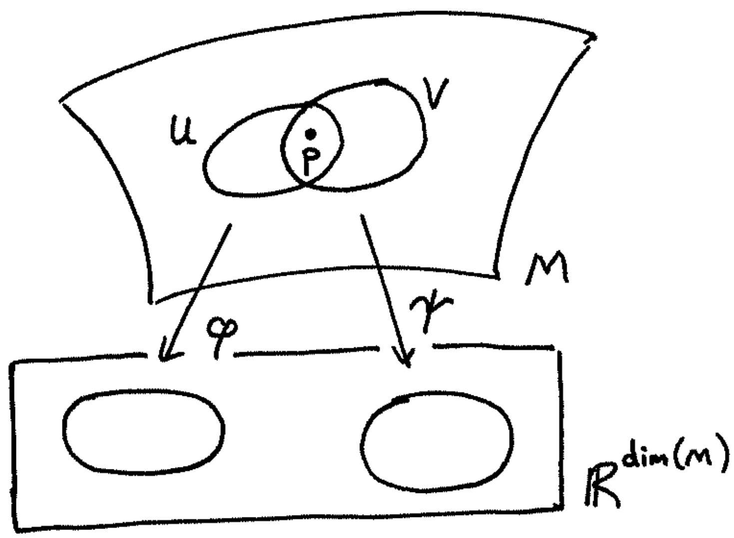
$$d \circ g^*\Big(\sum_{I_k}' a_{I_k} dx_{I_k}\Big) = \sum_{I_k}' d \circ g^*(a_{I_k} \wedge dx_{I_k})$$
$$= \sum_{I_k}' d(g^* a_{I_k} \wedge g^* dx_{I_k})$$
$$= \sum_{I_k}' \Big(dg^* a_{I_k} \wedge g^* dx_{I_k} + (-1)^0 g^* a_{I_k} \wedge d g^* dx_{I_k}\Big)$$
$$= \sum_{I_k}' g^* da_{I_k} \wedge g^* dx_{I_k} + g^*$$
$$= g^*\Big(\sum_{I_k}' da_{I_k} \wedge dx_{I_k}\Big)$$
$$= g^* \circ d\Big(\sum_{I_k}' a_{I_k} dx_{I_k}\Big).$$

$\boxed{\text{END}}$

Let M be a smooth manifold with boundary and charts $\{(\varphi, U)\}$. We define for $\omega \in \Omega^k(M)$ $d\omega \in \Omega^{k+1}(M)$ by

$$d\omega(p) = \varphi^* d(\varphi^{-1})^* \omega(p)$$

if $p \in U$.



This is well-defined: By the previous lemma

$$\begin{aligned}\psi^* d(\psi^{-1})^* \omega(p) &= \psi^* (\varphi \circ \psi^{-1})^* (\psi \circ \varphi^{-1})^* d(\psi^{-1})^* \omega(p) \\ &= \varphi^* d(\psi \circ \varphi^{-1})^* (\psi^{-1})^* \omega(p) \\ &= \varphi^* d(\varphi^{-1})^* \omega(p).\end{aligned}$$

Proposition:

If M is a smooth manifold with boundary, then for all k, $d:\Omega^k(M)\longrightarrow\Omega^{k+1}(M)$ is linear, if $\omega\in\Omega^k(M)$ and $\theta\in\Omega^\ell(M)$ then $d(\omega\wedge\theta)=d\omega\wedge\theta+(-1)^k\omega\wedge d\theta$, $d\circ d=0$, and if $f\in\Omega^0(M)$ then $df(p)=D_pf$ where $p\in M$.

proof:

The proposition holds for $d:\Omega^k(\mathbb{R}^n)\rightarrow\Omega^{k+1}(\mathbb{R}^n)$, and $d:\Omega^k(M)\longrightarrow\Omega^{k+1}(M)$ is defined locally as $d:\Omega^k(\mathbb{R}^n)\longrightarrow\Omega^{k+1}(\mathbb{R}^n)$.

$\boxed{\text{END}}$

Lemma:

Suppose $g:M\longrightarrow N$ is smooth. Then $d\circ g^*=g^*\circ d$.

proof:

Let (φ,U) be a chart on M containing

$p \in M$, and let (ψ, V) be a chart on N containing $g(p) \in N$.

If $\omega \in \Omega^k(N)$, then by the lemma on page 261 applied to the smooth map $\psi \circ g \circ \varphi^{-1}$, we have for $p \in M$

$$\begin{aligned}
dg^*\omega(p) &= \varphi^* d(\varphi^{-1})^* g^*\omega(p) \\
&= \varphi^* d(\varphi^{-1})^* g^* \psi^* (\psi^{-1})^* \omega(p) \\
&= \varphi^* d(\psi \circ g \circ \varphi^{-1})^* (\psi^{-1})^* \omega(p) \\
&= \varphi^* (\psi \circ g \circ \varphi^{-1})^* d(\psi^{-1})^* \omega(p) \\
&= g^* \psi^* d(\psi^{-1})^* \omega(p) \\
&= g^* d\omega(p)
\end{aligned}$$

so $d \circ g^* = g^* \circ d$.

$\boxed{\text{END}}$

Exercises:

① Let $\omega \in \Omega^2(\mathbb{R}^3)$ be

$$\omega(x,y,z) = 2\,dx \wedge dy + x^2\,dy \wedge dz + yz\,dx \wedge dz.$$

Find $d\omega \in \Omega^3(\mathbb{R}^3)$.

② Let $U, V \subseteq \mathbb{R}^n, \mathbb{H}^n$ be open and suppose $g: U \to V$ is smooth. Prove $dg^* dx_{I_k} = 0$, as was used in the lemma on page 261.

§ Partitions of Unity

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of a smooth manifold M. Then there are smooth functions $\{\rho_\beta : M \longrightarrow [0,1]\}_{\beta \in B}$ such that

(i) For any $p \in M$, $\left|\{\rho_\beta \mid \rho_\beta(p) \neq 0\}\right| < \infty$.

(ii) For any $p \in M$, $\sum_\beta \rho_\beta(p) = 1$.

(iii) For any β, there is some α such that $\rho_\beta(M - U_\alpha) = \{0\}$.

proof:

Let $M = \bigcup_{i=1}^{\infty} K_i$ for compact $K_i \subseteq K_{i+1} \subseteq M$. We may assume that for each i, $K_{i-2} \subseteq \mathrm{Int}(K_{i-1})$, or else that M is compact, in which case we replace $\{K_i\}_{i=1}^{\infty}$ with $K_1 = M$, and then the below proof applies formally.

We let $K_{-1} = K_0 = \emptyset$. For each $\alpha \in A$ and $p \in U_\alpha - K_{i-2}$, there is by the second lemma on page 95 some smooth $g_{\alpha,p} : M \to [0,1]$ and some open $B_{\alpha,p} \subseteq M$ containing p such that $g_{\alpha,p}(M - (U_\alpha - K_{i-2})) = \{0\}$ and $g_{\alpha,p}(B_{\alpha,p}) = \{1\}$.

Note that $\overline{B}_{\alpha,p} \subseteq U_\alpha - K_{i-2}$ and that $\bigcup_{\alpha,p} B_{\alpha,p}$ covers the set $\bigcup_{\alpha \in A}(U_\alpha - K_{i-2}) = M - K_{i-2}$, which contains the compact set $K_i - \mathrm{Int}(K_{i-1})$, thus there is a finite subcover $\{B_1^i, \ldots, B_{N_i}^i\} \subseteq \{B_{\alpha,p}\}_{\alpha,p}$ such that $\bigcup_k B_k^i$ contains $K_i - \mathrm{Int}(K_{i-1})$. Thus, $\bigcup_{i,k} B_k^i = M$.

For each B_k^i, we let $\alpha(i,k)$ and $p(i,k)$ be such that $B_k^i = B_{\alpha(i,k),p(i,k)}$, and we let $g_k^i = g_{\alpha(i,k),p(i,k)}$ so that $g_k^i(B_k^i) = \{1\}$ and $g_k^i((M - U_{\alpha(i,k)}) \cup K_{i-2}) = g_k^i(M - (U_{\alpha(i,k)} - K_{i-2})) = \{0\}$.

For $p \in M$, let $C_p \in \mathbb{N}$ be such that $p \in K_{i-2}$ when $i-2 \geq C_p$. Then $g^i_k(p) = 0$ if $i \geq C_p + 2$, so that

$$\left|\{g^i_k \mid g^i_k(p) \neq 0\}\right| \leq \left|\{g^i_k \mid i < C_p + 2\}\right| = \sum_{i=1}^{C_p+1} N_i < \infty.$$

Therefore, $\sum_{i,k} g^i_k : M \longrightarrow \mathbb{R}_{\geq 0}$ is a well-defined, smooth function that is nowhere 0 since any $p \in M$ is contained in some B^i_k.

For any i_0, k_0, we let $\rho_{i_0,k_0} : M \to [0,1]$ be

$$\rho_{i_0,k_0}(p) = \frac{g^{i_0}_{k_0}(p)}{\sum_{i,k} g^i_k(p)}$$

Then,

(i) $\left|\{\rho_{i_0,k_0} \mid \rho_{i_0,k_0}(p) \neq 0\}\right| \leq \sum_{i=1}^{C_p+1} N_i < \infty$.

(ii) $\sum_{i_0,k_0} \rho_{i_0,k_0}(p) = \dfrac{\sum_{i_0,k_0} g^{i_0}_{k_0}(p)}{\sum_{i,k} g^i_k(p)} = 1$.

(iii) For any i_0, k_0, $\rho_{i_0,k_0}(M - U_{\alpha(i_0,k_0)}) = \{0\}$. $\boxed{\text{END}}$

§ Orientability

Let V be an $\mathbb{R}$-vector space of dimension n. An orientation on V is a connected component of $\Lambda^n V - \{0\}$.

If M is a connected, smooth manifold of dimension n, then with $M_0 \subseteq \Lambda^n(TM)$ as on page 252, $\Lambda^n(TM) - M_0$ has exactly 1 or exactly 2 connected components. If it has 2, we say M is orientable. A choice of component of $\Lambda^n(TM) - M_0$ for orientable M is called an orientation. If an orientation is chosen, M is oriented.

We'll see soon that an orientation on M is a smooth choice of an orientation on each vector space T_pM, for $p \in M$.

A nonconnected manifold is oriented if all its components are.

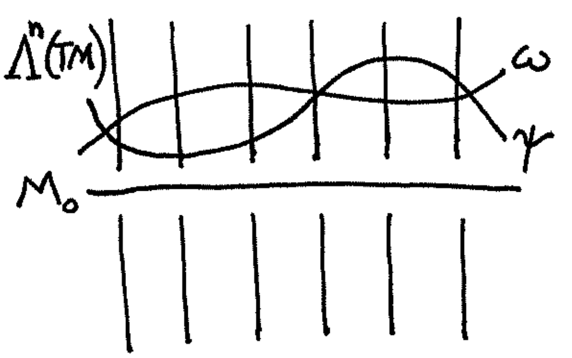
Let $\omega \in \Omega^n(M)$. Note that ω is nonvanishing, that is $\omega(p) \neq 0$ for all $p \in M$, if and only if $\omega(M) \subseteq \Lambda^n(TM) - M_0$.

If $\omega \in \Omega^n(M)$ is nonvanishing and M is connected, then $\omega(M)$ is contained in a connected component of $\Lambda^n(TM) - M_0$.

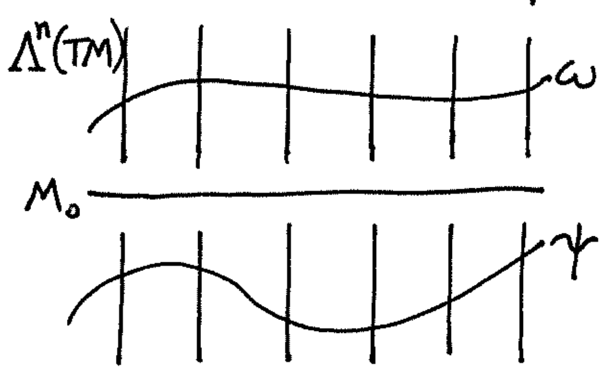
Lemma:

Suppose M is a smooth, connected, orientable manifold of dimension n, and that $\omega, \psi \in \Omega^n(M)$ are nonvanishing.

The same component of $\Lambda^n(TM) - M_0$ contains $\omega(M)$ and $\psi(M)$ if and only if $\omega = \lambda \psi$ for some $\lambda \in C^\infty(M)$ with $\lambda(p) > 0$ for all $p \in M$.



ω and ψ are pointwise positive multiples each other



ω and ψ are pointwise negative multiples of each other

<u>proof:</u>

This can be checked locally. If $U \subseteq M$ is a chart in a local trivialization for $\Lambda^n(TM)$, then for the bundle map $\pi : \Lambda^n(TM) \to M$ we have $\pi^{-1}(U) = U \times \mathbb{R}$, so that $\pi^{-1}(U)$ intersected with a connected component of $\Lambda^n(TM) - M_0$ is identified with either $U \times \mathbb{R}_{>0}$ or $U \times \mathbb{R}_{<0}$.

Locally, $\omega = a\,dx_1 \wedge \cdots \wedge dx_n$ and $\gamma = b\,dx_1 \wedge \cdots \wedge dx_n$ for nonvanishing $a, b \in C^\infty(U)$, so $\omega = \left(\frac{a}{b}\right)\gamma$ with $a/b \in C^\infty(U)$. Furthermore, $\left(\frac{a}{b}\right)(p) > 0$ for all $p \in M$ if and only if a and b are both negative, or both positive.

$\boxed{\text{END}}$

Proposition:

Let M be a smooth, connected manifold of dimension n. The following are equivalent.

I $\Lambda^n(TM) - M_0$ has 2 connected components.

II There are charts $\{(\hat{\varphi}, U)\}$ for M such that $\det\left(D_{\hat{\psi}(p)}(\hat{\varphi}\circ\hat{\psi}^{-1})\right) > 0$ for any $p \in M$ and any charts $\hat{\varphi}, \hat{\psi}$ with p in their domain.

III There is a nonvanishing $\omega \in \Omega^n(M)$.

IV $\Lambda^n(TM)$ is trivial.

proof:

III and IV are equivalent by 3, page 93.
IV implies I since $\Lambda^n(TM)$ being trivial means $\Lambda^n(TM) - M_0 = (M \times \mathbb{R}) - (M \times \{0\}) = M \times \mathbb{R}_{>0} \amalg M \times \mathbb{R}_{<0}$.

We are left to prove that I implies II and that II implies III.

To check I implies II, let $\mathcal{O}$ be a connected component of $\Lambda^n(TM) - M_0$. Given charts $\{(\varphi, U)\}$ for M, we can perhaps compose each φ with $\mathbb{R}^n \to \mathbb{R}^n$ given by $(x_1, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$ to create charts $\{(\hat{\varphi}, U)\}$ with $\hat{\varphi}^* dx_1 \wedge \cdots \wedge dx_n(U) \subseteq \mathcal{O}$.

With two such charts $(\hat{\varphi}, U)$ and $(\hat{\psi}, V)$, and with $p \in U \cap V$, $\hat{\varphi}^* dx_1 \wedge \cdots \wedge dx_n(U \cap V) \subseteq \mathcal{O}$ and $\hat{\psi}^* dx_1 \wedge \cdots \wedge dx_n(U \cap V) \subseteq \mathcal{O}$, so that each are contained in the same component of $\Lambda^n(U \cap V) - (U \cap V)_0$. Hence, there is some $\lambda \in C^\infty(U \cap V)$ with $\lambda > 0$ and $\lambda \hat{\varphi}^* dx_1 \wedge \cdots \wedge dx_n = \hat{\psi}^* dx_1 \wedge \cdots \wedge dx_n$.

Thus,

$$
\begin{aligned}
(dx_1 \wedge \cdots \wedge dx_n)(\hat{\psi}(p)) &= \left((\hat{\psi}^{-1})^* \lambda \hat{\varphi}^* dx_1 \wedge \cdots \wedge dx_n\right)(\hat{\psi}(p)) \\
&= (\lambda \circ \hat{\psi}^{-1})(\hat{\psi}(p)) \, (\hat{\varphi} \circ \hat{\psi}^{-1})^* (dx_1 \wedge \cdots \wedge dx_n)(\hat{\psi}(p)) \\
&= \lambda(p) \det\left(D_{\hat{\psi}(p)}(\hat{\varphi} \circ \hat{\psi}^{-1})\right)(dx_1 \wedge \cdots \wedge dx_n)(\hat{\psi}(p))
\end{aligned}
$$

Therefore $\lambda(p) \det\left(D_{\hat{\psi}(p)}(\hat{\varphi} \circ \hat{\psi}^{-1})\right) = 1$, so $\det\left(D_{\hat{\psi}(p)}(\hat{\varphi} \circ \hat{\psi}^{-1})\right) = \frac{1}{\lambda(p)} > 0$, proving II.

To check II implies III, let $\{(\hat{\varphi}, U)\}$ be charts for M with $\det\left(D_{\hat{\psi}(p)}(\hat{\varphi}\circ\hat{\psi}^{-1})\right)>0$. Let $\{\rho_\beta : M \longrightarrow [0,1]\}$ be a partition of unity subordinate to the cover of M by charts. That is, for any $p\in M$, $\left|\{\rho_\beta \mid \rho_\beta(p)\neq 0\}\right|<\infty$, $\sum_\beta \rho_\beta(p)=1$, and for any β, there is some chart $(\hat{\varphi}_\beta, U_\beta)$ with $\rho_\beta(M-U_\beta)=\{0\}$.

Since $\rho_\beta(M-U_\beta)=\{0\}$, $\rho_\beta\hat{\varphi}_\beta^* dx_1\wedge\cdots\wedge dx_n$ is a well-defined n-form on M, and we let $\omega=\sum_\beta \rho_\beta\hat{\varphi}_\beta^* dx_1\wedge\cdots\wedge dx_n \in \Omega^n(M)$.

For any $p\in M$, let $(\hat{\psi}, V)$ be a chart containing p. Regarding ω as its restriction to V, we have

$$(\hat{\psi}^{-1})^*\omega(\hat{\psi}(p)) = (\hat{\psi}^{-1})^*\Big(\sum_\beta \rho_\beta\, \hat{\varphi}_\beta^*\, dx_1\wedge\cdots\wedge dx_n\Big)(\hat{\psi}(p))$$

$$= \Big(\sum_\beta (\hat{\psi}^{-1})^*\rho_\beta\, \hat{\varphi}_\beta^*\, dx_1\wedge\cdots\wedge dx_n\Big)(\hat{\psi}(p))$$

$$= \Big(\sum_\beta (\rho_\beta\circ\hat{\psi}^{-1})(\hat{\psi}(p))\,(\hat{\varphi}_\beta\circ\hat{\psi}^{-1})^* dx_1\wedge\cdots\wedge dx_n\Big)(\hat{\psi}(p))$$

$$= \sum_\beta \rho_\beta(p)\det\big(D_{\hat{\psi}(p)}(\hat{\varphi}_\beta\circ\hat{\psi}^{-1})\big)\,dx_1\wedge\cdots\wedge dx_n(\hat{\psi}(p))$$

Since $\det\big(D_{\hat{\psi}(p)}(\hat{\varphi}_\beta\circ\hat{\psi}^{-1})\big)>0$ for all β, $\rho_\beta(p)\geq 0$ for all β, and $\rho_\beta(p)>0$ for at least one β (because $\sum_\beta \rho_\beta(p)=1$), we have that

$$\sum_\beta \rho_\beta(p)\det\big(D_{\hat{\psi}(p)}(\hat{\varphi}_\beta\circ\hat{\psi}^{-1})\big)>0$$

implying that $(\hat{\psi}^{-1})^*\omega(\hat{\psi}(p))\neq 0$, and because $\hat{\psi}$ is a locall diffeomorphism near p, $\omega(p)\neq 0$. That is, ω is nonvanishing, proving III.

$\boxed{\text{END}}$

Suppose M is a smooth, connected, oriented manifold of dimension n, with chosen connected component $\mathcal{O}_M^+ \subseteq \Lambda^n(TM) - M_0$. We say a nonvanishing $\omega \in \Omega^n(M)$ <u>determines the orientation</u> of M if $\omega(M) \subseteq \mathcal{O}_M^+$. Thus, ω determines the orientation if and only if $\lambda\omega$ does for any $\lambda \in C^\infty(M)$ with $\lambda > 0$.

We say $f \in \mathrm{Diff}(M)$ <u>preserves orientation</u> if for any $\omega \in \Omega^n(M)$ with $\omega(M) \subseteq \mathcal{O}_M^+$, we have $f^*\omega(M) \subseteq \mathcal{O}_M^+$. By the above paragraph, it suffices to check whether $f^*\omega(M) \subseteq \mathcal{O}_M^+$ for a single ω. Indeed, if $\lambda \in C^\infty(M)$ with $\lambda > 0$, then $f^*\lambda\omega = (\lambda \circ f) f^*\omega$ with $\lambda \circ f \in C^\infty(M)$ and $\lambda \circ f > 0$.

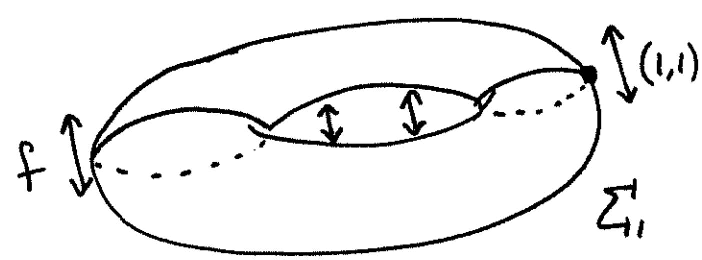
We say $f \in \mathrm{Diff}(M)$ <u>reverses orientation</u> if it does not preserve orientation. Equivalently, if for any (or for a single) $\omega \in \Omega^n(M)$ that determines the orientation of M, $f^*\omega$ does not determine the orientation of M.

Examples:

① $\det \in \Omega^n(\mathbb{R}^n)$ is nonvanishing, so $\mathbb{R}^n$ is an orientable manifold. If $g \in GL_n(\mathbb{R}) \leq \mathrm{Diff}(\mathbb{R}^n)$ then $g^*\det = \det(g)\det$ so that $\det$ and $g^*\det$ determine the same orientation of $\mathbb{R}^n$ if and only if $\det(g) > 0$. That is, g preserves orientation if and only if $\det(g) > 0$.

② Let $\Sigma_1 = \{(e^{i\theta}, e^{i\varphi}) \in \mathbb{C}^2 \mid \theta, \varphi \in \mathbb{R}\}$. Let $f \in \mathrm{Diff}(\Sigma_1)$ be $f(e^{i\theta}, e^{i\varphi}) = (e^{i\theta}, \overline{e^{i\varphi}})$.

We'll see in the next example that Σ_1 is orientable. Assuming that for now, in local coordinates near $(1,1) \in \Sigma_1$ we have $f^*(dx \wedge dy) = dx \wedge -dy = -dx \wedge dy$, so f reverses the orientation of Σ_1.



③ If G is a connected Lie group of dimension n, then any $\psi \in \Lambda^n(\mathfrak{g})$ determines an $\omega \in \Omega^n(G)$ by $\omega(g) = (D_g L_{g^{-1}})^* \psi$ for all $g \in G$. If $\psi \neq 0$, then ω is nonvanishing, so G is orientable.

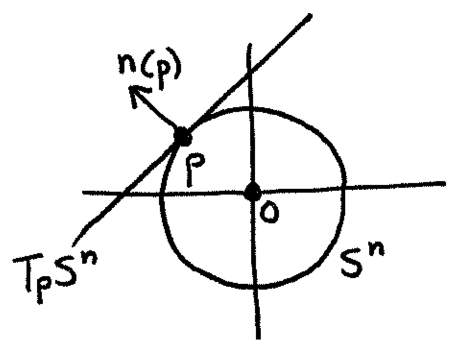
ω is left-invariant, meaning $(L_h)^* \omega = \omega$ for all $h \in G$. Indeed, if $g \in G$ then

$$
\begin{aligned}
(L_h)^* \omega(g) &= (D_g L_h)^* \omega(hg) \\
&= (D_g L_h)^* (D_{hg} L_{g^{-1}h^{-1}})^* \psi \\
&= (D_{hg} L_{g^{-1}h^{-1}} \circ D_g L_h)^* \psi \\
&= (D_g L_{g^{-1}})^* \psi \\
&= \omega(g).
\end{aligned}
$$

For $\psi \neq 0$, ω determines the orientation of G if and only if $\omega = (L_h)^* \omega$ does. Therefore, $L_h \in \mathrm{Diff}(G)$ preserves orientation of G for all $h \in G$.

④ Let $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ and let $a \in \mathrm{Diff}(\mathbb{R}^n)$ be $a(x) = -x$. Then a restricts to the antipodal map $a \in \mathrm{Diff}(S^n)$.

For $p \in S^n$, let $n(p) \in T_p\mathbb{R}^{n+1}$ be the unit norm vector orthogonal to T_pS^n, and pointing outward from $0 \in \mathbb{R}^{n+1}$.



Then $T_p\mathbb{R}^{n+1} = T_pS^n \oplus \mathbb{R}n(p)$ and $dn(p) \in \Lambda^1(T_p\mathbb{R}^{n+1})$ for $p \in S^n$.

Define $\omega \in \Omega^n(S^n)$ by requiring for each $p \in S^n$ that $(\omega \wedge dn)(p) = \det_{\mathbb{R}^{n+1}}(p)$. Then ω is nonvanishing, so S^n is orientable.

Note that $D_pa(T_pS^n) = T_{-p}S^n$, and that $D_pa(n(p)) = n(-p)$ so that $a^*dn(p) = dn(p)$ for $p \in S^n$.

Therefore, for $p \in S^n$,

$$
\begin{aligned}
(-1)^{n+1} \det_{\mathbb{R}^{n+1}}(p) &= \det(a) \det_{\mathbb{R}^{n+1}}(p) \\
&= \det(D_p a) \det_{\mathbb{R}^{n+1}}(p) \\
&= a^* \det_{\mathbb{R}^{n+1}}(p) \\
&= a^*(\omega \wedge dn)(p) \\
&= (a^*\omega \wedge a^* dn)(p) \\
&= (a^*\omega \wedge dn)(p).
\end{aligned}
$$

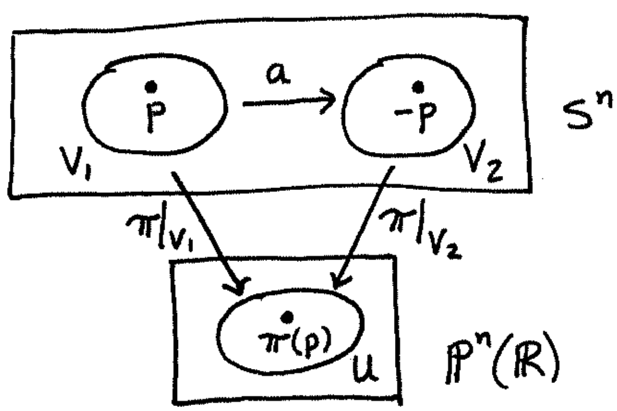
Thus, $\det_{\mathbb{R}^{n+1}}(p) = \left((-1)^{n+1} a^*\omega \wedge dn\right)(p)$, so by the definition of ω, $\omega = (-1)^{n+1} a^*\omega$. That is, ω and $a^*\omega$ are pointwise positive multiples in $\Omega^n(S^n)$ if and only if n is odd. That is, the antipodal map on S^n preserves orientation if and only if n is odd.

⑤ Let $\omega \in \Omega^n(S^n)$ and $a \in \mathrm{Diff}(S^n)$ be as in ④. If n is odd, then $a^*\omega = \omega$.

Let $\pi : S^n \longrightarrow {}_{\langle a \rangle}\backslash S^n = \mathbb{P}^n(\mathbb{R})$ be the quotient map. For any $p \in S^n$, we define $\psi \in \Omega^n(\mathbb{P}^n(\mathbb{R}))$ locally about $\pi(p)$ by taking an open neighborhood $U \subseteq \mathbb{P}^n(\mathbb{R})$ such that $\pi^{-1}(U) = V_1 \amalg V_2$ with $a(V_1) = V_2$ and $\pi|_{V_i} : V_i \longrightarrow U$ is a diffeomorphism for both i. Then we let $\psi(\pi(p)) = (\pi|_{V_i}^{-1})^*\omega$. This is well-defined since

$$(\pi|_{V_2}^{-1})^*\omega = (a \circ \pi|_{V_1}^{-1})^*\omega = (\pi|_{V_1}^{-1})^* a^*\omega = (\pi|_{V_1}^{-1})^*\omega.$$

Since $\pi|_{V_i}$ is a diffeomorphism and $\omega(p) \neq 0$, $\psi(\pi(p)) \neq 0$, so $\mathbb{P}^n(\mathbb{R})$ is orientable for odd n.



⑥ Let $\omega \in \Omega^n(S^n)$ and $a \in \mathrm{Diff}(S^n)$ be as in ④. If n is even, then $a^*\omega = -\omega$.

Let $\pi : S^n \longrightarrow \mathbb{P}^n(\mathbb{R})$ be as in ⑤. We claim $\mathbb{P}^n(\mathbb{R})$ is not orientable if n is even. Indeed, for the sake of a contradiction, assume $\psi \in \Omega^n(\mathbb{P}^n(\mathbb{R}))$ is nonvanishing. Then $\pi^*\psi \in \Omega^n(S^n)$ is nonvanishing since π is a local diffeomorphism.

By perhaps replacing ψ with $-\psi$, we may assume that there is some $\lambda \in C^\infty(S^n)$ with $\lambda > 0$ such that $\omega = \lambda \pi^*\psi$. Then

$$\begin{aligned} a^*\omega &= a^*(\lambda \pi^*\psi) \\ &= (\lambda \circ a)(a^*\pi^*\psi) \\ &= (\lambda \circ a)(\pi \circ a)^*\psi \\ &= (\lambda \circ a)\pi^*\psi \end{aligned}$$

Thus, $a^*\omega = \frac{\lambda \circ a}{\lambda}\omega$ and $\frac{\lambda \circ a}{\lambda} > 0$, contradicting that $a^*\omega = -\omega$. Therefore, $\mathbb{P}^n(\mathbb{R})$ is not orientable if n is even.

To relax the definitions on page 276 slightly, let M and N be smooth oriented manifolds of dimension n. Suppose M is oriented by a nonvanishing $\psi_M \in \Omega^n(M)$ and N by $\psi_N \in \Omega^n(N)$.

If $f: M \longrightarrow N$ is a diffeomorphism then $f^*\psi_N = \lambda \psi_M$ for some nonvanishing $\lambda \in C^\infty(M)$. We say f <u>preserves orientation</u>, and write $\sigma(f) = 1$, if $\lambda > 0$. We say f <u>reverses orientation</u>, and write $\sigma(f) = -1$, if $\lambda < 0$.

The definitions on page 276 apply to the special case that $M = N$, and both the domain and range of f are oriented by the same nonvanishing form.

<u>Lemma:</u>

If $f: M \longrightarrow N$ and $g: N \longrightarrow P$ are diffeomorphisms of oriented manifolds, then $\sigma(g \circ f) = \sigma(g)\sigma(f)$.

Exercises:

① Let M be a smooth manifold. If TM is trivial, prove M is orientable.

② Let M be an n-dimensional, smooth manifold with boundary, embedded in $\mathbb{R}^n$. Prove ∂M is orientable. (In particular, each Σ_g is orientable.)

③ Let M be a smooth, orientable manifold of dimension n. Let $f \in \mathrm{Diff}(M)$ and $\gamma_M \in \Omega^n(M)$ be nonvanishing. Show f preserves the orientation determined by γ_M if and only if it preserves the orientation determined by $-\gamma_M$.

④ Let M be a smooth, orientable, connected manifold. Let

$$\mathrm{Diff}^+(M) = \{ f \in \mathrm{Diff}(M) \mid f \text{ preserves orientation} \}$$

$\mathrm{Diff}^+(M)$ is well-defined by ③. Prove $\mathrm{Diff}^+(M) = \mathrm{Diff}(M)$, or that $\mathrm{Diff}^+(M)$ is an index-2 normal subgroup of $\mathrm{Diff}(M)$.

⑤ Prove the lemma on page 283.

⑥ Let $\Sigma_1^1 = \{(e^{i\theta}, e^{i\varphi}) \in \mathbb{C}^2 \mid \theta, \varphi \in \mathbb{R}\}$. By example ②, $f \in \text{Diff}(\Sigma_1^1)$ where $f(e^{i\theta}, e^{i\varphi}) = (e^{i\theta}, \overline{e^{i\varphi}})$ is orientation reversing. By example ③, $L_{(-1,1)} \in \text{Diff}(\Sigma_1^1)$ is orientation preserving. Thus, $g = f \circ L_{(-1,1)} \in \text{Diff}(\Sigma_1^1)$ is orientation reversing and $g^2 = 1$. Note that $\langle g \rangle \backslash \Sigma_1^1$ is the Klein bottle. Prove it is not orientable.

⑦ Suppose M is an orientable manifold. Let $\Gamma \leq \text{Diff}(M)$ act on M properly discontinuously and freely so that $\Gamma \backslash M$ is a manifold by the proposition on page 39.

Prove $\Gamma \backslash M$ is orientable if and only if $\Gamma \leq \text{Diff}^+(M)$.

§ Integration

Let $U \subseteq \mathbb{R}^n$, U^n be open, connected, and oriented. We let $i_U : U \longrightarrow \mathbb{R}^n$ be inclusion, so that $i_U' : U \to U$ is a diffeomorphism, where the range of i_U' is oriented by $dx_1 \wedge \cdots \wedge dx_n$. Thus, $\sigma(i_U') = 1$ if the orientation on U agrees with the standard orientation on $\mathbb{R}^n$, and $\sigma(i_U') = -1$ otherwise.

If $\omega \in \Omega^n(U)$, then $\omega = a\, dx_1 \wedge \cdots \wedge dx_n$ for some $a \in C^\infty(U)$, and we define

$$\int_U \omega = \sigma(i_U') \int_U a\, dx_1 \cdots dx_n$$

Example:

For $(a,b) \subseteq \mathbb{R}$, and $f \in C^\infty((a,b))$, we have $\int_{(a,b)} f\, dx = \int_a^b f\, dx$ if (a,b) is oriented by $dx \in \Omega^1((a,b))$, and $\int_{(a,b)} f\, dx = \int_b^a f\, dx = (-1)\int_a^b f\, dx$ if oriented by $-dx$.

Calculus change of variables formula

Let $U, V \subseteq \mathbb{R}^n, \mathbb{H}^n$ be open. If $f: V \to U$ is a diffeomorphism and $a \in C^\infty(U)$, then

$$\int_U a\, dx_1 \cdots dx_n = \int_V (a \circ f)\left|\det(Df)\right| dx_1 \cdots dx_n$$

Change of variables for forms on $\mathbb{R}^n$

Let $U, V \subseteq \mathbb{R}^n, \mathbb{H}^n$ be open. If $f: V \to U$ is a diffeomorphism and $\omega \in \Omega^n(U)$, then

$$\int_U \omega = \sigma(f) \int_V f^*\omega$$

proof:

The orientation on U is determined by $\sigma(i_U)\, dx_1 \wedge \cdots \wedge dx_n$, and the orientation on V is determined by $\sigma(i_V)\, dx_1 \wedge \cdots \wedge dx_n$. Note that the orientation on V is also determined by

$$\sigma(f)\, f^*\big(\sigma(i_u)\, dx_1\wedge\cdots\wedge dx_n\big) = \sigma(f)\sigma(i_u) f^*(dx_1\wedge\cdots\wedge dx_n)$$
$$= \sigma(f)\sigma(i_u)\det(Df)\, dx_1\wedge\cdots\wedge dx_n$$

so that for all $p\in V$, $\sigma(f)\sigma(i_u)\det(D_pf)$ and $\sigma(i_v)$ have the same sign. Therefore,

$$\sigma(i_u)\,|\det(D_pf)| = \sigma(f)\sigma(i_v)\det(D_pf).$$

If $\omega\in\Omega^n(U)$, then $\omega = a\,dx_1\wedge\cdots\wedge dx_n$ for some $a\in C^\infty(U)$ and

$$\begin{aligned}
\int_U \omega &= \sigma(i_u)\int_U a\, dx_1\cdots dx_n \\
&= \sigma(i_u)\int_V (a\circ f)\,|\det(Df)|\, dx_1\cdots dx_n \\
&= \sigma(f)\sigma(i_v)\int_V (a\circ f)\det(Df)\, dx_1\cdots dx_n \\
&= \sigma(f)\int_V (a\circ f)\det(Df)\, dx_1\wedge\cdots\wedge dx_n \\
&= \sigma(f)\int_V f^*(a\, dx_1\wedge\cdots\wedge dx_n) \\
&= \sigma(f)\int_V f^*\omega.
\end{aligned}$$

$\boxed{\text{END}}$

Let M be an oriented, smooth manifold of dimension n. Let $\gamma_M \in \Omega^n(M)$ be a nonvanishing form that determines the orientation of M. Let $\{(\varphi_i, U_i)\}$ be charts for M.

For each i, we orient $\varphi_i(U_i) \subseteq \mathbb{R}^n$ by $(\varphi_i^{-1})^* \gamma_M \in \Omega^n(\varphi_i(U_i))$. Thus, $\sigma(\varphi_i) = 1$ and for all i, j, $\sigma(\varphi_i \circ \varphi_j^{-1}) = \sigma(\varphi_i)\sigma(\varphi_j)^{-1} = 1$.

Let $\{\rho_i\}$ be a partition of unity subordinate to the cover $\{U_i\}$. That is, each $\rho_i : M \to [0,1]$ is smooth and

(i) $\left|\{\rho_i \mid \rho_i(p) \neq 0\}\right| < \infty$ for all $p \in M$.

(ii) $\sum_i \rho_i(p) = 1$ for all $p \in M$.

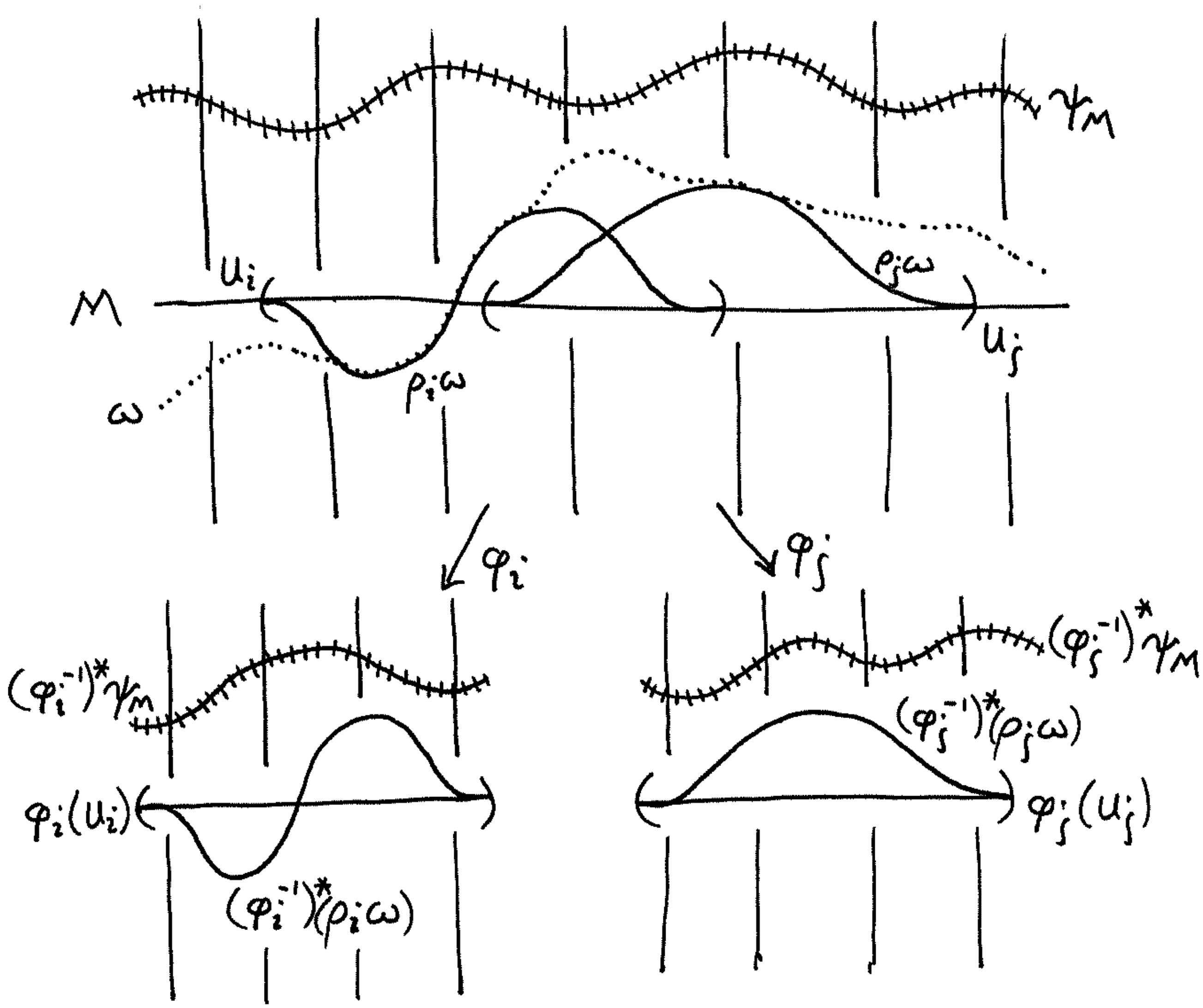
(iii) $\rho_i(M - U_i) = \{0\}$.

We say $\omega \in \Omega^k(M)$ is <u>compactly supported</u> if the closure of $p \in M$ for which $\omega(p) \neq 0$ is compact. We let $\Omega_c^k(M) \leq \Omega^k(M)$ be the vector subspace of compactly supported forms.

For any $\omega \in \Omega_c^n(M)$, we define $\int_M \omega$ as follows:

For any i, $\rho_i \omega \in \Omega_c^n(M)$ is zero outside U_i, by definition of ρ_i. Therefore, it's appropriate to regard $\int_M \rho_i \omega$ as $\int_{U_i} \rho_i \omega$, and to regard the latter integral as $\int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\rho_i \omega)$ using φ_i as an identification of U_i with $\varphi_i(U_i) \subseteq \mathbb{R}^n$ and of $\rho_i \omega$ with $(\varphi_i^{-1})^*(\rho_i \omega) \in \Omega^n(\varphi_i(U_i))$, the final integral above having been defined on page 286. Then since $\omega = \sum_i \rho_i \omega$, we define

$$\int_M \omega = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\rho_i \omega).$$



$$\int_M \omega = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\rho_i\omega)$$

The above definition of $\int_M \omega$ is independent of our choice of $\{\rho_i\}$. Indeed, if $\{\chi_i\}$ is another partition of unity subordinate to $\{U_i\}$, then

$$\sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\rho_i \omega) = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*\Big(\rho_i \sum_j \chi_j \omega\Big)$$

$$= \sum_{i,j} \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\rho_i \chi_j \omega)$$

$$= \sum_{i,j} \int_{\varphi_i(U_i \cap U_j)} (\varphi_i^{-1})^*(\rho_i \chi_j \omega)$$

$$= \sum_{i,j} \sigma(\varphi_i \circ \varphi_j^{-1}) \int_{\varphi_j(U_i \cap U_j)} (\varphi_i \circ \varphi_j^{-1})^* (\varphi_i^{-1})^*(\rho_i \chi_j \omega)$$

$$= \sum_{i,j} \int_{\varphi_j(U_i \cap U_j)} (\varphi_j^{-1})^*(\rho_i \chi_j \omega)$$

$$= \sum_{j,i} \int_{\varphi_j(U_j \cap U_i)} (\varphi_j^{-1})^*(\chi_j \rho_i \omega)$$

$$= \sum_j \int_{\varphi_j(U_j)} (\varphi_j^{-1})^*(\chi_j \omega).$$

M was oriented with γ_M. If we choose the alternate orientation, that determined by $-\gamma_M$, then we denote that oriented manifold by $-M$.

Lemma:

Let M be a smooth, oriented manifold of dimension n. If $\omega \in \Omega^n_c(M)$, then

$$\int_{-M} \omega = -\int_M \omega .$$

proof:

If we replace γ_M with $-\gamma_M$ in the definition of $\int_M \omega$ above, then we replace the orientation of each $\varphi_i(U_i) \subseteq \mathbb{R}^n$, that is we replace $(\varphi_i^{-1})^* \gamma_M$ with $-(\varphi_i^{-1})^* \gamma_M = (\varphi_i^{-1})^*(-\gamma_M)$. The result is that every summand $\int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\rho_i \omega)$ in $\int_M \omega$ is replaced with its negative.

$\boxed{\text{END}}$

Example

$$\int_a^b f\,dx = \int_{(a,b)} f\,dx = -\int_{-(a,b)} f\,dx = -\int_b^a f\,dx$$

Definition:

A refinement of charts $\{(\varphi_i, U_i)\}$ for M is a collection of charts $\{(\psi_j, V_j)\}$ for M such that for each j there is some $i(j)$ such that $V_j \subseteq U_{i(j)}$ and $\psi_j = \varphi_{i(j)}\big|_{V_j}$.

Lemma:

$\int_M \omega$ is independent of a refinement of charts used to determine $\int_M \omega$.

proof:

Let $\{(\psi_j, V_j)\}$ be a refinement of $\{(\varphi_i, U_i)\}$ with respective partitions of unity $\{\chi_j\}$ and $\{\rho_i\}$

subordinate to $\{V_j\}$ and $\{U_i\}$. Note that $\{\chi_j\rho_i\}$ is a partition of unity subordinate to $\{V_j\}$ since $\left|\{\chi_j\rho_i \mid \chi_j\rho_i(p)\neq 0\}\right|$ is finite for all $p\in M$, $\chi_j\rho_i(M-V_j)=\{0\}$, and

$$\sum_{i,j}\chi_j\rho_i(p)=\Big(\sum_j\chi_j(p)\Big)\Big(\sum_i\rho_i(p)\Big)=1.$$

Therefore,

$$\begin{aligned}
\int_M\omega &= \sum_i\int_{\varphi_i(U_i)}(\varphi_i^{-1})^*(\rho_i\omega)\\
&= \sum_i\int_{\varphi_i(U_i)}(\varphi_i^{-1})^*\Big(\sum_j\chi_j\rho_i\omega\Big)\\
&= \sum_{i,j}\int_{\varphi_i(U_i)}(\varphi_i^{-1})^*(\chi_j\rho_i\omega)\\
&= \sum_{i,j}\sigma(\varphi_i\circ\varphi_{i(j)}^{-1})\int_{\varphi_{i(j)}(U_{i(j)})}(\varphi_i\circ\varphi_{i(j)}^{-1})^*(\varphi_i^{-1})^*(\chi_j\rho_i\omega)\\
&= \sum_{i,j}\int_{\varphi_{i(j)}(U_{i(j)})}(\varphi_{i(j)}^{-1})^*(\chi_j\rho_i\omega)\\
&= \sum_{i,j}\int_{\psi_j(V_j)}(\psi_j^{-1})^*(\chi_j\rho_i\omega).
\end{aligned}$$

$\boxed{\text{END}}$

<u>Proposition:</u>

Let M be a smooth, oriented manifold of dimension n. Let $f \in \mathrm{Diff}(M)$ and let $\sigma(f) = 1$ if f preserves orientation and $\sigma(f) = -1$ if f reverses orientation.

For any $\omega \in \Omega^n_c(M)$, we have

$$\int_M f^*\omega = \sigma(f) \int_M \omega$$

where the integrals are determined by arbitrary charts for M.

<u>proof:</u>

By passing to a refinement of charts we may assume that $\{(\varphi_i, U_i)\}$ and $\{(\psi_i, V_i)\}$ are each charts for M and that $f(V_i) = U_i$.

We let $\{\rho_i\}$ be a partition of unity subordinate to $\{V_i\}$. Then $\{\rho_i \circ f^{-1}\}$ is a partition of unity subordinate to $\{U_i\}$.

For each i, $\varphi_i \circ f \circ \psi_i^{-1} : \psi_i(V_i) \longrightarrow \varphi_i(U_i)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ and $\sigma(\varphi_i \circ f \circ \psi_i^{-1}) = \sigma(\varphi_i)\sigma(f)\sigma(\psi_i)^{-1} = \sigma(f)$. Hence,

$$\int_M f^*\omega = \sum_i \int_{\psi_i(V_i)} (\psi_i^{-1})^* (\rho_i f^*\omega)$$

$$= \sum_i \int_{\psi_i(V_i)} (\psi_i^{-1})^* f^* \left((\rho_i \circ f^{-1}) \omega \right)$$

$$= \sum_i \int_{\psi_i(V_i)} (f \circ \psi_i^{-1})^* \left((\rho_i \circ f^{-1}) \omega \right)$$

$$= \sum_i \int_{\psi_i(V_i)} (\varphi_i \circ f \circ \psi_i^{-1})^* (\varphi_i^{-1})^* \left((\rho_i \circ f^{-1}) \omega \right)$$

$$= \sum_i \sigma(\varphi_i \circ f \circ \psi_i^{-1}) \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \left((\rho_i \circ f^{-1}) \omega \right)$$

$$= \sum_i \sigma(f) \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \left((\rho_i \circ f^{-1}) \omega \right)$$

$$= \sigma(f) \int_M \omega .$$

$\boxed{\text{END}}$

Corollary:

$\int_M \omega$ is independent of the charts chosen for M.

proof:

Apply the previous Proposition to the identity map on M.

$\boxed{\text{END}}$

If N is an embedded, oriented submanifold of M of dimension ℓ, then for any $\omega \in \Omega_c^{\ell}(M)$ we define

$$\int_N \omega = \int_N i^*\omega$$

where $i : N \longrightarrow M$ is inclusion. Note that for each $q \in N$, $D_q i : T_q N \to T_q M$ is inclusion so that $i^*\omega(q)(v_1, \ldots, v_\ell) = \omega(q)(v_1, \ldots, v_\ell)$. That is, $i^*\omega$ is simply the restriction of ω.

Integration in $\mathbb{R}^n$, sums, scaling, and pullbacks of forms are all linear. These were the operations used to define integration of forms, so we have the following

Proposition:

If N is an embedded, oriented submanifold of M of dimension ℓ, then

$$\int_N : \Omega_c^\ell(M) \longrightarrow \mathbb{R}$$

is linear.

In particular, if $N = M$ has dimension n then

$$\int_M : \Omega_c^n(M) \to \mathbb{R}$$

is linear.

Exercises:

① Let $U \subseteq \mathbb{R}^n$ be open, and suppose the orientation of U is determined by the nonvanishing form $\omega \in \Omega^n(U)$. Prove $\int_U \omega > 0$.

② Let M be a smooth, oriented, compact manifold of dimension n. Suppose the orientation of M is determined by the nonvanishing form $\omega \in \Omega^n(M)$. Prove $\int_M \omega > 0$.

③ Let K be a compact Lie group of dimension n, and let $\omega \in \Omega^n(K)$ be a nonvanishing left-invariant form as described in example ③ of the previous chapter. Let ω determine the orientation of K.

Since K is compact, if $f \in C^\infty(K)$ then $f\omega \in \Omega^n_c(K)$. Prove for any $g \in K$ that $\int_K (f \circ L_g)\omega = \int_K f\omega$.

④ Let $\langle\ ,\ \rangle$ be the standard inner product on $\mathbb{R}^n$, so that $\{e_1,\ldots,e_n\}$ is an orthonormal basis.

If $K \leq GL_n(\mathbb{R})$ is a compact Lie subgroup, show that for $v, w \in \mathbb{R}^n$, the equation

$$\langle v,w\rangle_K = \int_K \langle kv, kw\rangle \,\omega(k)$$

defines an inner product on $\mathbb{R}^n$. Show that $\langle gv, gw\rangle_K = \langle v,w\rangle_K$ for all $g \in K$.

⑤ Let $\{v_1,\ldots,v_n\}$ be an orthonormal basis for $\mathbb{R}^n$ with respect to $\langle\ ,\ \rangle_K$. Let $h \in GL_n(\mathbb{R})$ be defined by $hv_i = e_i$ for all i. Then $\langle v,w\rangle_K = \langle hv, hw\rangle$ for all $v,w \in \mathbb{R}^n$.

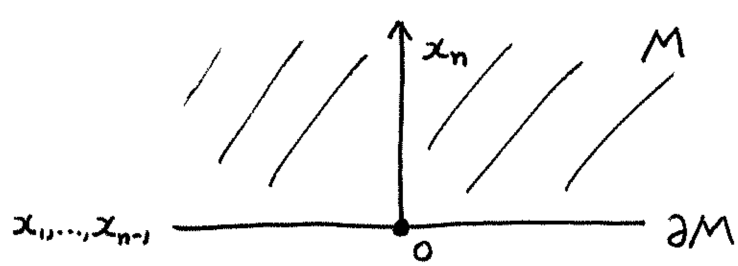
Prove that for all $k \in K$, and $v,w \in \mathbb{R}^n$, $\langle hkh^{-1}v, hkh^{-1}w\rangle = \langle v,w\rangle$ and thus that $hKh^{-1} \leq O(n)$.

In summary, any compact Lie subgroup of $GL_n(\mathbb{R})$ is conjugate to a subgroup of $O(n)$.

§ Stokes' Theorem

Let M be an oriented smooth manifold with boundary. Suppose the orientation of M is determined by a nonvanishing $\psi_M \in \Omega^n(M)$ where $n = \dim(M)$.

∂M is a manifold of dimension $n-1$, and we orient ∂M as follows: Any $p \in \partial M$ has local coordinates $x_1, \dots, x_n$ where $x_n = 0$ corresponds to ∂M, and $x_n \geq 0$ corresponds to M. In those coordinates near p, $\psi_M = a\, dx_1 \wedge \cdots \wedge dx_n$ for some smooth a, and near p we define $\psi_{\partial M} = (-1)^n a\, dx_1 \wedge \cdots \wedge dx_{n-1}$ so that $\psi_{\partial M} \in \Omega^{n-1}(\partial M)$ is nonvanishing. We let $\psi_{\partial M}$ determine the orientation on ∂M.



If $i:\partial M \to M$ is inclusion, then $i^*: \Omega_c^{n-1}(M) \to \Omega_c^{n-1}(\partial M)$ is restriction and for $\omega \in \Omega_c^{n-1}(M)$ we define $\int_{\partial M} \omega = \int_{\partial M} i^* \omega$.

Stokes' Theorem:

Let M be a smooth, oriented, n-dimensional manifold. If $\omega \in \Omega_c^{n-1}(M)$, then

$$\int_{\partial M} \omega = \int_M d\omega$$

proof:

Step 1: Forms on open subsets of $\mathbb{R}^n$ obey Stokes' Theorem.

Let $\omega \in \Omega_c^{n-1}(U)$ for open $U \subseteq \mathbb{R}^n$ with $\overline{U}$ compact and ω vanishing near $\partial \overline{U}$. For some $f_i \in C^\infty(U)$, $\omega = \sum_{i=1}^n f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$.

Therefore,

$$\begin{aligned}
d\omega &= d\left(\sum_{i=1}^{n} f_i\, dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n\right)\\
&= \sum_{i=1}^{n} d\left(f_i\, dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n\right)\\
&= \sum_{i=1}^{n} df_i\wedge dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n\\
&= \sum_{i=1}^{n}\left(\sum_{k=1}^{n}\frac{\partial f_i}{\partial x_k}dx_k\right)\wedge dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n\\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial f_i}{\partial x_k}dx_k\wedge dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n\\
&= \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}dx_i\wedge dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n\\
&= \sum_{i=1}^{n}(-1)^{i-1}\frac{\partial f_i}{\partial x_i}dx_1\wedge\cdots\wedge dx_n
\end{aligned}$$

If the orientation of $U\subseteq\mathbb{R}^n$ is determined by $(-1)^u dx_1\wedge\cdots\wedge dx_n$, then we have for $c>0$ with $f_i(x_1,\dots,x_n)=0$ if $|x_i|\geq c$, that

$$\int_U d\omega = \int_U \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \int_U \frac{\partial f_i}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \int_U (-1)^u \frac{\partial f_i}{\partial x_i}\, dx_1 \cdots dx_n$$

$$= (-1)^u \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^n} \frac{\partial f_i}{\partial x_i}\, dx_1 \cdots dx_n$$

$$= (-1)^u \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i}\, dx_i \right) dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

$$= (-1)^u \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(\int_{-c}^{c} \frac{\partial f_i}{\partial x_i}\, dx_i \right) dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

$$= (-1)^u \sum_{i=1}^{n} \int_{\mathbb{R}^{n-1}} 0 \; dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

$$= 0$$

Since $\partial U = \emptyset$ as $U \subseteq \mathbb{R}^n$ is open, $\int_{\partial U} \omega = 0$, so that $\int_U d\omega = \int_{\partial U} \omega$ for open $U \subseteq \mathbb{R}^n$.

Step 2: Forms on open subsets of $\mathbb{H}_n$ obey Stokes' Theorem.

Let $\omega \in \Omega_c^{n-1}(U)$ for open $U \subseteq \mathbb{H}_n$ with $\overline{U}$ compact and ω vanishing near $(\partial \overline{U}) - (\partial \mathbb{H}_n)$. For some $f_i \in C^\infty(U)$, $\omega = \sum_{i=1}^{n} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$. If $(-1)^u dx_1 \wedge \cdots \wedge dx_n$ determines the orientation on U, then as in Step 1,

$$\int_U d\omega = (-1)^u \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{H}^n} \frac{\partial f_i}{\partial x_i} \, dx_1 \cdots dx_n$$

$$= (-1)^u \sum_{i=1}^{n-1} (-1)^{i-1} \int_{\mathbb{H}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

$$+ (-1)^u (-1)^{n-1} \int_{\partial \mathbb{H}^n} \left(\int_0^\infty \frac{\partial f_n}{\partial x_n} dx_n \right) dx_1 \cdots dx_{n-1}$$

$$= 0 + (-1)^u (-1)^{n-1} \int_{\partial \mathbb{H}^n} \left(\int_0^c \frac{\partial f_n}{\partial x_n} dx_n \right) dx_1 \cdots dx_{n-1}$$

$$= (-1)^u (-1)^{n-1} \int_{\partial \mathbb{H}^n} \Big(f_n(x_1, \ldots, x_{n-1}, c) - f_n(x_1, \ldots, x_{n-1}, 0) \Big) dx_1 \cdots dx_{n-1}$$

$$= (-1)^u (-1)^n \int_{\partial \mathbb{H}^n} f(x_1, \ldots, x_{n-1}, 0) \, dx_1 \cdots dx_{n-1}$$

For $\tau : \partial\mathbb{H}^n \to \mathbb{H}^n$, the inclusion, we have $\tau^* dx_n = 0$, so

$$\tau^*\omega = \tau^*\left(\sum_{i=1}^n f_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right)$$

$$= \sum_{i=1}^n \tau^*\left(f_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right)$$

$$= \sum_{i=1}^n (f_i \circ \tau)\, \tau^* dx_1 \wedge \cdots \wedge \widehat{\tau^* dx_i} \wedge \cdots \wedge \tau^* dx_n$$

$$= (f_n \circ \tau)\, \tau^* dx_1 \wedge \cdots \wedge \tau^* dx_{n-1}$$

$$= f_n(x_1, x_2, \ldots, x_{n-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{n-1}.$$

Therefore,

$$\int_{\partial U} \omega = \int_{\partial U} \tau^*\omega$$

$$= \int_{\partial U} f_n(x_1, x_2, \ldots, x_{n-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{n-1}$$

$$= \int_{\partial\mathbb{H}^n} (-1)^u (-1)^n f_n(x_1, x_2, \ldots, x_{n-1}, 0)\, dx_1 \cdots dx_{n-1}$$

$$= \int_U d\omega$$

for $\omega \in \Omega_c^{n-1}(U)$ with $U \subseteq \mathbb{H}^n$ open.

Step 3:

For $\omega \in \Omega_c^{n-1}(M)$, let $\{(\varphi_i, U_i)\}$ be charts on M, so that $\{(\varphi_i, U_i \cap \partial M)\}$ are charts for ∂M. If $\{\rho_i\}$ is a partition of unity subordinate to $\{U_i\}$, then $\{\rho_i|_{\partial M}\}$ is a partition of unity subordinate to $\{U_i \cap \partial M\}$.

We have

$$
\begin{aligned}
\int_M d\omega &= \int_M d\left(\sum_i \rho_i \omega\right) \\
&= \int_M \sum_i d(\rho_i \omega) \\
&= \sum_i \int_M d(\rho_i \omega) \\
&= \sum_i \int_{U_i} d(\rho_i \omega) \\
&= \sum_i \int_{\partial U_i} \rho_i \omega \\
&= \int_{\partial M} \sum_i \rho_i \omega \\
&= \int_{\partial M} \omega .
\end{aligned}
$$

$\boxed{\text{END}}$

Some common applications of Stokes' Theorem include:

(i) If $\partial M = \emptyset$ and $\omega \in \Omega_c^{n-1}(M)$, then

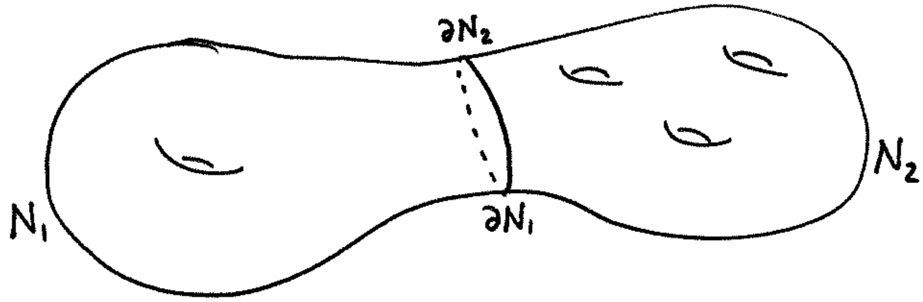
$$\int_M d\omega = \int_{\partial M} \omega = \int_{\emptyset} \omega = 0$$

(ii) If $\omega \in \Omega_c^{n-2}(M)$, then

$$\int_{\partial M} d\omega = \int_M dd\omega = \int_M 0 = 0.$$

(iii) If M is the union of two n-dimensional manifolds $N_1, N_2 \subseteq M$ with $N_1 \cap N_2 = \partial N_1 = \partial N_2$, then the orientation on M restricts to orientations on N_1 and N_2 and $\partial N_2 = -\partial N_1$. Therefore, $\int_{N_1} d\omega = -\int_{N_2} d\omega$ since

$$\int_{N_1} d\omega + \int_{N_2} d\omega = \int_{\partial N_1} \omega + \int_{-\partial N_1} \omega = \int_{\partial N_1} \omega - \int_{\partial N_1} \omega = 0.$$



Exercises:

① Let $I=[0,1]$ and let $dx \in \Omega^1(I)$ be pointwise the identity $T_pI = \mathbb{R} \to \mathbb{R}$.

Let M be an oriented manifold of dimension n, and suppose a nonvanishing $\psi_M \in \Omega^n(M)$ determines the orientation of M.

Suppose $\partial M = \emptyset$ so that $M \times I$ is a smooth manifold with $\partial(M\times I) = (M\times\{0\}) \amalg (M\times\{1\})$ up to diffeomorphism.

Orient $M\times I$ by having $\psi_M \wedge dx$ determine the orientation of $M\times I$, and then determine the orientation of $\partial(M\times I)$ as on page 302, thus determining orientations on $M\times\{0\}$ and $M\times\{1\}$. With the natural identification $(p,0)\mapsto(p,1)$ of $M\times\{0\}$ with $M\times\{1\}$, prove $M\times\{0\} = -(M\times\{1\})$ as oriented manifolds.

② Let M be a smooth, compact, oriented manifold of dimension n with $\partial M = \emptyset$. If N is a smooth manifold and $f: M \to N$ and $g: M\to N$ are smoothly homotopic, prove $\int_M f^*\omega = \int_M g^*\omega$ for any $\omega \in \Omega^n(N)$ with $d\omega = 0$.

③ Let $S^1 = \{ p \in \mathbb{R}^2 \mid \|p\| = 1 \}$. Orient S^1 as in example ④ of page 279. Let $\omega \in \Omega^1(\mathbb{R}^2)$ be $\omega(x,y) = xy^2 dx + x^2 y dy$

Find $\int_{S^1} \omega$.

§ Degree

Throughout this chapter, M is a compact smooth manifold, N is a connected smooth manifold, $\partial M = \partial N = \emptyset$ and $\dim(M) = \dim(N) = n$. In addition, aside from a brief discussion on "degree mod 2" on pages 342-3 we will assume that M and N are oriented by nonvanishing forms $\psi_M \in \Omega^n(M)$ and $\psi_N \in \Omega^n(N)$, respectively.

Before discussing degree, we'll begin by collecting a couple of lemmas that will be useful for us later.

Lemma:

Let $B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$. If $y \in B^n$ and $y \approx 0$, then there is some $h \in \mathrm{Diff}^+(B^n)$ such that $h(0) = y$, h restricted to a neighborhood of ∂B^n is the identity, and $h \simeq id_{B^n}$.

proof:

As in the second lemma on page 95, there is some $b \in C^\infty(B^n)$ and some $0 < \varepsilon << 1$ such that $b(x) = 1$ if $\|x\| < \varepsilon$ and $b(x) = 0$ if $\|x\| > 1 - \varepsilon$. We assume $\|y\| < \varepsilon$.

Define $Y \in \Gamma(TB^n)$ by $Y(x) = b(x) y$. For $t \in [0,1]$, we let $\theta_t^Y \in \mathrm{Diff}^+(B^n)$ be the associated flow. Then $\theta_1^Y(0) = y$, $\theta_1^Y(x) = x$ when $\|x\| > 1 - \varepsilon$, and $B^n \times [0,1] \longrightarrow B^n$ given by $(x,t) \longmapsto \theta_t^Y(x)$ is a homotopy between θ_1^Y and $\theta_0^Y = id_{B^n}$. Let $h = \theta_1^Y$.

$\boxed{\text{END}}$

Lemma:

Let N be a smooth manifold and choose $y, y' \in N$ such that $y \simeq y'$. Then there is some $h \in \mathrm{Diff}^+(N)$ such that $h(y) = y'$ and $h \simeq id_N$.

proof:

Identify a neighborhood $U \subseteq N$ of y with B^n such that y is identified with $0 \in B^n$. By the previous lemma, there is some $f \in \mathrm{Diff}^+(U)$ with $f(y) = y'$, f near ∂U is the identity, and $f \simeq id_U$.

Define $h \in \mathrm{Diff}^+(N)$ by $h|_U = f$ and $h|_{N-U} = id_{N-U}$.

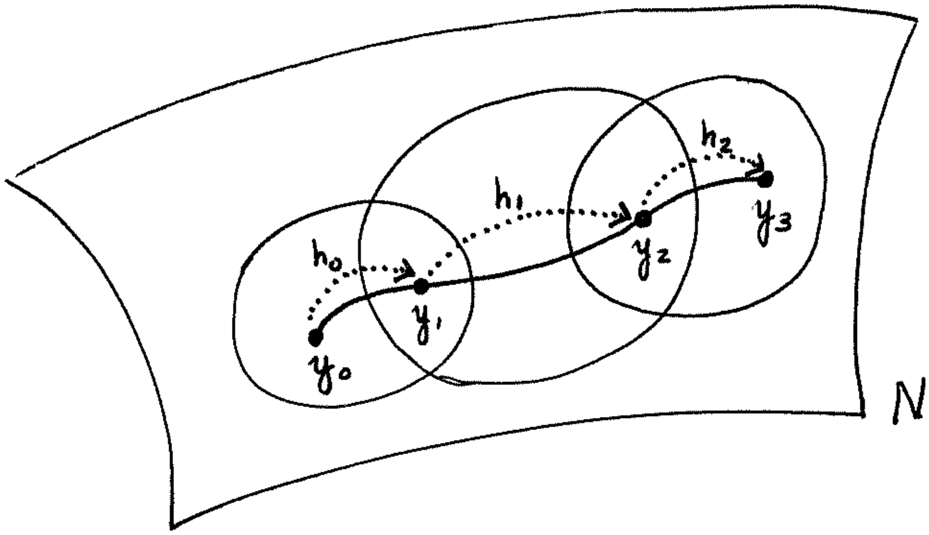
$\boxed{\text{END}}$

Lemma:

Let N be a smooth connected manifold and let $y, y' \in N$. Then there is some $h \in \mathrm{Diff}^+(N)$ such that $h(y) = y'$ and $h \simeq id_N$.

proof:

Choose a path in N from y to y'. By compactness of this interval, there are finitely many points $y_0 = y, y_1, \ldots, y_n = y'$ along the path, such that for each $0 \le i \le n-1$ there is some $h_i \in \mathrm{Diff}^+(N)$ with $h_i(y_i) = y_{i+1}$ and $h_i \simeq id_N$.

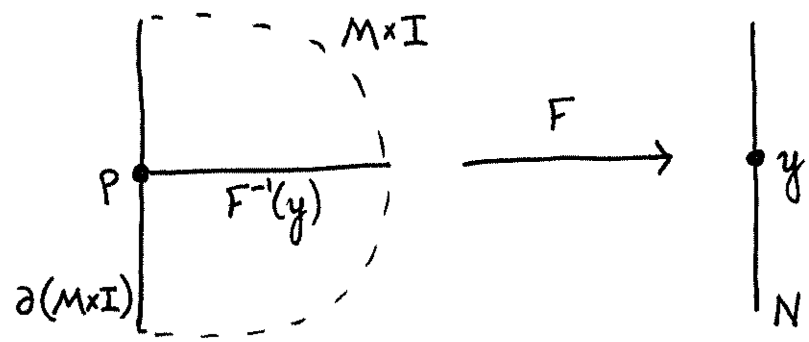


Let $h = h_{n-1} \circ h_{n-2} \circ \cdots \circ h_0$ so that $h(y_0) = y_n$ — that is $h(y) = y'$ — $h \in \mathrm{Diff}^+(N)$, and $h \simeq id_N$.

$\boxed{\text{END}}$

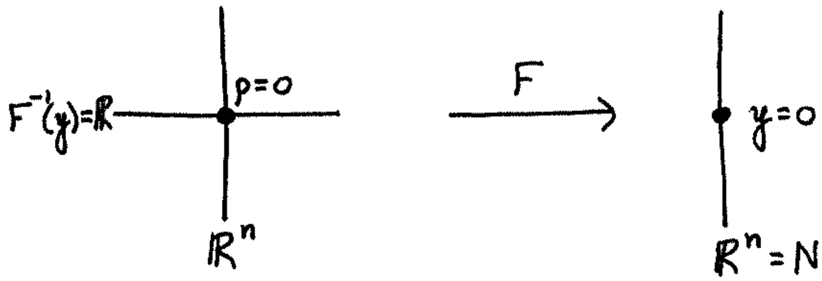
Lemma:

Suppose M and N are each smooth manifolds of dimension n. If $F : M \times I \longrightarrow N$ is smooth and $y \in N$ is a regular value of $F|_{\partial(M\times I)}$ and F, then any $p \in F|_{\partial(M\times I)}^{-1}(y)$ has an open neighborhood in $F^{-1}(y)$ that is diffeomorphic to a half-open interval with p as its closed endpoint.

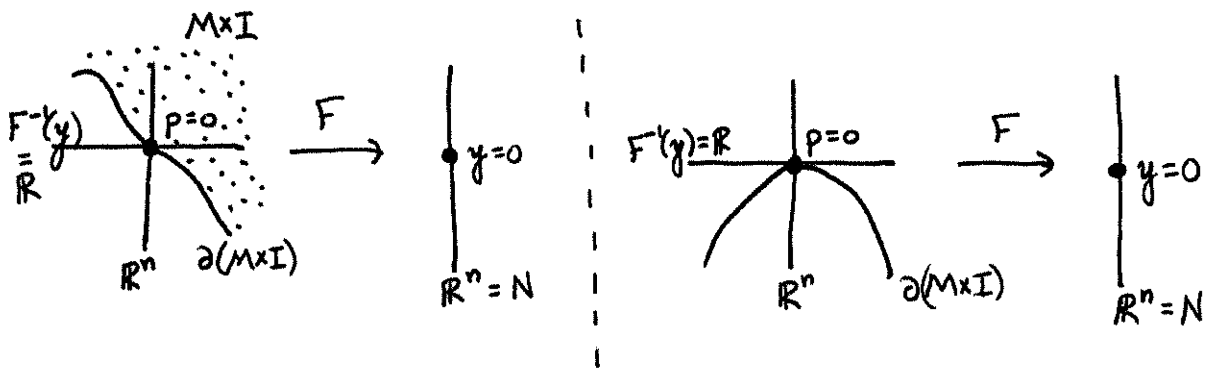


proof:

By the definition of a smooth function on a manifold with boundary, and the lemma on page 53, we may choose local coordinates such that $p \in M\times I$ corresponds to $0 \in \mathbb{R}^{n+1}$, $y \in N$ to $0 \in \mathbb{R}^n$, and $F: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ to projection.



Recall that $p \in \partial(M\times I)$. There are two cases. Either $\partial(M\times I) \pitchfork F^{-1}(y)$ or $\partial(M\times I) \not\pitchfork F^{-1}(y)$.



In the first case, $\partial(M\times I)$ intersects $F^{-1}(y)$ at p and $F^{-1}(y) - \partial(M\times I)$ is two open intervals, one in $M\times I$ and the other not, proving the lemma.

In the second case, we have a contradiction that y is a regular value of $F|_{\partial(M\times I)}$ since in this case $F^{-1}(y) \leq T_p(\partial(M\times I))$ so

$$\dim(D_pF(T_p(\partial(M\times I)))) = n-1 \neq n = \dim(T_yN).$$

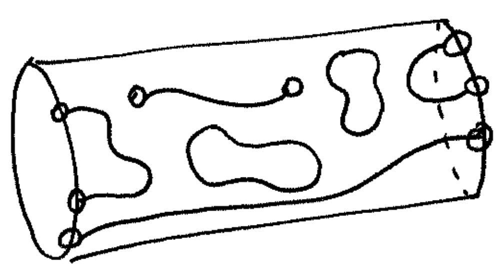
$\boxed{\text{END}}$

Lemma:

Suppose M and N are each smooth manifolds of dimension n, and that M is compact. If $F: M\times I \longrightarrow N$ is smooth and $y\in N$ is a regular value of $F|_{\partial(M\times I)}$ and F, then $F^{-1}(y)$ is a disjoint union of a finite collection of compact intervals in $\mathbb{R}$, $A_1,\dots,A_k$, and of a finite collection of circles, $C_1,\dots,C_\ell$. Furthermore, $\coprod_{i=1}^{k}\partial A_i = F^{-1}(y)\cap\partial(M\times I)$.

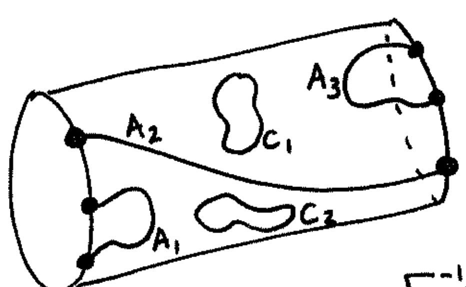
proof:

By the Regular Value Theorem, $F|_{M\times(0,1)}^{-1}(y)$ is a union of one-dimensional manifolds without boundary, so either open intervals $\mathring{A}_1,\dots,\mathring{A}_k$ or circles, $C_1,\dots,C_\ell$



$F|_{M\times(0,1)}^{-1}(y)\subseteq M\times(0,1)$

Since $M \times I$ and $\partial(M \times I)$ are compact, the closed sets $F^{-1}(y) \subseteq M \times I$ and $F|_{\partial(M\times I)}^{-1}(y) \subseteq \partial(M \times I)$ are compact. Furthermore, since y is a regular value of $F|_{\partial(M\times I)}$, $F|_{\partial(M\times I)}^{-1}(y)$ is a zero-dimensional manifold and thus is a finite set of points. Since $F^{-1}(y) = F|_{M\times(0,1)}^{-1}(y) \cup F|_{\partial(M\times I)}^{-1}(y)$ is compact, and by the previous lemma, we see that the endpoints of the $\mathring{A}_i$ bijectively correspond to the points in $F|_{\partial(M\times I)}^{-1}(y)$ and that there are only finitely many A_i and C_i.

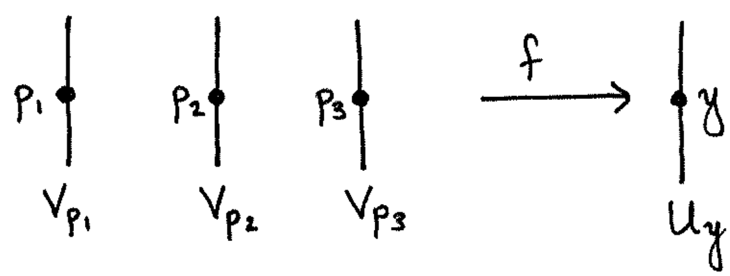


$F^{-1}(y) \subseteq \partial(M \times I)$

$\boxed{\text{END}}$

Lemma:

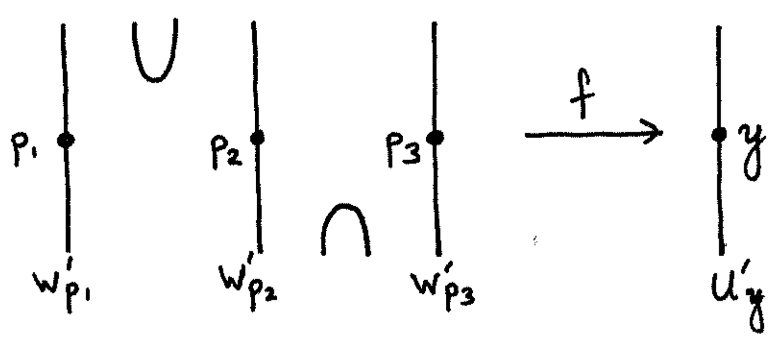
Let M and N be smooth manifolds of the same dimension. Suppose M is compact, $f: M \to N$ is smooth, and $y \in f(M) \subseteq N$ is a regular value of f. Then there is a open neighborhood $U_y \subseteq N$ of y and for each $p \in f^{-1}(y)$ an open neighborhood $V_p \subseteq M$ of p such that $f|_{V_p}: V_p \to U_y$ is a diffeomorphism for all p, and $f^{-1}(U_y) = \coprod_{p \in f^{-1}(y)} V_p$, and U_y and each V_p are connected.



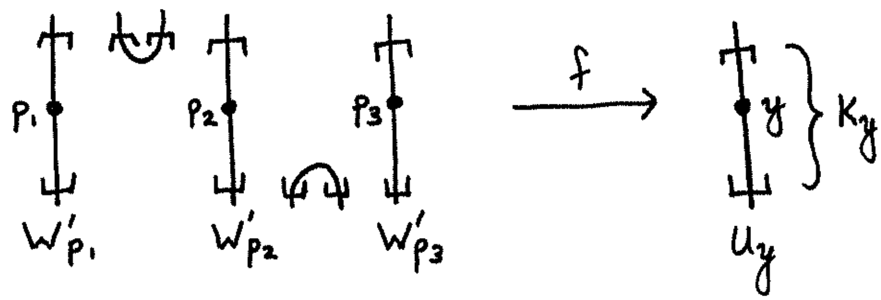
proof:

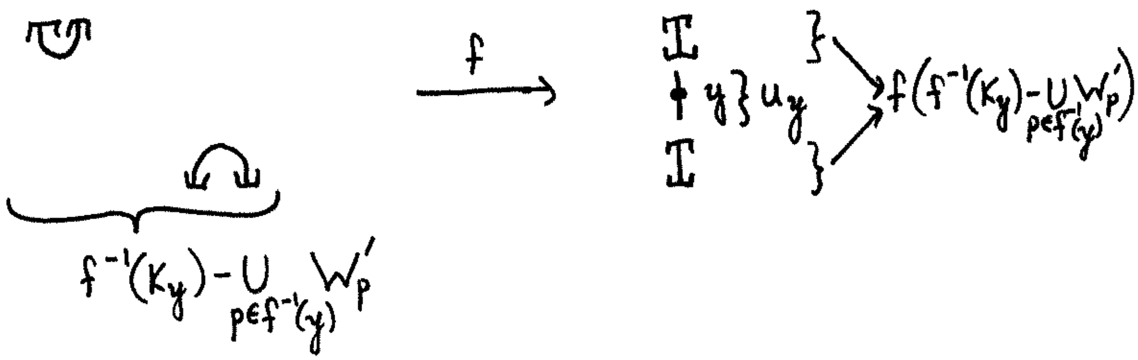
By the Regular Value Theorem, $f^{-1}(y)$ is a discrete set of points, necessarily finite by the compactness of M.

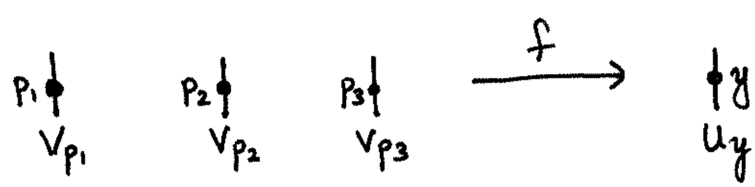
For each $p \in f^{-1}(y)$, $D_p f$ is surjective and hence bijective since M and N have the same dimension. Therefore, by the Inverse Function Theorem, there are open neighborhoods of each p, $W_p \subseteq M$, such that $f|_{W_p} : W_p \to f(W_p)$ is a diffeomorphism. Note $y \in f(W_p)$ for each p so that $\bigcap_{p \in f^{-1}(y)} f(W_p)$ is an open neighborhood of y. Name this set U_y' and let $W_p' = f|_{W_p}^{-1}(U_y')$ for each p so that $f|_{W_p'} : W_p' \to U_y'$ is a diffeomorphism. Note that we might not yet have $f^{-1}(U_y') = \coprod_{p \in f^{-1}(y)} W_p'$, only that $\coprod_{p \in f^{-1}(y)} W_p' \subseteq f^{-1}(U_y')$.



Choose a compact neighborhood $K_y \subseteq U_y'$ of y. Then $f^{-1}(K_y) \subseteq M$ is closed and hence compact since M is compact. Thus $f^{-1}(K_y) - \bigcup_{p \in f^{-1}(y)} W_p'$ is compact so $f\left(f^{-1}(K_y) - \bigcup_{p \in f^{-1}(y)} W_p'\right) \subseteq U_y'$ is a compact set not containing y. Hence, there is an open neighborhood $U_y \subseteq U_y' - f\left(f^{-1}(K_y) - \bigcup_{p \in f^{-1}(y)} W_p'\right)$ of y. We let $V_p = f|_{W_p'}^{-1}(U_y)$.







If we choose U_y connected, the lemma is satisfied.

$\boxed{\text{END}}$

Definition of degree:

Suppose M and N are smooth manifolds without boundaries and of the same dimension n. Let M be compact and N connected. Assume M and N are oriented, and that their orientations are determined by the nonvanishing forms $\gamma_M \in \Omega^n(M)$ and $\gamma_N \in \Omega^n(N)$, respectively.

Let $f: M \longrightarrow N$ be smooth. By Sard's Theorem, there exists a regular value $y \in N$ of f, and by the Regular Value Theorem, $f^{-1}(y)$ is a 0-dimensional embedded submanifold of compact M, and thus $f^{-1}(y)$ is finite.

If $p \in f^{-1}(y)$, then $D_p f$ is surjective by the definition of regular values, hence $D_p f$ is bijective since $\dim(M) = \dim(N)$.

Therefore, by the Inverse Function Theorem, there exist open, connected neighborhoods $V_p \subseteq M$ of $p \in f^{-1}(y)$ such that $f|_{V_p} : V_p \to f(V_p)$ is a diffeomorphism where $f(V_p) \subseteq N$ is an open, connected neighborhood of $y \in N$.

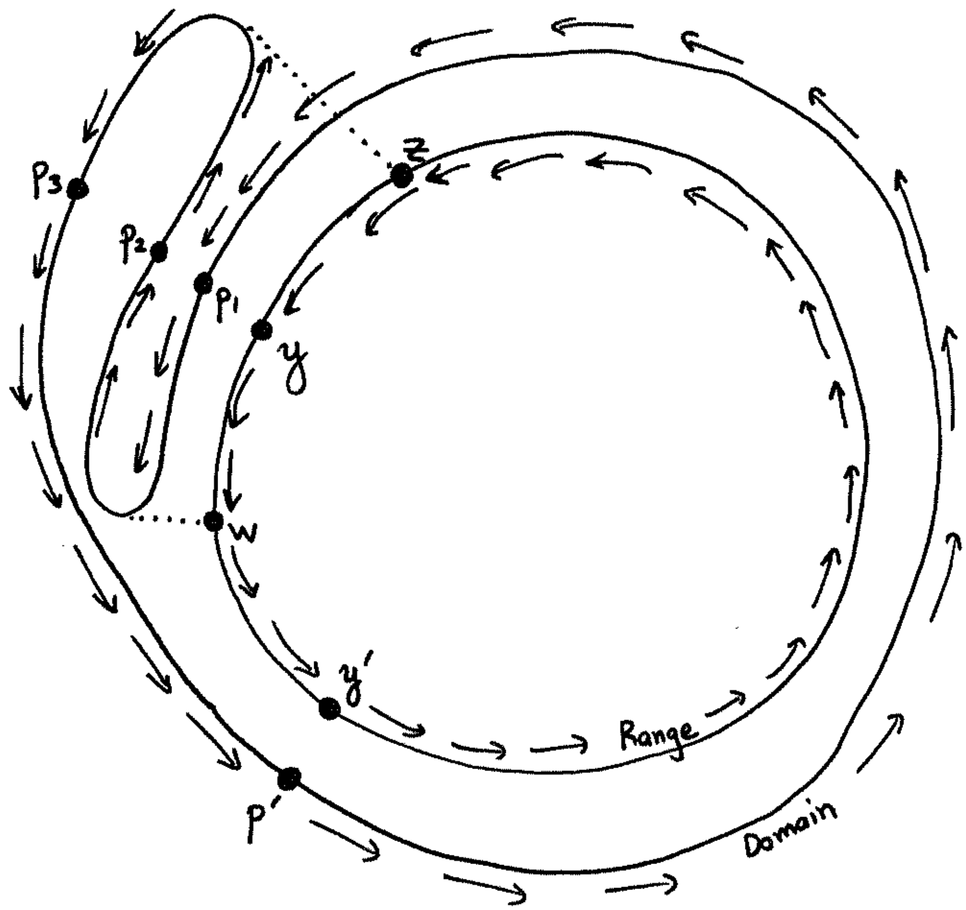
We let $\psi_M|_{V_p}$ and $\psi_N|_{f(V_p)}$ determine the orientations on V_p and $f(V_p)$, respectively. Then the <u>degree</u> of $f: M \to N$ is defined as

$$\deg(f) = \sum_{p \in f^{-1}(y)} \sigma(f|_{V_p})$$

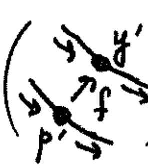
We will have to check that degree is well-defined with respect to the choice of regular value $y \in N$, and with respect to the choice of neighborhoods $V_p \subseteq M$ of the points $p \in f^{-1}(y)$. The former will be the subject of the next few pages, the latter is left as exercise 1 at the end of this chapter.

Example

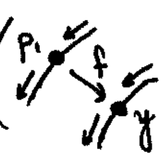
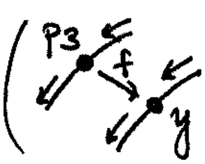
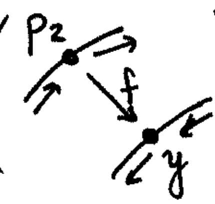
$f : S^1 \to S^1$ has degree equal to 1, where f is depicted below.



Critical values: z, w

For regular value y': $f^{-1}(y') = \{p'\}$ and $\sigma\Bigg($  $\Bigg) = 1$

For regular value y: $f^{-1}(y) = \{p_1, p_2, p_3\}$ and

$\sigma\Bigg($  $\Bigg) = \sigma\Bigg($  $\Bigg) = 1$ and $\sigma\Bigg($  $\Bigg) = -1.$

Lemma:

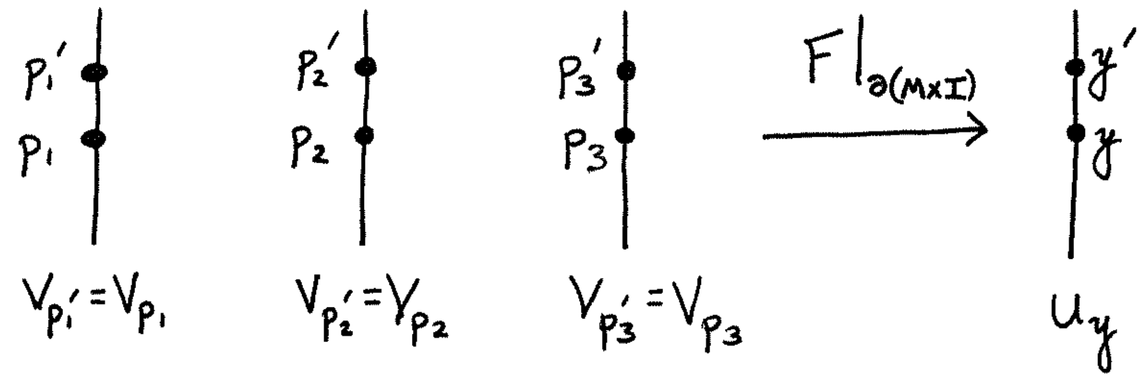
Let M and N be as in the definition of degree, and let $F: M\times I \to N$ be smooth. Suppose $y \in N$ is a regular value of $F|_{\partial(M\times I)}$ and choose, as in the Lemma on page 320, an open neighborhood $U_y \subseteq N$ of y and for each $p \in F|_{\partial(M\times I)}^{-1}(y)$ an open neighborhood $V_p \subseteq \partial(M\times I)$ such that $F|_{V_p}: V_p \to U_y$ is a diffeomorphism, $F|_{\partial(M\times I)}^{-1}(U_y) = \coprod_p V_p$, and U_y and each V_p are connected. Then

$$\sum_{p\in F|_{\partial(M\times I)}^{-1}(y)} \sigma\left(F|_{V_p}\right) = 0 .$$

proof:

Let $dx \in \Omega^1(I)$ be the standard nonvanishing 1-form on I. Then $\psi_M \wedge dx \in \Omega^{n+1}(M \times I)$ is nonvanishing so we let it determine the orientation of $M \times I$, and then, as on page 302, $\psi_M \wedge dx$ determines a nonvanishing $\psi_{\partial(M\times I)} \in \Omega^n(\partial(M \times I))$ that we use to determine the orientation of $\partial(M \times I)$.

By Sard's Theorem, there is some $y' \in U_y$ that is a regular value of both $F : M \times I \longrightarrow N$ and $F|_{\partial(M\times I)}$. Each V_p contains a unique $p' \in F|_{\partial(M\times I)}^{-1}(y')$ and we let $V_{p'} = V_p$.



Thus,

$$\sum_{p\in F|_{\partial(M\times I)}^{-1}(y)} \sigma\left(F|_{V_p}\right) = \sum_{p'\in F|_{\partial(M\times I)}^{-1}(y')} \sigma\left(F|_{V_{p'}}\right)$$

and it is the latter sum that we will show equals 0.

By the lemma on page 318, $F^{-1}(y')$ is a disjoint union of finitely many compact intervals $A_1, \ldots, A_\Delta$ and circles $C_1, \ldots, C_\ell$ and $\coprod_{i=1}^{\Delta} \partial A_i = F|_{\partial(M\times I)}^{-1}(y')$.

Since y' is a regular value of F, we have, for any $a \in A_i$, an exact sequence

$$0 \longrightarrow T_a A_i \longrightarrow T_a(M \times I) \xrightarrow{D_a F} T_{y'} N \longrightarrow 0$$

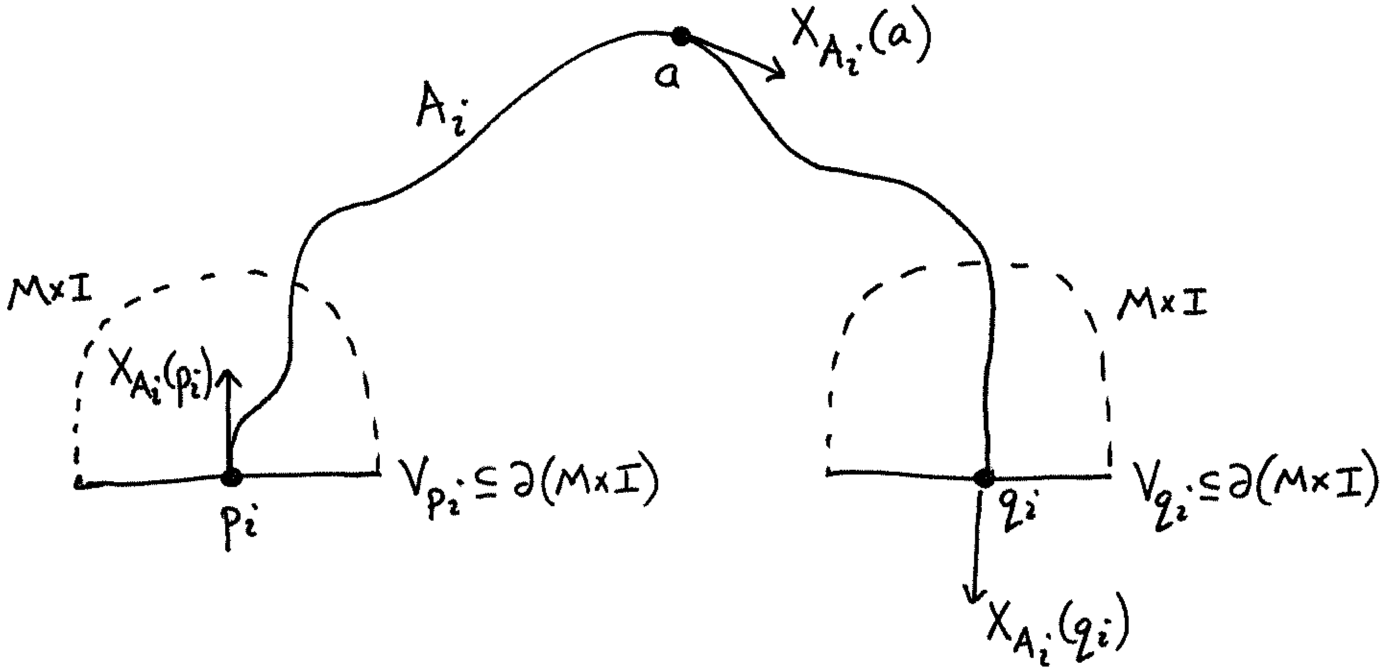
Therefore, there is a unique nonzero $\psi_{A_i}(a) \in \Lambda^1(T_a(M \times I))$ such that

$$\left(F^* \psi_N \wedge \psi_{A_i}\right)(a) = \left(\psi_M \wedge dx\right)(a).$$

Thus we have defined a nonvanishing $\psi_{A_i} \in \Omega^1(A_i)$.

Let $X_{A_i} \in \Gamma(TA_i)$ be the vector field defined by $\psi_{A_i}(a)\left(X_{A_i}(a)\right) = 1$. We see that $X_{A_i}(a) \neq 0$ for any $a \in A_i$ and thus X_{A_i} must point in to $M \times I$ at one point of $\partial A_i \subseteq \partial(M \times I)$ and out from

$M \times I$ at the other. We let $\partial A_i = \{p_i, q_i\}$ such that $X_{A_i}(p_i)$ points in and $X_{A_i}(q_i)$ points out.



In local coordinates for upper half space near $p_i \in \partial(M \times I) \subseteq M \times I$ we have

$V_{p_i} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\} = \mathbb{R}^n$ and

$X_{A_i}(p_i) = \sum_{k=1}^{n+1} b_k \frac{\partial}{\partial x_k}\Big|_{p_i}$ for some $b_k \in \mathbb{R}$ and

$b_{n+1} > 0$ since $X_{A_i}(p_i)$ points in to $M \times I$.

Thus $\psi_{A_i}(p_i) = \sum_{k=1}^{n+1} \frac{b_k}{\|b\|^2} dx_k$.

We claim $\left(F\big|_{\partial(M\times I)}^{*} \psi_N\right)(p_i) = \left(F^*\psi_N\right)(p_i)\Big|_{T_{p_i}(\partial(M\times I))}$.

Indeed, if in local coordinates for U_y we have $\psi_N(y') = \lambda dx_1 \wedge \cdots \wedge dx_n$ for $\lambda \in \mathbb{R}^\times$, then for $v_k \in T_{p_i}(M\times I)$ we have

$$\left(F^*\psi_N\right)(p_i)(v_1,\dots,v_n) = \psi_N(y')\left(D_{p_i}F(v_1),\dots, D_{p_i}F(v_n)\right)$$
$$= \lambda dx_1 \wedge \cdots \wedge dx_n \left(D_{p_i}F(v_1),\dots, D_{p_i}F(v_n)\right)$$

and if $w_k \in T_{p_i}(\partial(M\times I)) \leq T_{p_i}(M\times I)$ then

$$\left(F\big|_{\partial(M\times I)}^{*} \psi_N\right)(p_i)(w_1,\dots,w_n) = \psi_N(y')\left(D_{p_i}F\big|_{\partial(M\times I)}(w_1),\dots, D_{p_i}F\big|_{\partial(M\times I)}(w_n)\right)$$
$$= \lambda dx_1 \wedge \cdots \wedge dx_n \left(D_{p_i}F(w_1),\dots, D_{p_i}F(w_n)\right)$$

verifying our claim.

Note that we can also write

$$\left(F\big|_{\partial(M\times I)}^{*}\psi_N\right)(p_i) = \lambda \det\left(D_{p_i}F\big|_{\partial(M\times I)}\right) dx_1\wedge\cdots\wedge dx_n$$

so that if we let $\left(F^{*}\psi_N\right)(p_i) = \sum_{k=1}^{n+1} \xi_k\, dx_1\wedge\cdots\wedge \widehat{dx_k}\wedge\cdots\wedge\psi_{A_i}(p_i)$ for some $\xi_k \in \mathbb{R}$ then our verified claim implies $\xi_{n+1} = \lambda \det\left(D_{p_i}F\big|_{\partial(M\times I)}\right)$.

Next we claim $\left(F\big|_{\partial(M\times I)}^{*}\psi_N\wedge\psi_{A_i}\right)(p_i)$ equals $\left(F^{*}\psi_N\wedge\psi_{A_i}\right)(p_i)$. Indeed the former equals $\lambda\det\left(D_{p_i}F\big|_{\partial(M\times I)}\right) dx_1\wedge\cdots\wedge dx_n\wedge\psi_{A_i}(p_i)$ and the latter equals

$$\left(\sum_{k=1}^{n+1}\xi_k\, dx_1\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge\psi_{A_i}(p_i)\right)\wedge\psi_{A_i}(p_i)$$

$$=$$

$$\xi_{n+1}\, dx_1\wedge\cdots\wedge dx_n\wedge\psi_{A_i}(p_i)$$

verifying our claim.

The orientation of $M \times I$ near p_i is given in local coordinates as

$$
\begin{aligned}
(\psi_M \wedge dx)(p_i) &= \left(F^* \psi_N \wedge \psi_{A_i}\right)(p_i) \\
&= \left(F\big|_{\partial(M\times I)}^{*} \psi_N \wedge \psi_{A_i}\right)(p_i) \\
&= \lambda \det\left(D_{p_i} F\big|_{\partial(M\times I)}\right) dx_1 \wedge \cdots \wedge dx_n \wedge \psi_{A_i}(p_i) \\
&= \lambda \det\left(D_{p_i} F\big|_{\partial(M\times I)}\right) dx_1 \wedge \cdots \wedge dx_n \wedge \left(\sum_{k=1}^{n+1} \frac{b_k}{\|b\|^2} dx_k\right) \\
&= \frac{\lambda\, b_{n+1} \det\left(D_{p_i} F\big|_{\partial(M\times I)}\right)}{\|b\|^2} dx_1 \wedge \cdots \wedge dx_{n+1}
\end{aligned}
$$

Therefore, as on page 302,

$$
\psi_{\partial(M\times I)}(p_i) = \frac{(-1)^{n+1} \lambda\, b_{n+1} \det\left(D_{p_i} F\big|_{\partial(M\times I)}\right)}{\|b\|^2} dx_1 \wedge \cdots \wedge dx_n
$$

Where we recall that $b_{n+1} > 0$ since $X_{A_i}(p_i)$ points in to $M \times I$.

Note that

$$\left(F\big|_{V_{p_i}}^{*}\psi_N\right)(p_i) = \lambda \det\left(D_{p_i}F\big|_{V_{p_i}}\right) dx_1 \wedge \cdots \wedge dx_n$$
$$= \lambda \det\left(D_{p_i}F\big|_{\partial(M\times I)}\right) dx_1 \wedge \cdots \wedge dx_n$$

so that

$$\frac{(-1)^{n+1} b_{n+1}}{\|b\|^2}\left(F\big|_{V_{p_i}}^{*}\psi_N\right)(p_i) = \psi_{\partial(M\times I)}(p_i).$$

Thus, $F\big|_{V_{p_i}}^{*}\left(\psi_N\big|_{U_y}\right),\ \psi_{\partial(M\times I)}\big|_{V_p} \in \Omega^n(V_p)$ are pointwise positive (resp. negative) multiples of each other if $(-1)^{n+1} b_{n+1}$ is positive (resp. negative). That is,

$$\sigma\left(F\big|_{V_{p_i}}\right) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even.} \end{cases}$$

Similarly, in local coordinates near $q_i \in \partial A_i \subseteq \partial(M \times I)$, $X_{A_i}(q_i) = \sum_{k=1}^{n+1} c_k \frac{\partial}{\partial x_k}\Big|_{q_i}$ for some $c_k \in \mathbb{R}$ with $c_{n+1} < 0$ since $X_{A_i}(q_i)$ points out from $M \times I$ and

$$\psi_{\partial(M\times I)}(q_i) = \frac{(-1)^{n+1} c_{n+1}}{\|c\|^2} \left(F|_{V_{q_i}}{}^* \psi_N \right)(q_i)$$

and the sign of $\sigma(F|_{V_{q_i}})$ is determined by $(-1)^{n+1} c_{n+1}$ so that

$$\sigma(F|_{V_{q_i}}) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Therefore,

$$\sigma(F|_{V_{p_i}}) + \sigma(F|_{V_{q_i}}) = 0$$

for $\{p_i, q_i\} = \partial A_i$.

Now our lemma follows as

$$\sum_{p' \in F|_{\partial(M\times I)}^{-1}(y')} \sigma\left(F|_{V_{p'}}\right) = \sum_{p' \in \coprod_{i=1}^{\Delta} \partial A_i} \sigma\left(F|_{V_{p'}}\right)$$

$$= \sum_{i=1}^{\Delta} \sigma\left(F|_{V_{p_i}}\right) + \sigma\left(F|_{V_{q_i}}\right)$$

$$= \sum_{i=1}^{\Delta} 0$$

$$= 0.$$

$\boxed{\text{END}}$

Lemma:

Suppose $f: M \longrightarrow N$ and $g: M \longrightarrow N$ are smoothly homotopic, and that their degrees are computed with respect to a common regular value $y \in N$. Then

$$\deg(f) = \deg(g).$$

proof:

Let $F: M \times I \longrightarrow N$ be the homotopy between f and g such that $F|_{M \times \{0\}} = f$ and $F|_{M \times \{1\}} = g$.

We orient $M \times I$ using $\eta_M \wedge dx$, which induces an orientation on $\partial(M \times I)$, and as shown in ① page 310, $M \times \{0\} = -(M \times \{1\})$.

Hence,

$$\deg(f)-\deg(g)=\sum_{p\in f^{-1}(y)}\sigma\left(f|_{V_p}\right)-\sum_{p\in g^{-1}(y)}\sigma\left(g|_{V_p}\right)$$

$$=\sum_{p\in F|_{M\times\{0\}}^{-1}(y)}\sigma\left(F|_{V_p\times\{0\}}\right)-\sum_{p\in F|_{M\times\{1\}}^{-1}(y)}\sigma\left(F|_{V_p\times\{1\}}\right)$$

$$=\sum_{p\in F|_{M\times\{0\}}^{-1}(y)}\sigma\left(F|_{V_p\times\{0\}}\right)+\sum_{p\in F|_{-(M\times\{1\})}^{-1}(y)}\sigma\left(F|_{V_p\times\{1\}}\right)$$

$$=\sum_{p\in F|_{\partial(M\times I)}^{-1}(y)}\sigma\left(F|_{V_p\subseteq\partial(M\times I)}\right)$$

$$=0$$

by the previous lemma.

$\boxed{\text{END}}$

Theorem:

For $f: M \to N$ where f is a smooth map between smooth oriented manifolds without boundary, of the same dimension, with M compact and N connected, $\deg(f)$ is well-defined with respect to the choice of regular value in N used to compute $\deg(f)$.

proof:

Let $y, y' \in N$ be regular values of f. By the lemma on page 314, there is some $h \in \mathrm{Diff}^+(N)$ such that $h(y) = y'$ and $h \simeq id_N$. Then $h \circ f \simeq id \circ f = f$.

Also note that $(h \circ f)^{-1}(y') = f^{-1}(h^{-1}(y')) = f^{-1}(y)$.

and that y' is a regular value of $h \circ f$ since $D_p(h\circ f) = D_{f(p)}h \circ D_p f$ for $p \in (h\circ f)^{-1}(y') = f^{-1}(y)$.

By the previous lemma,

$$\sum_{p' \in f^{-1}(y')} \sigma\left(f|_{V_{p'}}\right) = \sum_{p \in (h\circ f)^{-1}(y')} \sigma\left((h\circ f)|_{V_p}\right)$$

$$= \sum_{p \in f^{-1}(y)} \sigma\left(h|_{f(V_p)}\right)\sigma\left(f|_{V_p}\right)$$

$$= \sum_{p \in f^{-1}(y)} \sigma\left(f|_{V_p}\right)$$

since $h \in \mathrm{Diff}^+(N)$.

$\boxed{\text{END}}$

Corollary:

For $f: M \to N$ and $g: M \to N$ as above, $f \simeq g$ implies $\deg(f) = \deg(g)$.

Examples:

① For $n \in \mathbb{Z}$, let $\chi_n \in \mathrm{Diff}(S^1)$ be defined as $\chi_n(e^{i\theta}) = e^{in\theta}$. Then $\deg(\chi_n) = n$ so if $n \neq k$ then $\chi_n \not\simeq \chi_k$. (See page 79 example ③.)

② If n is odd, $\mathbb{P}^n(\mathbb{R})$ is orientable as explained in example ⑤ on page 281.

With the notation from that example, $\pi: S^n \to \mathbb{P}^n(\mathbb{R})$ is the quotient map and

$$\deg(\pi) = \sigma(\pi|_{V_1}) + \sigma(\pi|_{V_2}) = 1 + 1 = 2.$$

Degree mod 2

Suppose $f: M \longrightarrow N$ is a smooth map between smooth manifolds without boundary and of the same dimension. Suppose M is compact and N is connected, but that perhaps M and N are not oriented or even orientable.

All of the proofs from this chapter carry over in this context, except that we could not determine whether the terms $\sigma(f|_{V_p})$ in the definition of degree are $+1$ or -1. In characteristic 2, there is no ambiguity, and in that case $\deg(f) = |f^{-1}(y)| \pmod 2$ for a regular value $y \in N$ of f. The proofs in this chapter reduce in this context to proofs that $\deg(f) \bmod 2$ is well-defined with respect to choice of a regular value $y \in N$, and that $f \simeq g$ implies $\deg(f) = \deg(g) \pmod 2$.

Example:

For any n, and any $x \in \mathbb{P}^n(\mathbb{R})$, there are 2 points in the preimage of x under the quotient map $\pi : S^n \longrightarrow \mathbb{P}^n(\mathbb{R})$. Thus, $\deg(\pi) = 0 \pmod 2$.

We now prove a generalization of the change of variables formula with respect to integration and diffeomorphisms.

Theorem:

Let $f: M \longrightarrow N$ be a smooth map between smooth manifolds of the same dimension, n, and without boundary. Suppose M and N are oriented, M is compact, and N is connected.

If $\omega \in \Omega_c^n(N)$ then

$$\int_M f^*\omega = \deg(f) \int_N \omega$$

proof:

As in the lemma on page 320, let $y \in N$ be a regular value of f with an open neighborhood $U_y \subseteq N$ such that U_y is connected, $f^{-1}(U_y) = \coprod_{p \in f^{-1}(y)} V_p$, and $f|_{V_p} : V_p \longrightarrow U_y$ is a diffeomorphism for all $p \in f^{-1}(y)$.

Given $y_k \in N$, let $h_k \in \mathrm{Diff}^+(N)$ be such that $h_k(y) = y_k$ and $h_k \simeq id_N$ as in the lemma on page 314.

Since ω is compactly supported, there are finitely many h_k such that ω is supported on $\bigcup_{k=1}^{m} h_k(U_y)$.

Let $\{\rho_k\}$ be a partition of unity subordinate to $\{h_k(U_y)\}_{k=1}^m$. Since $f \simeq h_k \circ f$ and $d\rho_k\omega \in \Omega^{n+1}(N) = 0$, we have by exercise ② of page 310 that

$$\int_M f^*\omega = \int_M f^*\left(\sum_{k=1}^m \rho_k\omega\right)$$

$$= \sum_{k=1}^m \int_M f^*\rho_k\omega$$

$$= \sum_{k=1}^m \int_M (h_k\circ f)^*\rho_k\omega$$

$$= \sum_{k=1}^m \int_M f^* h_k^* \rho_k\omega$$

Note that $\rho_k\omega$ vanishes outside of $h_k(U_y)$, so $h_k^*\rho_k\omega$ outside U_y, so $f^*h_k^*\rho_k\omega$ outside $\coprod_{p\in f^{-1}(y)} V_p$.

Therefore,

$$\int_M f^*\omega = \sum_{k=1}^{m} \int_M f^* h_k^* \rho_k \omega$$

$$= \sum_{k=1}^{m} \sum_{p \in f^{-1}(y)} \int_{V_p} f^* h_k^* \rho_k \omega$$

$$= \sum_{k=1}^{m} \sum_{p \in f^{-1}(y)} \sigma\left(f|_{V_p}\right) \int_{U_y} h_k^* \rho_k \omega$$

$$= \sum_{k=1}^{m} \deg(f)\, \sigma\left(h_k|_{U_y}\right) \int_{h_k(U_y)} \rho_k \omega$$

$$= \deg(f) \sum_{k=1}^{m} \int_{h_k(U_y)} \rho_k \omega$$

$$= \deg(f) \int_N \omega .$$

$\boxed{\text{END}}$

Exercises:

① Let $f: M \longrightarrow N$ be a smooth map between smooth oriented manifolds of the same dimension and without boundaries. Let M be compact and N connected.

For $p \in f^{-1}(y)$ where $y \in N$ is a regular value of f, let $V_p, W_p \subseteq M$ be connected open neighborhoods of p. Prove $\sigma(f|_{V_p}) = \sigma(f|_{W_p})$.

② Let $f: M \to N$ be as above. Prove that if f is not surjective then $\deg(f) = 0$ and thus if N is not compact and $\omega \in \Omega_c^{\dim(N)}(N)$ then $\int_M f^*\omega = 0$. (For example, if $a\,dx \in \Omega_c^1(\mathbb{R})$ then $\int_{S^1} f^*(a\,dx) = 0$ for any smooth $f: S^1 \to \mathbb{R}$.)

③ Suppose $f(x) \in \mathbb{C}[x]$ is monic and has no roots. Prove the Fundamental Theorem of Algebra by showing $f = 1$. Use for $S^1 \subseteq \mathbb{C}$ the functions

$$F_1 : S^1 \times I \longrightarrow S^1, \quad F_1(x,t) = \frac{f(tx)}{|f(tx)|} \quad \text{and}$$

$$F_2 : S^1 \times I \longrightarrow S^1, \quad F_2(x,t) = \frac{t^n f\left(\frac{x}{t}\right)}{\left|t^n f\left(\frac{x}{t}\right)\right|}.$$

§ De Rham Cohomology

Suppose M is a smooth manifold and let $k \geq 0$. Motivated by Stokes' Theorem, we ask for which $\omega \in \Omega^k(M)$ is it true that $\omega = d\theta$ for some $\theta \in \Omega^{k-1}(M)$.

First note that if $\omega = d\theta$, then $d\omega = dd\theta = 0$, so we only need to consider ω in this question for which $d\omega = 0$. This brings us to cohomology.

Let $Z^k(M) \leq \Omega^k(M)$ be the kernel of $d : \Omega^k(M) \to \Omega^{k+1}(M)$, and let $B^k(M) \leq \Omega^k(M)$ be the image of $d : \Omega^{k-1}(M) \to \Omega^k(M)$. Because $d \circ d = 0$, $B^k(M) \leq Z^k(M)$ and we define the <u>k-th De Rham cohomology group of M</u> as the vector space $H^k(M) = Z^k(M) / B^k(M)$. In the above, by definition, $B^0(M) = 0$.

Note, for example, that $H^k(M)=0$ implies that for any $\omega\in\Omega^k(M)$ with $d\omega=0$, there is some $\theta\in\Omega^{k-1}(M)$ such that $d\theta=\omega$.

We can replace forms in the above with compactly supported forms:

- $Z^k_c(M)\leq\Omega^k_c(M)$ the kernel of $d:\Omega^k_c(M)\to\Omega^{k+1}_c(M)$
- $B^k_c(M)\leq\Omega^k_c(M)$ the image of $d:\Omega^{k-1}_c(M)\to\Omega^k_c(M)$
- $B^0_c(M)=0$
- $H^k_c(M)=Z^k_c(M)/B^k_c(M)$.

The vector space $H^k_c(M)$ is the <u>k-th compactly supported De Rham cohomology of M.</u>

Note that if M is compact then $H^k(M)=H^k_c(M)$ for all k, and if $k>\dim(M)$ then $\Omega^k(M)=\Omega^k_c(M)=0$ so $H^k(M)=H^k_c(M)=0$.

Proposition:

$$H^0(\mathbb{R}) = \mathbb{R} \quad \text{and} \quad H^k(\mathbb{R}) = 0 \text{ if } k \neq 0.$$

proof:

Since $\mathbb{R}$ is 1-dimensional, $H^k(\mathbb{R}) = 0$ if $k \geq 2$.

To see that $H^0(\mathbb{R}) = \mathbb{R}$, let $f \in Z^0(\mathbb{R}) \subseteq \Omega^0(\mathbb{R})$ so that $f'dx = df = 0$. That is, f' is a constant function, and since any constant function $f \in \Omega^0(\mathbb{R})$ has $df = 0$, we see that $Z^0(\mathbb{R}) = \mathbb{R}$. By definition, $B^0(\mathbb{R}) = 0$, so $H^0(\mathbb{R}) = Z^0(\mathbb{R}) / B^0(\mathbb{R}) = \mathbb{R}/0 = \mathbb{R}$.

To see that $H^1(\mathbb{R}) = 0$, let $f\,dx \in Z^1(\mathbb{R}) = \Omega^1(\mathbb{R})$. Then $\int_0^x f(t)\,dt \in \Omega^0(\mathbb{R})$ and $d\left(\int_0^x f(t)\,dt\right) = f\,dx$ so $f\,dx \in B^1(\mathbb{R})$. Hence, $Z^1(\mathbb{R}) = B^1(\mathbb{R})$ and $H^1(\mathbb{R}) = Z^1(\mathbb{R}) / B^1(\mathbb{R}) = 0$.

END

Lemma:

Let M be an n-dimensional smooth oriented manifold without boundary. Then

$$\int_M : H_c^n(M) \longrightarrow \mathbb{R}$$

is a well-defined linear map where $\int_M [\omega] = \int_M \omega$.

proof:

For $\omega + d\theta \in [\omega]$ with $\theta \in \Omega_c^{n-1}(M)$, we have by Stokes' Theorem

$$\int_M \omega + d\theta = \int_M \omega + \int_M d\theta = \int_M \omega + \int_{\partial M} \theta = \int_M \omega + \int_{\emptyset} \theta = \int_M \omega .$$

END

Proposition:

$H^1_c(\mathbb{R}) = \mathbb{R}$ and $H^k_c(\mathbb{R}) = 0$ if $k \neq 1$.

proof:

If $f \in Z^0_c(\mathbb{R})$ then $f' dx = df = 0$ so f is a compactly supported constant function on $\mathbb{R}$. Thus, $f = 0$, so $Z^0_c(\mathbb{R}) = 0$ and $H^0_c(\mathbb{R}) = 0$.

For $H^1_c(\mathbb{R})$, if $b \in \Omega^0_c(\mathbb{R})$ with $b \geq 0$ and $b \neq 0$ then $\int_{\mathbb{R}} [b dx] = \int_{\mathbb{R}} b dx \neq 0$ so $\int_{\mathbb{R}} : H^1_c(\mathbb{R}) \longrightarrow \mathbb{R}$ is a nontrivial linear function, and hence surjective. To see that it's injective as well, let $f dx \in Z^1_c(\mathbb{R}) = \Omega^1_c(\mathbb{R})$ be such that $\int_{\mathbb{R}} f dx = 0$. Choose $z \in \mathbb{R}$ less than the support of f so that $\int_z^x f(t) dt \in \Omega^0_c(\mathbb{R})$ and $d\left(\int_z^x f(t) dt\right) = f dx$. Thus $f dx \in B^1_c(\mathbb{R})$, so $\int_{\mathbb{R}} : H^1_c(\mathbb{R}) \rightarrow \mathbb{R}$ is bijective.

END

The two previous propositions are a special case of the following theorem, which will be expanded on in Math 6520.

Poincaré Duality:

Let M be an n-dimensional smooth oriented manifold without boundary. Then for $0 \le k \le n$,

$$H^k(M) \cong H_c^{n-k}(M)$$

Cohomology can reduce difficult questions about manifolds to more tractable problems in linear algebra. It does this by turning manifolds into vector spaces, and by turning smooth maps into linear functions.

Lemma:

Let $f: M \to N$ be a smooth map of smooth manifolds, and let $k \geq 0$. Then there is a well-defined linear map

$$f^* : H^k(N) \longrightarrow H^k(M)$$

where $f^*([\omega]) = [f^*\omega]$.

proof:

Suppose $\omega + d\theta \in [\omega] \in \Omega^k(N)$ where $\theta \in \Omega^{k-1}(N)$. By the lemma on page 263, $f^* d = d f^*$, so

$$[f^*(\omega + d\theta)] = [f^*\omega + f^* d\theta] = [f^*\omega + d f^*\omega] = [f^*\omega].$$

$\boxed{\text{END}}$

Note for $id \in \mathrm{Diff}(M)$, that $id^* = id$.

The above lemma remains true for compactly supported De Rham cohomology if we require f to be proper, so that $f^*\omega$ will be compactly supported if ω is.

Lemma:

$\bigoplus_{k=0}^{\dim(M)} H^k(M)$ can be endowed with the structure of a graded algebra using the following cup product as a multiplication:

$$[\omega] \cup [\theta] = [\omega \wedge \theta]$$

proof:

We check that the cup product is well-defined. By the lemma on page 263,

$$\begin{aligned}(\omega + d\psi) \wedge (\theta + d\varphi) &= \omega \wedge \theta + d\psi \wedge \theta + \omega \wedge d\varphi + d\psi \wedge d\theta \\ &= \omega \wedge \theta + d(\psi \wedge \theta) \pm \psi \wedge d\theta \pm d(\omega \wedge \varphi) \pm d\omega \wedge \varphi + d(\psi \wedge d\theta) \pm d\psi \wedge d^2\theta \\ &= \omega \wedge \theta + d(\psi \wedge \theta + \omega \wedge \varphi).\end{aligned}$$

$\boxed{\text{END}}$

Let $V, U \subseteq \mathbb{R}^n$ be open and $\xi : V \to U$ a diffeomorphism. Let $\Xi = id_{\mathbb{R}} \times \xi : \mathbb{R} \times V \to \mathbb{R} \times U$.

For $f_{I_{k-1}}, g_{I_k} \in C^\infty(\mathbb{R} \times U)$ we have

$$\sum_{I_{k-1}}{}' f_{I_{k-1}}(t,x)\, dt \wedge dx_{I_{k-1}} + \sum_{I_k}{}' g_{I_k}(t,x)\, dx_{I_k} \in \Omega^k(\mathbb{R} \times U),$$

and any smooth k-form on $\mathbb{R} \times U$ has this form.

We define $P : \Omega^k(\mathbb{R} \times U) \longrightarrow \Omega^{k-1}(U)$ by

$$P\Big(\sum_{I_{k-1}}{}' f_{I_{k-1}}(t,x)\, dt \wedge dx_{I_{k-1}} + \sum_{I_k}{}' g_{I_k}(t,x)\, dx_{I_k}\Big)$$

$$= \sum_{I_{k-1}}{}' \Big(\int_0^t f_{I_{k-1}}(s,x)\, ds\Big) dx_{I_{k-1}}$$

<u>Lemma:</u>

For any diffeomorphism $\xi : V \to U$ and any $\omega \in \Omega^k(\mathbb{R} \times U)$, $P \Xi^* \omega = \Xi^* P \omega$.

proof:

Let $\omega = \sum_{I_{k-1}} f_{I_{k-1}}(t,x)\, dt \wedge dx_{I_{k-1}} + \sum_{I_k} g_{I_k}(t,x)\, dx_{I_k}$.

Then

$$\Xi^* \omega = \sum_{I_{k-1}} \left(f_{I_{k-1}} \circ \Xi\right) \Xi^* dt \wedge \Xi^* dx_{I_{k-1}} + \sum_{I_k} \left(g_{I_k} \circ \Xi\right) \Xi^* dx_{I_k}$$

$$= \sum_{I_{k-1}} \left(f_{I_{k-1}} \circ \Xi\right) dt \wedge \Xi^* dx_{I_{k-1}} + \sum_{I_k} \left(g_{I_k} \circ \Xi\right) \Xi^* dx_{I_k}$$

so that

$$P \Xi^* \omega = \sum_{I_{k-1}} \left(\int_0^t f_{I_{k-1}}(s, \xi(y))\, ds \right) \Xi^* dx_{I_{k-1}}$$

where $y \in V$ with $\xi(y) = x$.

We also have that

$$\Xi^* P\omega = \Xi^* \left(\sum_{I_{k-1}} \left(\int_0^t f_{I_{k-1}}(s,x)\, ds \right) dx_{I_{k-1}} \right)$$

$$= \sum_{I_{k-1}} \left(\int_0^t f_{I_{k-1}}(s, \xi(y))\, ds \right) \Xi^* dx_{I_{k-1}}$$

$\boxed{\text{END}}$

Lemma:

For a smooth manifold M and any $k \geq 0$, there is a linear function

$$P: \Omega^k(\mathbb{R}\times M) \longrightarrow \Omega^{k-1}(M)$$

such that if $\xi \in \mathrm{Diff}(M)$ and $\Xi = id_{\mathbb{R}} \times \xi \in \mathrm{Diff}(\mathbb{R}\times M)$ then $P\,\Xi^* = \Xi^* P$ and such that P is as defined on page 358 for open sets $M \subseteq \mathbb{R}^n$.

proof:

Let $\omega \in \Omega^k(\mathbb{R}\times M)$. If (φ, U) is a chart on M containing $p \in M$, we define

$$P\omega(p) = \Phi^* P(\Phi^{-1})^* \omega(p)$$

This is well-defined because if p is contained in the chart (ψ, V) then

$$\begin{aligned}\Psi^* P(\Psi^{-1})^* \omega(p) &= \Psi^* (\Phi \circ \Psi^{-1})^* (\Psi \circ \Phi^{-1})^* P (\Psi^{-1})^* \omega(p) \\ &= \Phi^* P (\Psi \circ \Phi^{-1})^* (\Psi^{-1})^* \omega(p) \\ &= \Phi^* P(\Phi^{-1})^* \omega(p).\end{aligned}$$

For $\xi \in \mathrm{Diff}(M)$, we let (φ, U) be a chart containing p and (ψ, V) a chart containing $\psi(p)$.

If $\omega \in \Omega^k(\mathbb{R}\times M)$, then by the previous lemma,

$$
\begin{aligned}
P\Xi^*\omega(p) &= \Phi^* P(\Phi^{-1})^*\Xi^*\omega(p) \\
&= \Phi^* P(\Phi^{-1})^*\Xi^*\Psi^*(\Psi^{-1})^*\omega(p) \\
&= \Phi^* P(\Psi\circ\Xi\circ\Phi^{-1})^*(\Psi^{-1})^*\omega(p) \\
&= \Phi^*(\Psi\circ\Xi\circ\Phi^{-1})^* P(\Psi^{-1})^*\omega(p) \\
&= \Xi^*\Psi^* P(\Psi^{-1})^*\omega(p) \\
&= \Xi^* P\omega(p)
\end{aligned}
$$

so $P\Xi^* = \Xi^* P$.

END

Lemma:

Let $\pi : \mathbb{R}\times M \longrightarrow M$ be projection.
For $a \in \mathbb{R}$, let $i_a : M \rightarrow \mathbb{R}\times M$ be $i_a(p) = (a,p)$.
Then

$$dP\omega + Pd\omega = \omega - \pi^* i_0^* \omega$$

proof:

Let (φ, U) be a chart containing p. Let x be the local coordinates in U so that

$$\omega\big|_{\mathbb{R}\times U}(t,x) = \sum_{I_{k-1}} f_{I_{k-1}}(t,x)\, dt \wedge dx_{I_{k-1}} + \sum_{I_k} g_{I_k}(t,x)\, dx_{I_k}$$

for some $f_{I_{k-1}}, g_{I_k} \in C^\infty(\mathbb{R}\times U)$. Thus, in $\mathbb{R}\times U$ we have

$$d(P\omega) = d\left(\sum_{I_{k-1}}' \left(\int_0^t f_{I_{k-1}}(s,x)\,ds\right) dx_{I_{k-1}}\right)$$

$$= \sum_{I_{k-1}}' d\left(\int_0^t f_{I_{k-1}}(s,x)\,ds\right) dx_{I_{k-1}}$$

$$= \sum_{I_{k-1}}' \left(f_{I_{k-1}} dt + \sum_{i=1}^{n} \frac{\partial \int_0^t f_{I_{k-1}}(s,x)\,ds}{\partial x_i} dx_i \right) \wedge dx_{I_{k-1}}$$

and

$$P(d\omega) = P\left(\sum_{I_{k-1}}' df_{I_{k-1}}(t,x)\,dt \wedge dx_{I_{k-1}} + \sum_{I_k}' dg_{I_k}(t,x)\,dx_{I_k}\right)$$

$$= P\Bigg(\sum_{I_{k-1}}' \sum_{i=1}^{n} \frac{\partial f_{I_{k-1}}(t,x)}{\partial x_i} dx_i \wedge dt \wedge dx_{I_{k-1}}$$

$$+ \sum_{I_k}' \left(\frac{\partial g_{I_k}(t,x)}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial g_{I_k}(t,x)}{\partial x_i} dx_i\right) \wedge dx_{I_k}\Bigg)$$

$$= P\Bigg(-\sum_{I_{k-1}}' \sum_{i=1}^{n} \frac{\partial f_{I_{k-1}}(t,x)}{\partial x_i} dt \wedge dx_i \wedge dx_{I_{k-1}}$$

$$+ \sum_{I_k}' \frac{\partial g_{I_k}(t,x)}{\partial t} dt \wedge dx_{I_k}\Bigg)$$

$$= -\sum_{I_{k-1}}' \sum_{i=1}^{n} \left(\int_0^t \frac{\partial f_{I_{k-1}}(s,x)}{\partial x_i} ds\right) dx_i \wedge dx_{I_{k-1}} + \sum_{I_k}' \left(\int_0^t \frac{\partial g_{I_k}(s,x)}{\partial s} ds\right) dx_{I_k}$$

so that

$$dP\omega + Pd\omega = \sum_{I_{k-1}}' f_{I_{k-1}}(t,x)\,dt\wedge dx_{I_{k-1}} + \sum_{I_k}' \left(\int_0^t \frac{\partial g_{I_k}(s,x)}{\partial s}\,ds\right) dx_{I_k}.$$

Furthermore,

$$\begin{aligned}\pi^* i_0^* \omega &= \pi^* i_0^* \Big(\sum_{I_{k-1}}' f_{I_{k-1}}(t,x)\,dt\wedge dx_{I_{k-1}} + \sum_{I_k}' g_{I_k}(t,x)\,dx_{I_k} \Big) \\ &= \pi^* \Big(\sum_{I_k}' g_{I_k}(0,x)\,dx_{I_k} \Big) \\ &= \sum_{I_k}' g_{I_k}(0,x)\,dx_{I_k}\end{aligned}$$

so that

$$\omega - \pi^* i_0^* \omega = \sum_{I_{k-1}}' f_{I_{k-1}}(t,x)\,dt\wedge dx_{I_{k-1}} + \sum_{I_k}' \big(g(t,x) - g(0,x)\big)\,dx_{I_k}$$

and hence, $dP\omega + Pd\omega = \omega - \pi^* i_0^* \omega$ as desired.

$\boxed{\text{END}}$

Poincaré Lemma:

Let M be a smooth manifold, and let $k \geq 0$. Then $H^k(\mathbb{R} \times M) \cong H^k(M)$. In particular, $H^k(\mathbb{R}^n) = H^k(\mathbb{R}^0)$ is $\mathbb{R}$ if $k=0$ and 0 if $k \neq 0$.

proof:

For $[\omega] \in H^k(M)$,

$$i_0^* \pi^* [\omega] = [i_0^* \pi^* \omega] = [(\pi \circ i_0)^* \omega] = [id^* \omega] = [\omega].$$

For $[\omega] \in H^k(\mathbb{R} \times M)$, the previous lemma yields

$$\pi^* i_0^* [\omega] = [\omega - dP\omega - Pd\omega] = [\omega - dP\omega - P(0)] = [\omega].$$

Therefore, $\pi^* : H^k(M) \longrightarrow H^k(M \times \mathbb{R})$ and $i_0^* : H^k(M \times \mathbb{R}) \longrightarrow H^k(M)$ are inverses.

$\boxed{\text{END}}$

Proposition:

Suppose $f, g : M \to N$ are smooth maps between smooth manifolds and that $f \simeq g$. Then $f^* = g^* : H^k(N) \to H^k(M)$ for all k.

proof:

Let $F : I \times M \to N$ be the smooth homotopy between f and g. We will see in the next chapter that we may extend F to a smooth map $F : \mathbb{R} \times M \to N$, where we still have $F \circ i_0 = f$ and $F \circ i_1 = g$. (See exercise #3, page 380.)

As in the proof of the Poincaré Lemma, $i_1^* \pi^* = (\pi \circ i_1)^* = id^* = id$, so that both i_0^* and i_1^* are the inverse of π^* and thus $i_0^* = i_1^*$. Therefore,

$$f^* = (F \circ i_0)^* = i_0^* F^* = i_1^* F^* = (F \circ i_1)^* = g^*.$$

$\boxed{\text{END}}$

We say that the smooth manifolds M and N are <u>smoothly homotopy equivalent</u> $(M \simeq N)$ if there are smooth maps $f: M \to N$ and $g: N \to M$ with $f \circ g \simeq id_N$ and $g \circ f \simeq id_M$.

<u>Examples:</u>

① $\mathbb{R}^n \simeq \{0\}$. Let $f: \mathbb{R}^n \to \{0\}$ be constant and $g: \{0\} \to \mathbb{R}^n$ inclusion. Then $f \circ g = id_{\{0\}}$ and $g \circ f = f \simeq id_{\mathbb{R}^n}$ by $F: \mathbb{R}^n \times I \to \mathbb{R}^n$, $F(p,t) = tp$.

② $(\mathbb{R}^n - \{0\}) \simeq S^{n-1}$. Let $f: \mathbb{R}^n - \{0\} \to S^{n-1}$ be $f(p) = \frac{p}{\|p\|}$ and $g: S^{n-1} \to \mathbb{R}^n - \{0\}$ inclusion. Then $f \circ g = id_{S^{n-1}}$ and $g \circ f = f \simeq id_{\mathbb{R}^n \times \{0\}}$ by $F: (\mathbb{R}^n \times \{0\}) \times I \to \mathbb{R}^n \times \{0\}$, $F(p,t) = tp + (1-t)\frac{p}{\|p\|}$.

Proposition:

If $M \simeq N$, then $H^k(M) = H^k(N)$ for all $k \geq 0$.

proof:

Let $f: M \longrightarrow N$ and $g: N \longrightarrow M$ be such that $f \circ g \simeq id$ and $g \circ f \simeq id$. By the previous proposition,

$$f^* g^* = (g \circ f)^* = id^* = id$$

and

$$g^* f^* = (f \circ g)^* = id^* = id$$

so that $f^*: H^k(N) \longrightarrow H^k(M)$ and $g^*: H^k(M) \longrightarrow H^k(N)$ are inverses.

$\boxed{\text{END}}$

In Math 6520, you'll see the following result which relates to the first chapter of this text.

Proposition:

Let Σ_g be a closed, orientable, smooth surface of genus g. Then

$$\sum_{k=0}^{2} (-1)^k \dim\left(H^k(\Sigma_g)\right) = \chi(\Sigma_g).$$

Exercises:

① Let ℓ be the number of connected components of a smooth manifold M. Prove $H^0(M) = \mathbb{R}^\ell$.

② Let M be an n-dimensional, orientable, smooth manifold without boundary. Prove $H^n_c(M)$ is nontrivial.

Use Poincaré Duality to prove ③-⑤.

③ $H^n_c(\mathbb{R}^n) = \mathbb{R}$ and $H^k(\mathbb{R}^n) = 0$ if $k \neq n$.

④ $H^0(\Sigma_g) = H^2(\Sigma_g)$ and $H^1(\Sigma_g) = \mathbb{R}^{2g}$. (Use previous proposition)

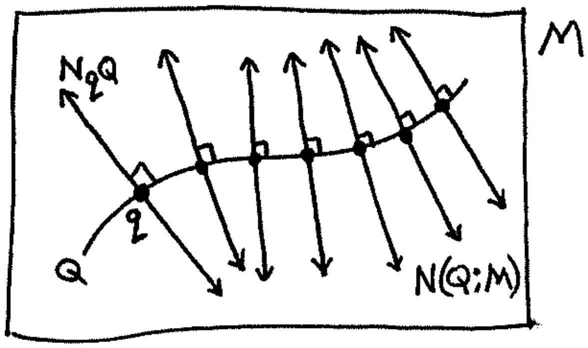
⑤ $S^n \not\cong \mathbb{R}^k$.

§ Tubular Neighborhoods

Suppose Q and M are smooth manifolds, and that $Q \subseteq M$ is an embedded submanifold. We assume M is embedded in $\mathbb{R}^n$.

For $q \in Q$, we have $T_qQ \leq T_qM \leq T_q\mathbb{R}^n = \mathbb{R}^n$. We let $N_qQ = \{ v \in T_qM \mid \langle v, T_qQ \rangle = 0 \}$, the <u>normal space</u> of Q at q, so that $T_qQ \oplus N_qQ$ equals T_qM.

We let $N(Q;M) \to Q$ be the subbundle of $TQ \to Q$ given by vectors normal to M. That is, for $q \in Q$, the fiber over q in $N(Q;M)$ is N_qQ.



$N(Q;M)$ is the <u>normal bundle</u> of Q in M.
It has rank $\dim(M) - \dim(Q)$ and dimension $\dim(M)$.

If Q_0 is the union of zero vectors in the fibers of $N(Q,M) \to Q$, then Q_0 is an embedded submanifold of $N(Q,M)$ that is diffeomorphic to Q.

Tubular Neighborhood Theorem:

Suppose Q is an embedded submanifold of M. Then there is an open $W \subseteq M$ containing Q and an open $V \subseteq N(Q;M)$ containing Q_0, and a diffeomorphism $f: W \longrightarrow V$ such that if $q \in Q$ and $0_q \in T_q Q$ is the zero vector, then $f(q) = 0_q$ so that $f|_Q : Q \longrightarrow Q_0$ is a diffeomorphism.

Furthermore, there is a submersion $F: W \longrightarrow Q$ such that $F|_Q = id_Q$.

proof:

To simplify this proof, we will assume that Q is compact and $M = \mathbb{R}^n$.

Let $g: N(Q;\mathbb{R}^n) \to \mathbb{R}^n$ be $g(v_q) = v_q + q$ for $v_q \in N_qQ$. Then $g(0_q) = q$

Recall that $\mathbb{R}^n = N_qQ \oplus T_qQ$ and note that $D_{0_q}g = id_{N_qQ} \oplus id_{T_qQ}$ so that $D_{0_q}g$ is bijective. By the Inverse Function Theorem, there is an open $U_q \subseteq N(Q;\mathbb{R}^n)$ containing 0_q such that $g|_{U_q}: U_q \to g(U_q)$ is a diffeomorphism.

For $n \in \mathbb{N}$, let $V_n = \{v_q \in N(Q;\mathbb{R}^n) \mid \|v_q\| < 1/n\}$. We claim $g|_{V_N}$ is injective for some N. Indeed, if not, then for each $n \in \mathbb{N}$ choose distinct $v_{q_n}, w_{p_n} \in V_n$ such that $g(v_{q_n}) = g(w_{p_n})$. By the compactness of Q, we may assume $q_n \to q$ and $p_n \to p$ for some $q, p \in Q$.

Thus, $v_{q_n} \to 0_q$ and $w_{p_n} \to 0_p$ so

$$q = g(0_q) = g(0_p) = p$$

so for $n >> 0$, $v_{q_n}, w_{p_n} \in U_q$ and

$$g|_{U_q}(v_{q_n}) = g|_{U_q}(w_{p_n}).$$

Since $g|_{U_q}$ is bijective, $v_{q_n} = w_{p_n}$, which is our desired contradiction. Thus, there is some N such that $g|_{V_N}$ is injective.

By the compactness of Q, there are $\{q_1, \dots, q_k\} \subseteq Q$ such that $Q \subseteq \bigcup_{i=1}^{k} U_{q_i}$, so that $g|_{\bigcup_{i=1}^{k} U_{q_i}} : \bigcup_{i=1}^{k} U_{q_i} \to g(\bigcup_{i=1}^{k} U_{q_i})$ is a proper, local diffeomorphism that is defined on an open set in $N(Q; \mathbb{R}^n)$ containing Q. (See exercise #2 on page 380.)

Let $V = V_N \cap \left(\bigcup_{i=1}^{k} U_{q_i} \right)$ so that $g|_V : V \to g(V)$ is a bijective, proper, local diffeomorphism. Then $g(V) \subseteq \mathbb{R}^n$ is an embedded submanifold. Let $W = g(V)$ and $f = g|_V^{-1}$.

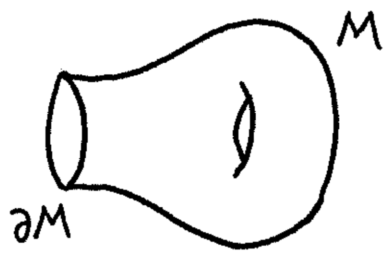
If $b : N(Q;\mathbb{R}^n) \to Q$ is the bundle projection, then let $F = b \circ g|_V^{-1} : W \to Q$. F is a submersion since b and $g|_V^{-1}$ are, and if $q \in Q$, then

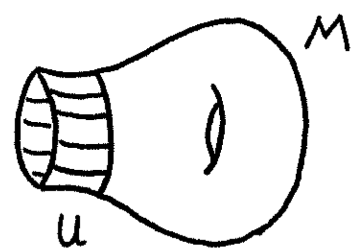
$$F(q) = b \circ g|_V^{-1}(q) = b(0_q) = q.$$

$\boxed{\text{END}}$

Collar Neighborhood Theorem:

Suppose M is a smooth manifold with boundary. Then there is an open $U \subseteq M$ containing ∂M such that U is diffeomorphic to $\partial M \times [0,1)$.



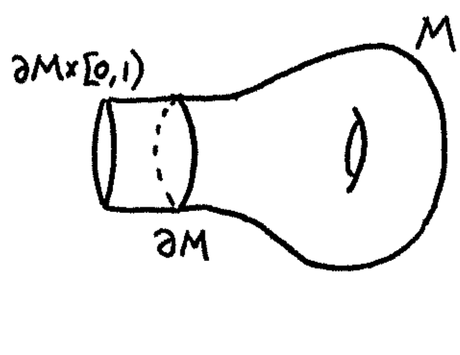


proof:

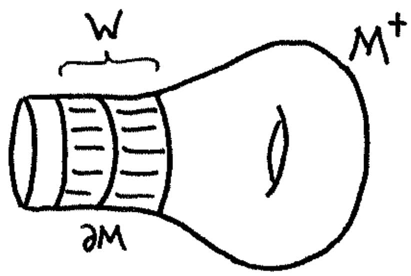
It suffices to prove for the case ∂M is connected. See exercise #2 on page 380.

Let M^+ be the smooth manifold

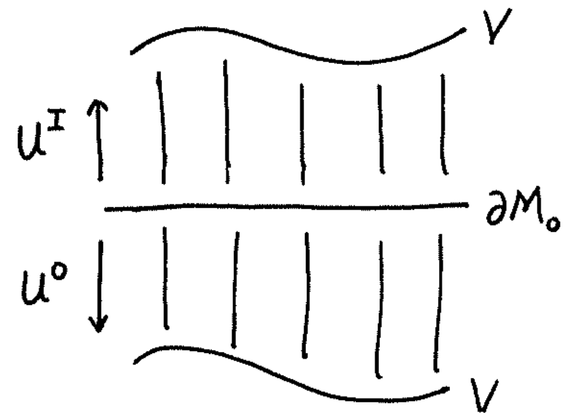
$$M \amalg \big(\partial M \times [0,1)\big) \Big/ \begin{array}{c} p \sim (p,0) \\ \text{for } p \in \partial M \end{array}$$



Note that ∂M is embedded in M and in $\partial M \times [0,1)$, so ∂M is embedded in M^+. Therefore, by the Tubular Neighborhood Theorem, there are open $W \subseteq M^+$ and $V \subseteq N(\partial M; M^+)$, containing ∂M and ∂M_0 respectively, with a diffeomorphism $f: W \to V$ such that $f|_{\partial M}: \partial M \to \partial M_0$ is an identification.



Let $U^I, U^O \subseteq N(\partial M; M^+)$ be the sets of inward and outward normal vectors, respectively, with respect to ∂M in M.



The set $(U^I \cup \partial M_0) \cap V$ contains a smooth manifold of the form $\mathcal{O} = \bigcup_{p \in \partial M} (\{p\} \times [0, r(p)))$ for some $r \in C^\infty(\partial M)$. Let $U = f^{-1}(\mathcal{O}) \subseteq M$, so that U is an open neighborhood of ∂M that is diffeomorphic to $\mathcal{O}$.

Last, note that $\mathcal{O}$ is diffeomorphic to $\partial M \times [0,1)$. Indeed, if $p \in \partial M$ and $t \in [0, r(p))$, then

$$\begin{aligned} \mathcal{O} &\longrightarrow \partial M \times [0,1) \\ (p,t) &\longmapsto \left(p, \frac{t}{r(p)}\right) \end{aligned}$$

is a diffeomorphism.

$\boxed{\text{END}}$

Exercises:

① Prove that $g|_{\bigcup_{i=1}^{k} U_{q_i}}$ from the last paragraph of page 375 is proper.

② Prove that the Collar Neighborhood Theorem for connected ∂M implies the general Collar Neighborhood Theorem.

③ As used in the proof on page 366, show that if M and N are smooth manifolds, and if $F : I \times M \longrightarrow N$ is smooth, then there is a smooth $\hat{F} : \mathbb{R} \times M \longrightarrow N$ such that $\hat{F}|_{I \times M} = F$.

§ Sard's Theorem

In this appendix we will show that the set of critical values of a smooth function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has measure zero in $\mathbb{R}^m$. Since a smooth manifold can be assumed to have a countable collection of charts, and since diffeomorphisms preserve sets of measure zero, we will be justified in stating that the critical values between smooth manifolds have measure zero.

Again, in this appendix, $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is smooth. We let $C \subseteq \mathbb{R}^n$ be the set of $p \in \mathbb{R}^n$ such that $D_p f$ is not surjective, so that $f(C)$ is the set of critical values of f. Thus, our goal is to show that $f(C)$ has measure zero in $\mathbb{R}^m$.

For $k \in \mathbb{N}$, we let $C_k \subseteq C$ be the set of all $p \in C$ such that all the ℓ^{th} partial derivatives of the coordinate functions of f equal 0 at p if $\ell \leq k$. Note that if $K \in \mathbb{N}$, then $C = C_K \cup \left(\bigcup_{k=0}^{K-1} (C_k - C_{k+1}) \right)$ where C_0 is taken to be C. Thus, our goal will be to show that for some $K \in \mathbb{N}$,

$$f(C_K) \cup \left(\bigcup_{k=0}^{K-1} f(C_k - C_{k+1}) \right)$$

has measure zero in $\mathbb{R}^m$.

<u>Lemma</u>: If $K \in \mathbb{N}$ is large enough to imply that $n - m(K+1) < 0$, then $f(C_K)$ has measure zero in $\mathbb{R}^m$.

proof:

Let $\Delta \subseteq \mathbb{R}^n$ be a coordinate cube with edge lengths equal to 1. Since Δ is compact, the norms of the $(K+1)^{st}$ derivatives of f are bounded on Δ. Thus, if $p \in C_K \cap \Delta$ and $h \in \mathbb{R}^n$ is such that $p+h \in \Delta$, then by Taylor's Theorem there is some $R(p,h) \in \mathbb{R}^m$ and some constant $\lambda > 0$ depending only on f and Δ such that $f(p+h) = f(p) + R(p,h)$ and $\|R(p,h)\| \leq \lambda \|h\|^{K+1}$.

Let $r \in \mathbb{N}$ and subdivide Δ into cubes $\Delta_i \subseteq \Delta$ for $1 \leq i \leq r^n$ of edge length $\frac{1}{r}$. If for some Δ_i we have $p \in \Delta_i \cap C_K$, then for any $p+h \in \Delta_i$ we have $\|h\| \leq \frac{\sqrt{n}}{r}$ and thus $\lambda \|h\|^{K+1} = \lambda \left(\frac{\sqrt{n}}{r}\right)^{K+1}$ so $f(\Delta_i \cap C_K)$

is contained in a cube of edge length $2\lambda\left(\frac{\sqrt{n}}{r}\right)^{K+1}$ and measure $\left(2\lambda\left(\frac{\sqrt{n}}{r}\right)^{K+1}\right)^m$. Therefore, $f(\Delta \cap C_K) = \bigcup_{i=1}^{r^n} f(\Delta_i \cap C_K)$ has its measure bounded above by

$$r^n\left(2\lambda\left(\frac{\sqrt{n}}{r}\right)^{K+1}\right)^m = \alpha\, r^{n-m(K+1)}$$

where α is a constant depending only on f, Δ, and n.

Since $n-m(K+1)<0$, we see that the measure of $f(\Delta \cap C_K)$ converges to zero as $r \longrightarrow \infty$. But the measure of $f(\Delta \cap C_K)$ is independent of r, so it equals zero.

As $\mathbb{R}^n$ is covered by countably many such cubes $\Delta \subseteq \mathbb{R}^n$, we see that the measure of $f(C_K)$ equals zero.

$\boxed{\text{END}}$

<u>Lemma</u>: If the critical values of any smooth function $\mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{m-1}$ have measure zero in $\mathbb{R}^{m-1}$, then $f(C - C_1)$ has measure zero in $\mathbb{R}^m$.

<u>proof</u>:

If $m=1$, then $p \in C$ implies $p \in C_1$. That is, $C - C_1 = \emptyset$ so $f(C - C_1) = \emptyset$ has measure zero. We are left to consider the case when $m \geq 2$.

Let $p \in C - C_1$ so that, perhaps after permuting the coordinates of $\mathbb{R}^n$ and $\mathbb{R}^m$, we may assume $\frac{\partial \pi_1 \circ f}{\partial x_1}(p) \neq 0$.

Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be defined for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as $h(x) = (\pi_1 \circ f(x), x_2, \dots, x_n)$. Thus,

$$D_p h = \left(\begin{array}{c|ccc} \frac{\partial \pi_1 \circ f}{\partial x_1}(p) & * & \cdots & * \\ \hline 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{array}\right)$$

so $\det(D_p h) = \frac{\partial \pi_1 \circ f}{\partial x_1}(p) \neq 0$ and $D_p h$ is an isomorphism which implies by the Inverse Function Theorem that there is some open $U \subseteq \mathbb{R}^n$ containing p such that $h(U) \subseteq \mathbb{R}^n$ is open and $h|_U : U \to h(U)$ is a diffeomorphism.

Let $g : h(U) \to \mathbb{R}^m$ be defined as $g = f \circ (h|_U)^{-1}$. Since $(h|_U)^{-1}$ is a diffeomorphism, the critical values of g are those of $f|_U$, which contains $f(U \cap C)$ and thus $f(U \cap (C - C_1))$.

Recall that $h(x_1, x_2, \ldots, x_n) = (\pi_1 \circ f(x_1, \ldots, x_n), x_2, \ldots, x_n)$ so that if $(t, x_2, \ldots, x_n) \in h(U) \subseteq \mathbb{R}^n$, then there is some $x_1 \in \mathbb{R}$ such that $(x_1, x_2, \ldots, x_n) \in U$ and $\pi_1 \circ f(x_1, x_2, \ldots, x_n) = t$. Therefore, if we let $x = (x_1, x_2, \ldots, x_n)$, then

$$\begin{aligned} g(t, x_2, \ldots, x_n) &= f \circ (h|_U)^{-1}(\pi_1 \circ f(x), x_2, \ldots, x_n) \\ &= f(x) \\ &= (\pi_1 \circ f(x), \ldots, \pi_m \circ f(x)) \\ &\in \{t\} \times \mathbb{R}^{m-1} \end{aligned}$$

That is, $g\big((\{t\} \times \mathbb{R}^{n-1}) \cap h(U)\big) \subseteq \{t\} \times \mathbb{R}^{m-1}$.

We now let $g_t = g|_{(\{t\} \times \mathbb{R}^{n-1}) \cap h(U)}$.

Let $CV_t \subseteq \{t\} \times \mathbb{R}^{m-1}$ be the critical values of g_t. We would like to conclude that CV_t has measure zero in $\{t\}\times\mathbb{R}^{n-1}$ for all t, and by covering $(\{t\}\times\mathbb{R}^{n-1})\cap h(U)$ by countably many open sets diffeomorphic to $\mathbb{R}^{n-1}$, we can conclude as we wish using the assumption in the statement of this lemma.

Now we claim that the critical values of g are $\bigcup_{t\in\mathbb{R}} CV_t$. Indeed, from the above we see that

$$D_{(t,x_2,\dots,x_n)}g = \left(\begin{array}{c|c} 1 & 0 \cdots\cdots 0 \\ \hline * & \\ \vdots & D_{(x_2,\dots,x_n)}g_t \\ * & \end{array}\right)$$

so that $D_{(t,x_2,\dots,x_n)}g$ is surjective exactly when $D_{(x_2,\dots,x_n)}g_t$ is, verifying our claim that the critical values of g are $\bigcup_{t\in\mathbb{R}} CV_t$.

Since each CV_t has measure zero in $\{t\} \times \mathbb{R}^{m-1}$, we have by Fubini's Theorem that the critical values of g have measure zero in $\mathbb{R}^m$. As noted earlier, the critical values of g contain $f(U \cap (C - C_1))$, so this set too has measure zero.

To complete the proof, note that there is a countable collection of $p \in C - C_1$ as above, with corresponding open U, such that all the sets U cover $C - C_1$. Thus, $f(C - C_1)$ has measure zero.

$\boxed{\text{END}}$

Lemma: If the critical values of any smooth function $\mathbb{R}^{n-1} \to \mathbb{R}^m$ have measure zero in $\mathbb{R}^m$, then $f(C_k - C_{k+1})$ has measure zero in $\mathbb{R}^m$ for any $k \in \mathbb{N}$.

proof:

Let $p \in C_k - C_{k+1}$. Then $\dfrac{\partial^{k+1} \pi_r \circ f}{\partial x_{j(1)} \cdots \partial x_{j(k+1)}}(p) \neq 0$ for some $r \in \{1, \ldots, m\}$ and some $j : \{1, \ldots, k+1\} \to \{1, \ldots, n\}$ while all k^{th} partial derivatives of the coordinate functions of f vanish at p.

After possibly permuting the coordinates of $\mathbb{R}^n$ and $\mathbb{R}^m$, we may assume that

$$\frac{\partial}{\partial x_1}\left(\frac{\partial^k \pi_1 \circ f}{\partial x_{j(1)} \cdots \partial x_{j(k)}}\right)(p) \neq 0 \quad \text{and} \quad \frac{\partial^k \pi_1 \circ f}{\partial x_{j(1)} \cdots \partial x_{j(k)}}(p) = 0.$$

Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be defined for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as $h(x) = \left(\dfrac{\partial^k \pi_1 \circ f}{\partial x_{j(1)} \cdots \partial x_{j(k)}}(x), x_2, \ldots, x_n\right)$.

Thus,

$$D_p h = \left(\begin{array}{c|ccc} \frac{\partial}{\partial x_1}\left(\frac{\partial^k \pi_1 \circ f}{\partial x_{j(1)} \cdots \partial x_{j(k)}}\right)(p) & * & \cdots & * \\ \hline 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{array}\right)$$

so there is an open set $U \subseteq \mathbb{R}^n$ containing p such that $h(U) \subseteq \mathbb{R}^n$ is open and $h|_U : U \to h(U)$ is a diffeomorphism.

Let $g = f \circ (h|_U)^{-1} : h(U) \longrightarrow \mathbb{R}^m$,

and let $g_0 = g|_{(\{0\} \times \mathbb{R}^{n-1}) \cap h(U)}$.

Note that for $q \in C_k \cap U$,

$\dfrac{\partial^k \pi_1 \circ f}{\partial x_{j(1)} \cdots \partial x_{j(k)}}(q) = 0$ so that $h(C_k \cap U) \subseteq \{0\} \times \mathbb{R}^{n-1}$

and for $q \in C_k \cap U$ we have $g_0 \circ h(q) = g \circ h(q) = f(q)$.
That is, $g_0 \circ h(C_k \cap U) = f(C_k \cap U)$.

Furthermore, the values in $g_0 \circ h(C_k \cap U)$ are critical values of g_0 since if $q \in C_k \cap U$ then $q \in C_1$ so that $D_q f = 0$ and hence

$$\left(\begin{array}{c|c} * & \\ \vdots & D_{h(q)} g_0 \\ * & \end{array}\right) = D_{h(q)} g = D_{h(q)} f \circ (h|_U)^{-1} = D_q f \circ (D_{h(q)} h|_U)^{-1} = 0.$$

Covering the domain of g_0 by countably many sets diffeomorphic to $\mathbb{R}^{n-1}$ and applying the assumption from the statement of the lemma yields that $f(C_k \cap U) = g_0 \circ h(C_k \cap U)$ has measure zero. Then covering $C_k - C_{k+1}$ with countably many U as above yields that $f(C_k - C_{k+1})$ has measure zero.

END

Proof of Sard's Theorem:

Recall that $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is smooth. If $n=0$ and $f(C)\neq\emptyset$, then $f(C)$ is a point and $m\geq 1$, so $f(C)$ has measure zero in $\mathbb{R}^m$.

Now assume the theorem is true for any smooth map $f:\mathbb{R}^{n-1}\longrightarrow\mathbb{R}^{\ell}$ where $\ell\geq 0$ is arbitrary. For $K\in\mathbb{N}$ with $n-m(K+1)<0$ we have

$$f(C)=f(C_K)\cup\left(\bigcup_{k=0}^{K-1} f(C_k-C_{k+1})\right)$$

where we take C_0 to be C. That this set has measure zero is the content of the previous three lemmas.

$\boxed{\text{END}}$

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