# The Plane of Complex Numbers

In this chapter we'll introduce the complex numbers as a plane of numbers. Each complex number will be identified by a number on a "real axis" and a number on an "imaginary axis". This description of the complex numbers is analogous to the description of  $\mathbb{R}^2$  using Cartesian coordinates. We'll see that this description of the complex numbers is well suited for understanding addition of complex numbers, and not as ideal for understanding multiplication.

### A Once imaginary number

A very popular complex number is the number i. If you square i, the result is -1. That is,  $i^2 = -1$ . The letter i is used to write this number because it was once thought of as an "imaginary" number, a number that shouldn't really exist. We no longer think of i as an imaginary number. It's regarded as one of the most important numbers in mathematics. That it was thought of as a mysterious number at some point in history is not surprising. Almost every number was thought of as mysterious or nonexistent at some point. The reason there is no Roman numeral for 0 or for -1 is because 0 and -1were not known to exist in the time of the ancient Romans.

## Complex numbers

The *complex numbers* is the set of objects of the form x + iy where x and y are real numbers. The set of complex numbers is denoted by the single letter  $\mathbb{C}$ . To repeat the previous two sentences using set notation:

$$\mathbb{C} = \{ x + iy \mid x, y \in \mathbb{R} \}$$

#### Examples.

• 5 + i3 is a complex number. It is a number of the form x + iy, where x = 5 and y = 3.

- 4 + i7 is a complex number.
- $6 + i8 \in \mathbb{C}$ .
- $9 + i9 \in \mathbb{C}$ .

Any complex number is identified by two coordinates of real numbers. For example, the complex number 5 + i3 is identified by the two real numbers 5 and 3.

If x + iy is a complex number, then we'll call x the real part of x + iy. The real number y is the *imaginary part* of x + iy.

#### Examples.

- The real part of the complex number 2 + i4 is 2. The imaginary part is 4.
- $9 + i2 \in \mathbb{C}$  has real part 9 and imaginary part 2.

For two complex numbers to be equal, both their real parts and their imaginary parts must be equal.

#### Examples.

- $\bullet \ 7 + i4 = 7 + i4$
- $7 + i4 \neq 4 + i7$
- $7 + i4 \neq 3 + i4$
- $7 + i4 \neq 7 + i5$

There is a simpler way that we often write some complex numbers, as the examples below illustrate.

#### Examples.

• Complex numbers of the form x + i0 are usually just written as x. For example, we would write 3 instead of 3 + i0, and -7 instead of -7 + i0.

• Complex numbers of the form 0 + iy are usually written as iy, For example, 0 + i4 is written as i4, and 0 + i9 is the same number as i9.

• If the imaginary part of a complex number is negative, then we usually write the minus sign in front of the letter *i*. For example, we write 3 - i5 instead of 3 + i(-5), and the complex number -2 + i(-6) is usually written as -2 - i6.

• The number i is the preferred way of writing the complex number 0 + i(1).

• 0 + i(-1) is written as -i.

• The complex number 0 + i0 is called the complex number *zero*. We write it simply as 0.

## **Real numbers**

Any real number is a complex number. If  $x \in \mathbb{R}$ , then  $x = x + i0 \in \mathbb{C}$ . For example,  $1 = 1 + i0 \in \mathbb{C}$  and  $-\sqrt{2} = -\sqrt{2} + i0 \in \mathbb{C}$ . This means that the set of real numbers is a subset of the set of complex numbers. It's the set of complex numbers whose imaginary parts equal 0. To repeat,  $\mathbb{R} \subseteq \mathbb{C}$ .

While the real numbers are a subset of the complex numbers, there are very many complex numbers that are not real numbers. Any complex number of the form x + iy where  $y \neq 0$  is not a real number. For example, 2 + i3 and i = 0 + i(1) are not real numbers.

## **Imaginary numbers**

An *imaginary number* is any complex number whose real part equals 0. That is, an imaginary number is a complex number of the form 0 + iy. For example, i3 is an imaginary number. So is -i6 = i(-6).

The only imaginary number that is also a real number is the number 0. Both the real part and the imaginary part of 0 are 0.

## The Complex numbers are a plane

Because each complex number is identified by a pair of real number coordinates—its real part and its imaginary part— $\mathbb{C}$  is drawn as a plane. We draw the plane  $\mathbb{C}$  much as we draw the plane  $\mathbb{R}^2$ . There is a horizontal axis that looks like the line of real numbers, and there is a vertical axis that looks like the line of real numbers turned on its side. When we had looked at  $\mathbb{R}^2$ we had called the horizontal and vertical axes the *x*- and *y*-axes respectively. When we draw  $\mathbb{C}$ , we call the horizontal axis the *real axis*. It's made up of the set of real numbers, which is a subset of  $\mathbb{C}$ . We call the vertical axis the *imaginary axis*. It's made up of the set of imaginary numbers, which is also a subset of  $\mathbb{C}$ .



Any complex number is the sum of a real number and an imaginary number. In the picture above, the number 2+i3 is drawn as a point in  $\mathbb{C}$ . Its horizontal position is determined by its real part, 2. Its vertical position is determined by the imaginary number i3.

\* \* \* \* \* \* \* \* \* \* \* \*

## Addition

You can add two complex numbers in essentially the same way that you can add two vectors. Just add their coordinates, their real parts and their imaginary parts.

#### Examples.

• The sum of the complex numbers 3 + i5 and 2 + i3 is the number

$$[3+i5] + [2+i3] = (3+2) + i(5+3)$$
$$= 5+i8$$

• The sum of the complex numbers 1 - i2 and 4 + i is the number

$$[1 - i2] + [4 + i] = (1 + 4) + i(-2 + 1)$$
$$= 5 - i$$

• The sum of a real number x and an imaginary number iy is x + iy.

## Geometry of addition

Addition in the plane of complex numbers works essentially the same way as addition in the plane of vectors does. For example, addition by 1 + i2would add 1 to the real part of every complex number, and it would add 2 to the imaginary part. Addition by 1 + i2 would shift complex numbers to the right by 1 and up by 2.



Addition by 1+12

## Subtraction

We subtract two complex numbers by subtracting their real parts and subtracting their imaginary parts.

#### Examples.

• The difference of 3 + i5 and 2 + i3 is

$$[3+i5] - [2+i3] = (3-2) + i(5-3)$$
$$= 1 + i2$$

• The difference of -3 + i2 and 4 + i5 is [-3 + i2] - [4 + i5] = (-3 - 4) + i(2 - 5)= -7 - i3

## Norms

The norm of a complex number is very similar to the norm of a vector. It's the distance between the complex number and the number 0. The only difference between the norm of a vector in the plane and the norm of a complex number is that the norm of a vector is indicated by surrounding a vector with double vertical lines—such as ||(2,3)||—whereas norms of complex numbers are indicated by surrounding the number with single vertical lines—exactly as we do when we write the absolute value of a real number, such as |5|. Otherwise, norms of vectors in the plane and norms of complex numbers are determined in the same way. Namely, the norm of a complex number x + iy is

$$|x+iy| = \sqrt{x^2 + y^2}$$

Example.

• 
$$|2 - i7| = \sqrt{2^2 + (-7)^2} = \sqrt{4 + 49} = \sqrt{53}$$



Any real number is a complex number. The norm of the complex (and real) number x = x + i0 is exactly the same as the absolute value of the real number x. The number |x| is the distance between x and 0.

#### Example.

• 
$$|-3+i0| = \sqrt{(-3)^2 + 0^2} = \sqrt{9} = 3 = |-3|$$
  
\* \* \* \* \* \* \* \* \* \* \* \* \* \* \* \*

## Multiplication

Writing complex numbers using Cartesian coordinates—in the form x+iy is not as well suited for multiplication as it is for addition, and it's even worse for division. We can say something about it though.

To multiply two complex numbers, just treat the number i as a variable, and multiply using the distributive law. Then use the rule that  $i^2 = -1$  to write the answer in the form x + iy.

#### Examples.

• Using the distributive law, and treating i as a variable, we see that the product of the complex numbers 2 + i3 and 4 + i2 is

$$(2+i3)(4+i2) = 2(4) + 2(i2) + i3(4) + i3(i2)$$
$$= 8 + i16 + i^{2}6$$

Now using the rule that  $i^2 = -1$ , we have that

$$(2+i3)(4+i2) = 8+i16-6$$
  
= 2+i16

• 
$$(3+i4)(2+i5) = 6+i23+i^220 = -14+i23$$

## Geometry of multiplication

The two examples above describe how to multiply any pair of complex numbers, but they don't give much insight into what it looks like to multiply every number in the complex plane simultaneously by a single complex number. For example, we know that adding the number 1 + i2 shifts every complex number to the right 1 and up 2, but it's not clear at this point how to describe the transformation of the plane of complex numbers that occurs when we multiply by 1 + i2.

It turns out that multiplication and division of complex numbers are better understood using polar coordinates for complex numbers. That will be the topic of the next chapter.

## Multiplicative inverse

The multiplicative inverse of  $x + iy \in \mathbb{C}$ , as long as  $x + iy \neq 0$ , is given by the formula

$$\frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Example.

• The multiplicative inverse of 2 + i3 is

 $\frac{5}{2}$ 

$$\frac{1}{2+i3} = \frac{2}{2^2+3^2} - i\frac{3}{2^2+3^2} = \frac{2}{13} - i\frac{3}{13}$$

# Division

To divide two complex numbers, as long as the denominator does not equal 0, first find the multiplicative inverse of the denominator, and then multiply.

## Example.

• To find the quotient  $\frac{5+i7}{2+i3}$ , first find  $\frac{1}{2+i3}$ , which we know from the previous example equals  $\frac{2}{13} - i\frac{3}{13}$ . Then

$$\frac{+i7}{+i3} = (5+i7)\left(\frac{1}{2+i3}\right)$$
$$= (5+i7)\left(\frac{2}{13}-i\frac{3}{13}\right)$$
$$= \frac{10}{13}-i\frac{15}{13}+i\frac{14}{13}-i^2\frac{21}{13}$$
$$= \frac{10}{13}-i\frac{15}{13}+i\frac{14}{13}+\frac{21}{13}$$
$$= \frac{31}{13}-i\frac{1}{13}$$

No one would blame you if you didn't look forward to dividing complex numbers that are written in Cartesian coordinates, as 5 + i7 and 2 + i3 are. In the next chapter we'll see that complex numbers can be divided much more easily if they are written in polar coordinates.

\* \* \* \* \* \* \* \* \* \* \*

## **Rules of numbers**

Writing x + iy every time we want to write a complex number makes for an awful lot of writing. We don't want to have to write 4 symbols every time we write about a complex number, so we often write complex numbers using a single letter, usually z or w. For example, if we write that  $z \in \mathbb{C}$ , then z might equal 5 + i7 or -3 + i2.

If you have two real numbers, then they are either equal to each other, or one is greater than the other. Complex numbers do not have any analogous rule. It doesn't make sense to speak of one complex number as being greater than or less than another complex number. With this exception, complex numbers satisfy the same basic rules of algebra that real numbers do. These rules are listed below. In each of the rules,  $z, w, u \in \mathbb{C}$ .

#### Rules of addition.

- (z+w) + u = z + (w+u) (Law of associativity)
- z + w = w + z (Law of commutativity)
- z + 0 = z (Law of identity)
- -z + z = 0 (Law of inverses)

#### Rules of multiplication.

- (zw)u = z(wu) (Law of associativity)
- zw = wz (Law of commutativity)
- z1 = z (Law of identity)
- If  $z \neq 0$  then  $\frac{1}{z}z = 1$  (Law of inverses)

#### Distributive Law.

• z(w+u) = zw + zu (Distributive Law)

# Exercises

For #1-12, find the given values

1.) real part of $2 + i5$	7.) $(2+i5) - (1+i3)$
2.) real part of $-2 + i3$	8.) $(7+i6) - (2+i4)$
3.) imaginary part of $3 + i4$	9.) $ 4+i3 $
4.) imaginary part of $5 - i2$	10.) $ 2+i5 $
5.) $(2+i6) + (3+i7)$	11.) $(2+i5)(4+i3)$
6.) $(3+i4) + (5+i2)$	12.) $(6+i3)(2+i4)$

Match the complex numbers in #13-16 with the correct letter drawn in the complex plane below.



All further exercises in this chapter have nothing to do with complex numbers. Match the functions with their graphs.

 17.)  $\sqrt[3]{x}$  18.)  $\sqrt[3]{x} - 2$  19.)  $\sqrt[3]{x+1}$  

 20.)  $\sqrt[3]{x-2}$  21.)  $x^3$  22.)  $\sqrt[3]{-x}$  

 23.)  $-\sqrt[3]{x}$  24.)  $\sqrt[3]{x} + 3$ 



Match the functions with their graphs.

25.) 
$$\sin(x)$$
  
26.)  $\cos(x)$   
27.)  $f(x) = \begin{cases} \sin(x) & \text{if } x < 0; \\ \cos(x) & \text{if } x \ge 0. \end{cases}$   
28.)  $g(x) = \begin{cases} \cos(x) & \text{if } x < 0; \\ \sin(x) & \text{if } x \ge 0. \end{cases}$ 





C.)







For #29-41, find the domain of the given equation. Recall that a function of the form  $\sqrt{f(x)}$  must have  $f(x) \ge 0$ , that a function of the form  $\log_a (f(x))$  must have that f(x) > 0, and that a function of the form  $\frac{g(x)}{f(x)}$  must have that  $f(x) \ne 0$ .

29.)  $\sqrt{x} = 3x - 7$ 30.)  $x^{15} - 3x^7 - 2x^2 + 3 = 0$ 31.)  $\frac{x^2 - 3x + 5}{2 - x} = x - 8$ 32.)  $\log_e(x + 2) = 5$ 33.)  $e^{2x - 4} = x + 7$ 34.)  $(x + 1)^3 = 3x$ 35.)  $\sqrt{-x} = x^2 - 3x + 4$ 36.)  $\frac{2}{x^2 - 1} = 3x$ 37.)  $\sqrt[3]{x} = 3x + 4$ 38.)  $(x + 3)^2 = 2x + 5$ 39.)  $\frac{2x - 1}{3} = 3x - 4$ 40.)  $\log_e(x) = x^2 - \log_e(x - 7)$ 41.)  $\frac{13}{x - 1} = 2x + 5$  For #42-65, decide which of the equations have no solution. Recall that equations of the form  $f(x)^2 = c$  or  $\sqrt{f(x)} = c$  will not have a solution if c < 0. Equations of the form  $e^{f(x)} = c$  will not have a solution if  $c \le 0$ . Equations of the form  $\sin(f(x)) = c$  or  $\cos(f(x)) = c$  will not have solutions if  $c \notin [-1, 1]$ . Equations of the form  $af(x)^2 + bf(x) + c = 0$  will not have solutions if  $b^2 - 4ac < 0$ .

42.)  $e^{2x-3} = 4$ 54.)  $\sin(x) = 0$ 55.)  $(2x-3)^2 = 4$ 43.)  $\cos(x+1) = 3$ 56.)  $\sqrt{x+1} = 5$ 44.)  $\cos(x) = \frac{1}{4}$ 45.)  $\sin(3x) = -2$ 57.)  $\log_e(x+2) = -8$ 58.)  $(x+2)^3 = 5$ 46.)  $\tan(x) = -37$ 59.)  $\sqrt{3x-5} = 0$ 47.)  $(2x+1)^2 = -2$ 48.)  $e^{3x} = 0$ 60.)  $\log_e(-x) = 0$ 61.)  $\sqrt{7x-4} = -1$ 49.)  $\cos(x) = -4$ 50.)  $\sin(x+2) = \frac{3}{2}$ 62.)  $\tan(x) = 78$ 51.)  $(x+3)^2 = 0$ 63.)  $\log_{e}(3x) = 4$ 64.)  $(3-x)^3 = -4$ 52.)  $e^{x+2} = -1$ 53.)  $\cos(x)^2 + \cos(x) + 5 = 0$ 65.)  $\log_{e}(x)^{2} + \log_{e}(x) - 8 = 0$