Unions of Solutions

We'll begin this chapter by discussing unions of sets. Then we'll focus our attention on unions of sets that are solutions of polynomial equations.

Unions

If B is a set, and if C is a set, then $B \cup C$ (spoken as "B union C") is the set of every object that is either in B or in C. That is,

$$B \cup C = \{\, a \mid a \in B \text{ or } a \in C \,\}$$

Examples.

- $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$
- $\{a, b, c\} \cup \{7, 8, 9\} = \{a, b, c, 7, 8, 9\}$
- $\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, 4, 5, \dots\}$
- $\{1\} \cup \{1, 2, 3\} = \{1, 2, 3\}$
- If B is a set, then $B \cup B = B$.
- If $C \subseteq B$, then $C \cup B = B$.

Unions of solutions

Claim: Suppose that S is the set of solutions of the equation $p(x, y) = 0$ and that V is the set of solutions of the equation $q(x, y) = 0$. Then $S \cup V$ is the set of solutions of the equation $p(x, y)q(x, y) = 0$.

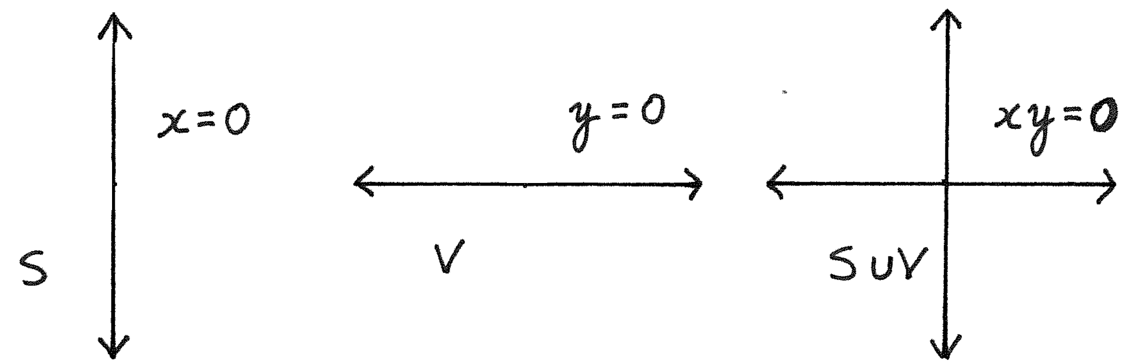
We'll look at the proof of this claim at the end of the chapter. Before then, we'll look at three examples that illustrate the claim.

Examples.

- The equation $x = 0$ has as its set of solutions all of those points in the plane whose x-coordinates equal 0. That is, if S is the set of solutions of $x = 0$, then S is the y-axis.

If V is the set of solutions of the equation $y = 0$, that is if V is the set of all points whose y-coordinates equal 0, then V is the x-axis.

The equation $xy = 0$ has $S \cup V$ as its set of solutions. That's because in order for the product of two numbers to equal 0, at least one of those two numbers must be zero. That is, if a point in the plane is a solution of $xy = 0$, either its x-coordinate equals 0 (which means that point is in S, the y-axis) or its y-coordinate equals 0, (which means that the point is in V, the x-axis). To say that the solutions of $xy = 0$ are points either in S or in V is to say that the solutions of $xy = 0$ are points in $S \cup V$.



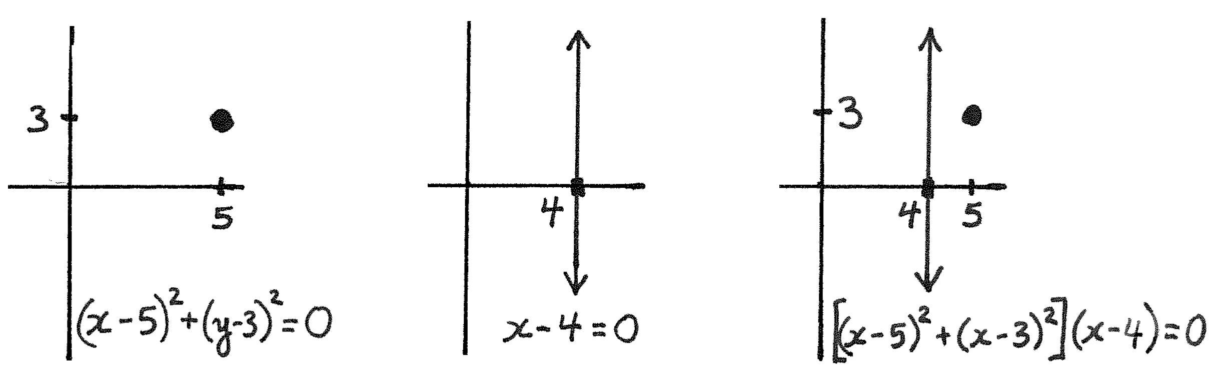
• We saw in exercise #14 from the chapter on Polynomial Equations that the equation $(x-5)^2+(x-3)^2 = 0$ has a single solution, the point $(5, 3)$.

The equation $x - 4 = 0$ is the equation for a vertical line. It's equivalent to the equation $x = 4$. The solutions are the vertical line of points whose x-coordinates equal 4.

The claim above states that the set of solutions of the equation

$$\left[(x-5)^2 + (x-3)^2\right](x-4) = 0$$

is the union of the single point $(5, 3)$ and the vertical line that crosses the x-axis at the number 4.



• The equation $y = x$ has as its solutions the line of slope 1 that passes through the point $(0, 0)$. The solutions of the equation $y = -x$ are a line of slope -1 that passes through the point $(0, 0)$.

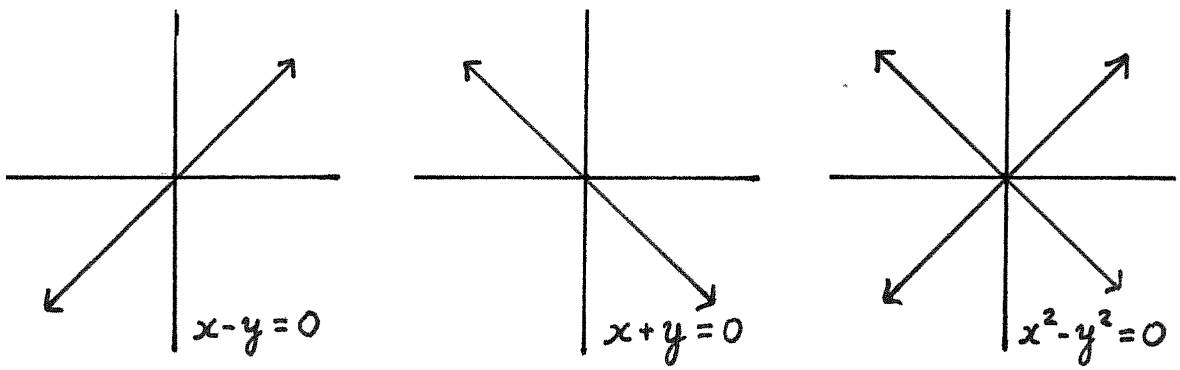
To use the claim from the beginning of this chapter, we need equations that have a zero on one side of the equal sign (they need to look like $p(x, y) = 0$), so we will rewrite the equations $y = x$ and $y = -x$ as $x - y = 0$ and $x + y = 0$, respectively. Now the claim states that the set of solutions of

$$(x - y)(x + y) = 0$$

an equation that is perhaps written better as

$$x^2 - y^2 = 0$$

is the union of the lines of slope 1 and -1 that contain the point $(0, 0)$.



Now let's return to the proof of the claim from the beginning of this chapter.

Proof: The claim was that if S is the set of solutions of the equation $p(x, y) = 0$, and V is the set of solutions of the equation $q(x, y) = 0$, then $S \cup V$ is the set of solutions of the equation $p(x, y)q(x, y) = 0$. Let's see why this is true.

Saying that a point (α, β) is in $S \cup V$ is the same thing as saying that either $(\alpha, \beta) \in S$ or $(\alpha, \beta) \in V$. That's the same as saying that either $p(\alpha, \beta) = 0$ or $q(\alpha, \beta) = 0$ (because S is the set of solutions of $p(x, y) = 0$, and V is the set of solutions of $q(x, y) = 0$). And that's the same as saying that

$$p(\alpha, \beta)q(\alpha, \beta) = 0$$

because the only way to multiply two numbers and get 0, is if at least one of the two numbers you multiplied was 0. That $p(\alpha, \beta)q(\alpha, \beta) = 0$ is what it means for (α, β) to be a solution of the equation $p(x, y)q(x, y) = 0$.

We have shown that there is no difference between points in $S \cup V$ and solutions of $p(x, y)q(x, y) = 0$. They are the same. ■

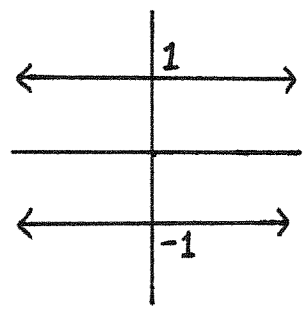
Exercises

For #1-4, write the polynomial equation asked for in the form

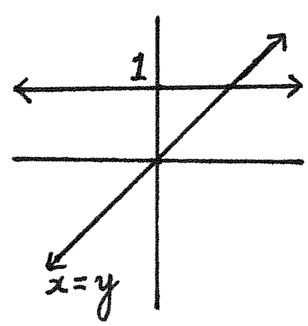
$$Ax^2 + Bxy + y^2 + Dx + Ey + F = 0$$

so that the coefficient of the y^2-term equals 1.

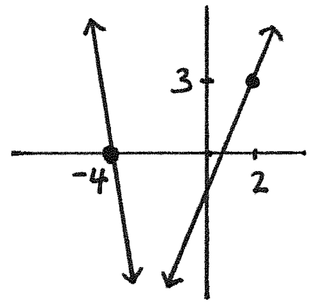
1.) Give an equation in two variables whose set of solutions is every point in the plane that has either a y-coordinate equal to 1, or a y-coordinate equal to -1.



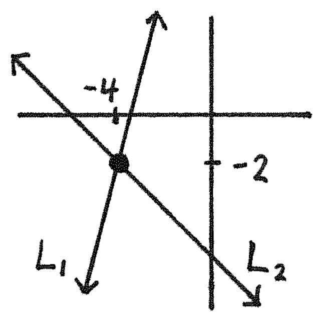
2.) Give an equation whose set of solutions is every point in the plane that has either a y-coordinate equal to 1, or whose x-coordinate equals its y-coordinate.



3.) Give an equation whose set of solutions is every point in the plane that either is contained in the line of slope 2 that passes through the point $(2, 3)$, or is contained in the line of slope -3 that passes through the point $(-4, 0)$.



4.) Let L_1 be a line in the plane that has slope 5. Let L_2 be a line in the plane that has slope -1. Suppose that both L_1 and L_2 contain the point $(-4, -2)$. Give an equation whose set of solutions is $L_1 \cup L_2$.



For #5-8, give the inverses of the following matrices. Notice that these matrices, in order, are the flip over the y-axis, the flip over the x-axis, the flip over the $x = y$ line, and a diagonal matrix.

5.) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

6.) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

7.) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

8.) $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ $\quad$ ($a \neq 0$ and $d \neq 0$.)

Find the solutions of the following equations.

9.) $\sqrt{x^5 - 3x^4 - 3x^2 + 5x + 6} = -12$

10.) $e^{x^3-2} + 1 = 5$