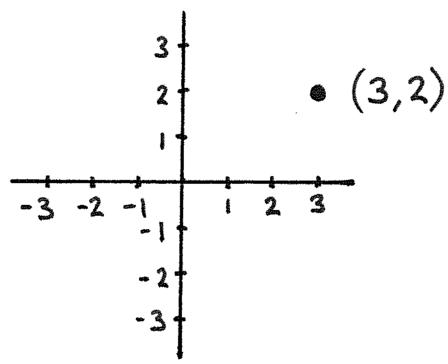
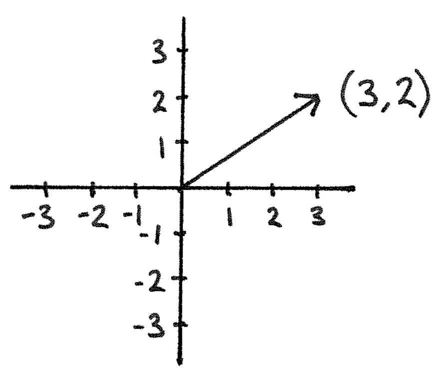
The Plane of Vectors

$\mathbb{R}^2$ — or the *x-y plane*, or *the plane* as we'll sometimes refer to it informally — is the set of all pairs of real numbers. For example, the pair of numbers $(3, 2)$ is an object in the plane. It's the point whose x-coordinate equals 3, and whose y-coordinate equals 2.



The objects in $\mathbb{R}^2$ are often called *points*, and they are often drawn as dots in $\mathbb{R}^2$, as is demonstrated above.

To complicate matters just a little, points in $\mathbb{R}^2$ are also sometimes called *vectors*, and instead of being drawn as dots, they are drawn as arrows that emanate from $(0, 0)$ and terminate at the point the arrow is meant to denote. For example, below is the point $(3, 2)$ drawn as a vector, using an arrow that emanates from $(0, 0)$ and that terminates at $(3, 2)$.

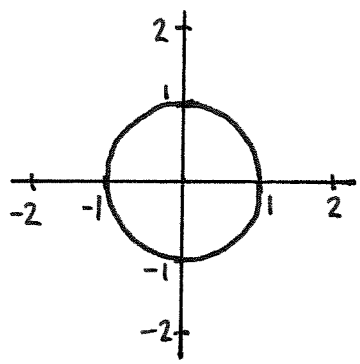


It's important to note that the two pictures above are two different visual representations of exactly the same thing, the pair of numbers $(3, 2)$. Which of these two pictures a person chooses to draw to represent $(3, 2)$ is largely a matter of personal taste.

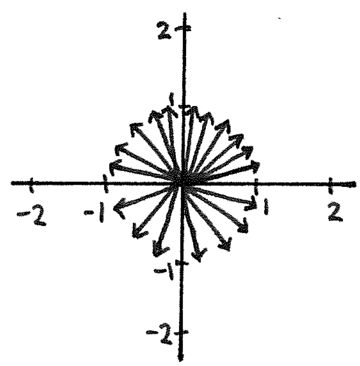
As we've discussed, $\mathbb{R}^2$ and "the plane" mean the same thing, and objects in the plane are interchangeably called "points" or "vectors". There is another style choice in talking about vectors in $\mathbb{R}^2$. Some write vectors (or points) as a row of numbers, so that $(3, 2)$ is the vector in $\mathbb{R}^2$ whose x-coordinate equals 3 and whose y-coordinate equals 2. These rows of numbers are often called *row vectors*. Some prefer to write vectors (or points) in $\mathbb{R}^2$ as a column of numbers, so that $\binom{3}{2}$ is the vector whose x-coordinate equals 3 and whose y-coordinate equals 2. These columns of numbers are sometimes called *column vectors*. It mostly doesn't matter whether you prefer to write vectors as rows or columns, and we'll write vectors interchangeably as rows and columns.

Drawing sets of points in the plane

If you want to draw a few vectors at the same time, you can draw them as arrows. If you're drawing very many vectors at the same time though, it's easiest to think of them as points, and to draw them as dots. For example, if you drew one dot for each of the infinitely many points in the plane whose distance from $(0, 0)$ equals 1, then the result would be the circle below.



This is the circle of a radius 1 centered at the point $(0, 0)$. The picture would be a bit more cluttered if we tried to draw all of the points in the circle as arrows simultaneously.



$$* \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad *$$

Multiplying a row vector and a column vector

If you have a row vector and a column vector, you can put the row on the left, the column on the right, and multiply them to create a number. Use the following recipe:

$$(a, b)\begin{pmatrix} u \\ w \end{pmatrix} = au + bw$$

For example,

$$(1, 2)\begin{pmatrix} -3 \\ 4 \end{pmatrix} = 1(-3) + 2(4) = -3 + 8 = 5$$

$$* \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad *$$

Vector addition

To add two vectors in $\mathbb{R}^2$, add each of their coordinates.

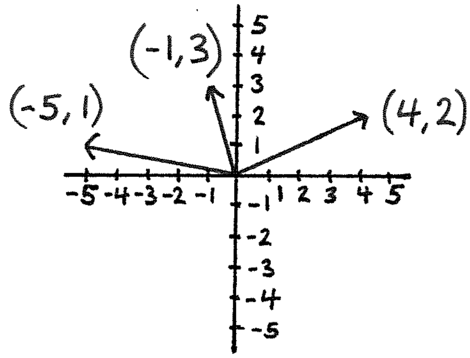
Example. Writing an example using row vectors:

$$(-5, 1) + (4, 2) = (-5 + 4, 1 + 2) = (-1, 3)$$

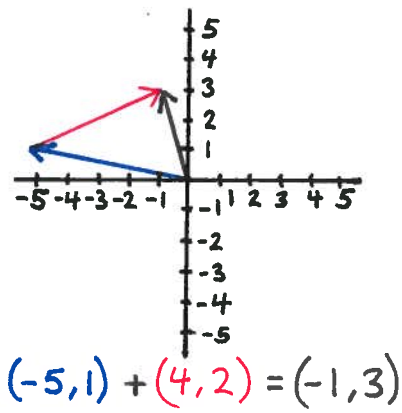
Writing the same example using column vectors:

$$\begin{pmatrix} -5 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 + 4 \\ 1 + 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

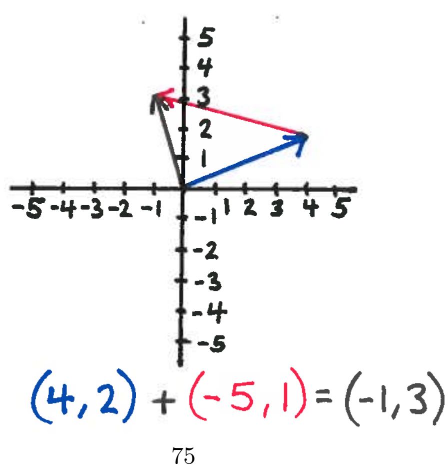
The vectors involved in this example are, written as rows, $(-5, 1)$, $(4, 2)$, and $(-1, 3)$. All three of these vectors are drawn below as arrows.



There's an interesting geometry involved in adding vectors. Notice that if we begin with the blue arrow $(-5, 1)$, and then we place a red parallel copy of the vector $(4, 2)$ on the end of that, the red arrowhead terminates at the point $(-1, 3)$.



Adding vectors is commutative. The order in which you add vectors doesn't matter. Below is a picture of $(4, 2) + (-5, 1) = (-1, 3)$.



Example. Adding the vector $(0,0)$ to any other vector doesn't change the vector: $(a,b)+(0,0)=(a+0,b+0)=(a,b)$. We added $(0,0)$ to (a,b), and we got back (a,b).

For numbers, adding the number 0 doesn't change anything: $x+0=0$. Because adding the vector $(0,0)$ doesn't change vectors, we call $(0,0)$, by analogy, the *zero vector*.

The zero vector is also called *the origin* in the plane.

Vector subtraction

To subtract two vectors in $\mathbb{R}^2$, subtract each of their coordinates.

Examples.

- As an example using row vectors,
$$(2,-4)-(-1,3)=(2-(-1),-4-3)=(3,-7)$$
- $(a,b)-(a,b)=(0,0)$

* * * * * * * * * * * * *

Addition functions

Suppose that (a,b) is a row vector. The *addition function of the vector* (a,b) is the function $A_{(a,b)}:\mathbb{R}^2\to\mathbb{R}^2$ where
$$A_{(a,b)}(x,y)=(x,y)+(a,b)$$
We use the name $A_{(a,b)}$ for this function because the A reminds us that we are adding vectors, and the subscript on $A_{(a,b)}$ reminds us of exactly which vector we are adding, the vector (a,b).

A word about notation. Notice in the equation above, that we put the vector (x,y) into the function $A_{(a,b)}$, and we wrote that as $A_{(a,b)}(x,y)$ rather than as $A_{(a,b)}\Big((x,y)\Big)$, with one pair of parentheses around the row vector (x,y) as we normally write row vectors, and another larger pair of parentheses around the row vector that are meant to convey that we are putting (x,y) into the function A, similar to how we use parentheses for $f(x)$ to say that we are putting a number x into the function f. We probably should write

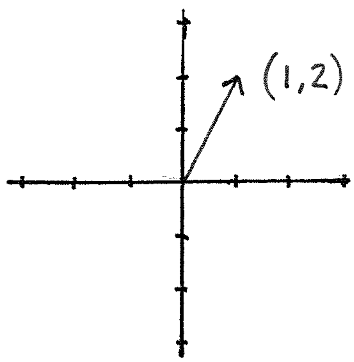
$A_{(a,b)}\Big((x, y)\Big)$ to denote that we are putting the vector (x, y) into the function $A_{(a,b)}$, but that's just too many parentheses to write at once, so nobody ever does. Instead, we leave out one of the pairs of parentheses and just write $A_{(a,b)}(x, y)$ to mean that we've put the vector (x, y) into the function $A_{(a,b)}$.

Example. Let's look at the addition function of the vector $(1, 2)$, the function $A_{(1,2)} : \mathbb{R}^2 \to \mathbb{R}^2$ where $A_{(1,2)}(x, y) = (x, y) + (1, 2)$. Then,

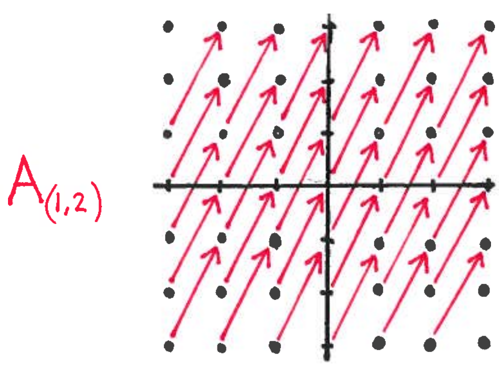
- $A_{(1,2)}(2, 4) = (2, 4) + (1, 2) = (3, 6)$
- $A_{(1,2)}(7, 1) = (7, 1) + (1, 2) = (8, 3)$
- $A_{(1,2)}(-6, 0) = (-5, 2)$.

This is a pretty simple function. We just add the same vector, in this example $(1, 2)$, to every vector that we put into the function $A_{(1,2)}$.

We can draw a visual representation of the addition function of the vector $(1, 2)$. The vector $(1, 2)$ has an x-coordinate of 1 and a y-coordinate of 2. This vector points to the right 1 unit and up 2 units.

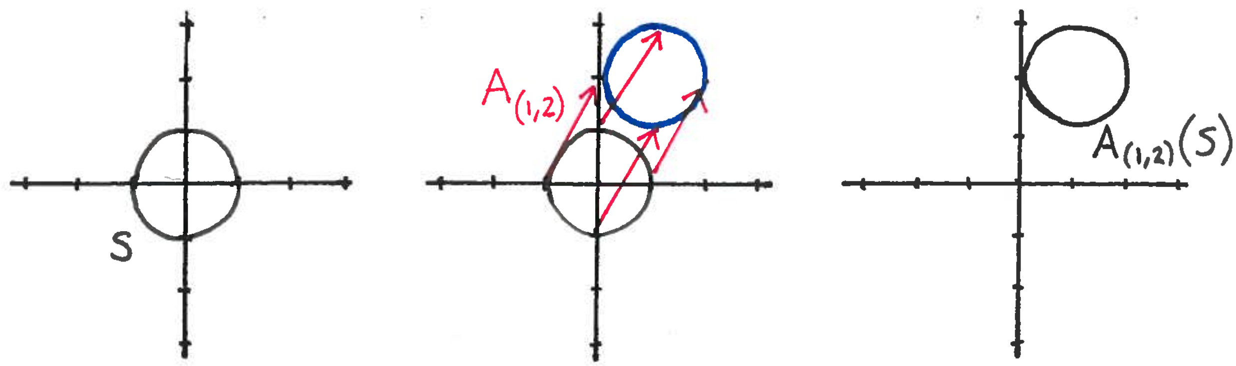


Adding the vector $(1, 2)$ to every vector in the plane, which is what the function $A_{(1,2)}$ does, has the effect of moving every point in the plane to the right one unit and up two units.



Instead of looking at what the function $A_{(1,2)}$ does to every vector in the plane, and instead of just checking one vector at a time as this example began, we will at times be interested in how the function $A_{(1,2)}$ affects a subset of the plane.

If S is the circle of points in the plane at distance 1 from $(0,0)$, then we let $A_{(1,2)}(S)$ be the set of all of the outputs of $A_{(1,2)}$ that come from using points of the circle S as inputs. That is, the points in $A_{(1,2)}(S)$ are what we get by feeding points from S into the function $A_{(1,2)}$.



Written using set notation

$$A_{(1,2)}(S) = \{\, A_{(1,2)}(c,d) \mid (c,d) \in S \,\}$$

or equivalently

$$A_{(1,2)}(S) = \{\, (c,d) + (1,2) \mid (c,d) \in S \,\}$$

The set $A_{(1,2)}(S)$ is called the *image of S under $A_{(1,2)}$*.

Scalar multiplication

In linear algebra, real numbers are often called *scalars*. To multiply a scalar and a vector, multiply every coordinate in the vector by the scalar.

Example.

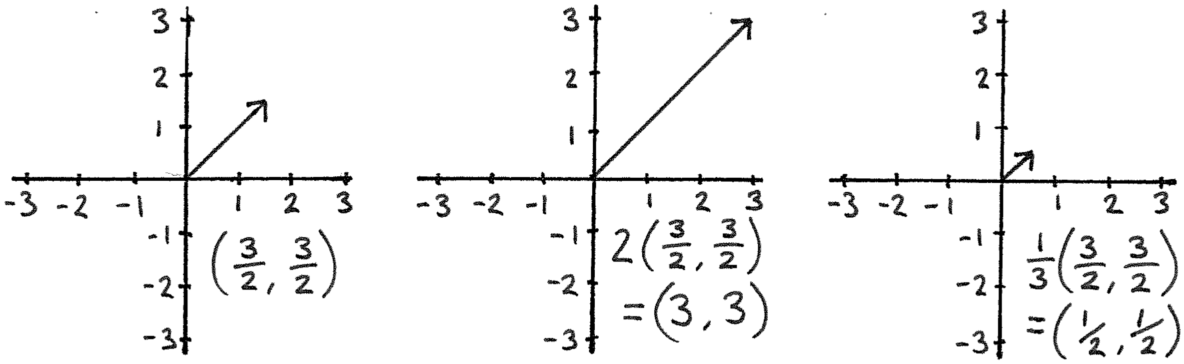
Written as rows, the scalar 2 times the vector $(7,-3)$ is

$$2(7,-3) = (2(7), 2(-3)) = (14,-6)$$

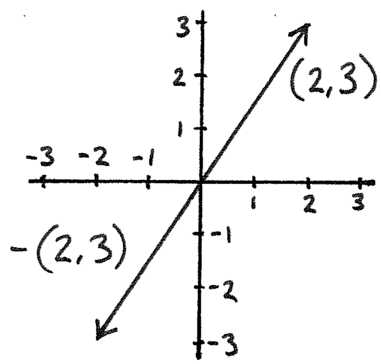
Written with column vectors:

$$2\begin{pmatrix} 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 2(7) \\ 2(-3) \end{pmatrix} = \begin{pmatrix} 14 \\ -6 \end{pmatrix}$$

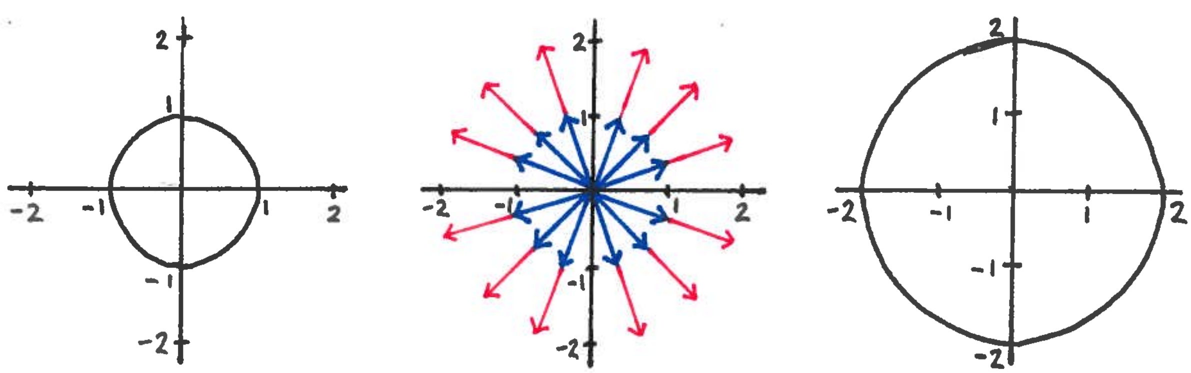
Scalar multiplication has the effect of scaling vectors, in the sense that multiplying a vector by the scalar 2 makes it twice as long. Multiplying a vector by $\frac{1}{3}$ makes it one-third as long.



Multiplying a vector by the scalar -1 keeps the vector the same length, but it points the vector backwards. Notice that -1 times that vector $(2, 3)$, what we would write more simply as $-(2, 3)$, is just $((-1)2, (-1)3) = (-2, -3)$.



If we multiply every point in the circle of radius 1 centered at the point $(0, 0)$ by the scalar 2, then all of the points would be twice as far from $(0, 0)$. The result would be a circle of radius 2 around $(0, 0)$.



Inverse addition functions

Just as the inverse of adding the number 7 is adding the number -7 (or subtracting 7), the inverse of adding the vector (a, b) is adding the vector $-(a, b)$. That is, $A_{(a,b)}^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is the function $A_{-(a,b)} : \mathbb{R}^2 \to \mathbb{R}^2$.

We can check that these functions are inverses. Just use the algebra of adding and subtracting vectors to verify that

$$A_{-(a,b)} \circ A_{(a,b)}(x, y) = (x, y) \quad \text{and that} \quad A_{(a,b)} \circ A_{-(a,b)}(x, y) = (x, y)$$

Maybe even better than checking the algebra though, the reason $A_{(a,b)}$ and $A_{-(a,b)}$ are inverses is because $A_{(a,b)}$ moves points in the plane in the direction of the vector (a, b), while $A_{-(a,b)}$ moves points in the direction of $-(a, b)$, which is exactly backwards of the direction of (a, b). If you move points, and then you move them back, it's as if you've done nothing. That's what inverse functions are.

* * * * * * * * * * * * *

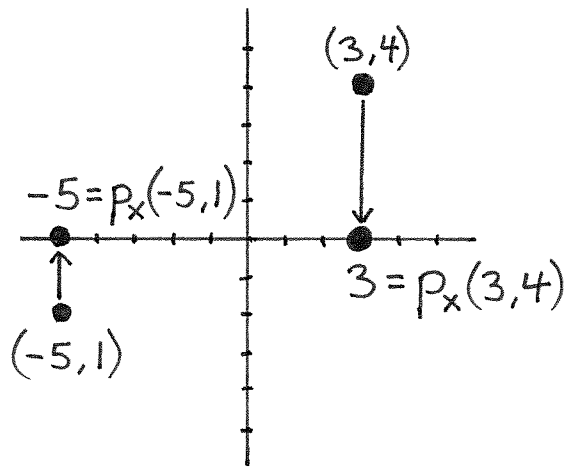
Projections

There are two *projection functions* that we will use. The first is $p_X : \mathbb{R}^2 \to \mathbb{R}$ where $p_X(x, y) = x$. This function is sometimes called *projection onto the x-axis.* You input a vector into p_X, and the output is its x-coordinate.

Examples.

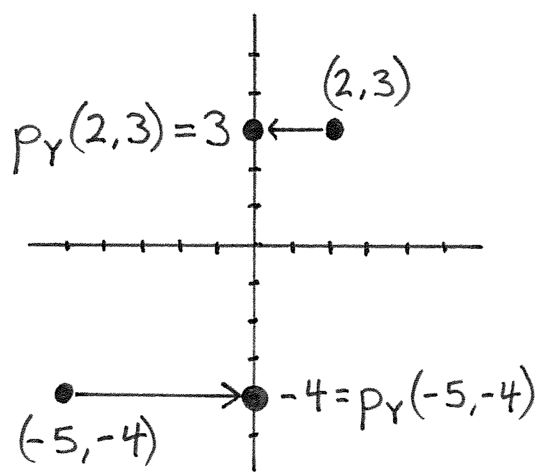
- $p_X(2, 3) = 2$
- $p_X(-1, 7) = -1$
- $p_X(3, 4) = 3$

We visually think of projection onto the x-axis as moving points in the plane directly up or down until they determine a number on the x-axis. That number is the x-coordinate of the point we started with.



The second projection function that we need is *projection onto the y-axis.* It's the function $p_Y : \mathbb{R}^2 \to \mathbb{R}$ where $p_Y(x, y) = y$. Thus, $p_Y(2, 3) = 3$ and $p_Y(4, -7) = -7$.

Visually, p_Y moves points in the plane directly left or right until they determine a number on the y-axis. That number is the y-coordinate of the vector.



Exercises

Multiplying a row vector and a column vector has a lot of different names in mathematics and the sciences. Some call this the *dot product*, some call it the *inner product*, others prefer the *scalar product*. We'll just stick with calling it multiplying a row vector and a column vector. For #1-6, find the number that each of these products equals.

1.) $(1,1)\begin{pmatrix}3\\4\end{pmatrix}$

2.) $(-8,0)\begin{pmatrix}2\\\sqrt{2}\end{pmatrix}$

3.) $(2,-4)\begin{pmatrix}0\\0\end{pmatrix}$

4.) $(2,-6)\begin{pmatrix}5\\\frac{1}{2}\end{pmatrix}$

5.) $\left(\frac{-3}{10},6\right)\begin{pmatrix}\frac{2}{3}\\-7\end{pmatrix}$

6.) $(-1,-3)\begin{pmatrix}2\\6\end{pmatrix}$

For #7-16, perform the algebra of vectors required, either addition of vectors, subtraction of vectors, or scalar multiplication.

7.) $(3,4)+(5,6)$

8.) $8(-3,4)$

9.) $-3(3,-2)$

10.) $\left(\frac{2}{3},-\frac{4}{5}\right)+\left(\frac{3}{5},\frac{1}{4}\right)$

11.) $(2,8)-(4,9)$

12.) $\frac{1}{8}(-16,32)$

13.) $\left(\frac{3}{5},\frac{2}{3}\right)-\left(\frac{7}{8},-\frac{1}{2}\right)$

14.) $(2,3)-(2,3)$

15.) $0(3,4)$

16.) $(5,7)+(0,0)$

For #17-24, find the appropriate output of the addition functions.

17.) $A_{(1,2)}(5,7)$

18.) $A_{(2,7)}(-3,10)$

19.) $A_{-(4,5)}(2,6)$

20.) $A^{-1}_{(2,4)}(0,-5)$

21.) $A_{(3,8)}(2,5)$

22.) $A^{-1}_{(3,8)}(2,5)$

23.) $A_{(7,1)}(0,0)$

24.) $A_{-(2,3)}(-1,5)$

For #25-32, find the appropriate output of the projection functions.

25.) $p_X(2,4)$

26.) $p_X\big(-\frac{2}{3},\sqrt{2}\big)$

27.) $p_Y(3,7)$

28.) $p_Y(2,0)$

29.) $p_X(\sqrt{3},8)$

30.) $p_Y\big(\frac{3}{5},\frac{5}{6}\big)$

31.) $p_Y(-3,-9)$

32.) $p_X(7,2)$

For #33-40, match the number on the left to the set on the right that it is an object in.

33.) 3	A.) $(-\infty, 5)$
34.) 27	B.) $[5, 10)$
35.) -4	C.) $[10, \infty)$
36.) 0	
37.) 5	
38.) 10	
39.) 2	
40.) 7	

Let's look at the following piecewise defined function

$$f(x) = \begin{cases} x^2 & \text{if } x \in (-\infty, 5); \\ -\frac{3}{2} & \text{if } x \in [5, 10); \quad \text{and} \\ 2x + 4 & \text{if } x \in [10, \infty). \end{cases}$$

The function $f(x)$ is comprised of three pieces. The first is $f(x) = x^2$ as long as $x \in (-\infty, 5)$. The second is $f(x) = -\frac{3}{2}$ when $x \in [5, 10)$. The third is $f(x) = 2x + 4$ for any $x \in [10, \infty)$.

For #41-48, use your answers from #33-40 to match the number, x, on the left to the piece of f on the right that you would use to determine $f(x)$.

41.) 3	A.) $f(x) = x^2$ as long as $x \in (-\infty, 5)$
42.) 27	B.) $f(x) = -\frac{3}{2}$ when $x \in [5, 10)$
43.) -4	C.) $f(x) = 2x + 4$ for any $x \in [10, \infty)$
44.) 0	
45.) 5	
46.) 10	
47.) 2	
48.) 7	

Use the piecewise defined function $f(x)$ given above, and your answers from #41-48 to find the following values.

49.) $f(3)$

50.) $f(27)$

51.) $f(-4)$

52.) $f(0)$

53.) $f(5)$

54.) $f(10)$

55.) $f(2)$

56.) $f(7)$

What are the roots of the quadratic polynomials in #57 and #58?

57.) $-3x^2 - 4x + 7$

58.) $-x^2 + 6x - 7$

And now just a few logarithm questions. Each of the numbers below is an integer. Which integers are they?

59.) $\log_e(e^2)$

60.) $\log_e(e^{-7})$

61.) $\log_2(2^3)$

62.) $\log_2(8)$

63.) $\log_2(16)$

64.) $\log_5(25)$