Trigonometric Identities

We have seen several identities involving trigonometric functions. These are often called *trigonometric identities*. The rest of this page and the beginning of the next page list the trigonometric identities that we've encountered.

Pythagorean identity

$$\cos(x)^2 + \sin(x)^2 = 1$$

Definition of tan, sec, csc, cot

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Half rotation identities

$$\cos(x + \pi) = -\cos(x)$$

$$\sin(x + \pi) = -\sin(x)$$

Quarter rotation identities

$$\sin(x + \tfrac{\pi}{2}) = \cos(x)$$

$$\cos(x - \tfrac{\pi}{2}) = \sin(x)$$

Sum formulas

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

Double angle formulas

$$\cos(2x) = \cos(x)^2 - \sin(x)^2$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

Even identities

$$\cos(-x) = \cos(x)$$

$$\sec(-x) = \sec(x)$$

Odd identities

$$\sin(-x) = -\sin(x)$$

$$\tan(-x) = -\tan(x)$$

$$\csc(-x) = -\csc(x)$$

$$\cot(-x) = -\cot(x)$$

Period identities

$\cos(x+2\pi) = \cos(x)$

$\sin(x+2\pi) = \sin(x)$

$\sec(x+2\pi) = \sec(x)$

$\csc(x+2\pi) = \csc(x)$

$\tan(x+\pi) = \tan(x)$

$\cot(x+\pi) = \cot(x)$

Cofunction identities

$\sin(\frac{\pi}{2}-x) = \cos(x)$

$\cos(\frac{\pi}{2}-x) = \sin(x)$

$\tan(\frac{\pi}{2}-x) = \cot(x)$

$\cot(\frac{\pi}{2}-x) = \tan(x)$

$\sec(\frac{\pi}{2}-x) = \csc(x)$

$\csc(\frac{\pi}{2}-x) = \sec(x)$

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There are 31 trigonometric identities listed above. We've seen explanations for why the first 27 are true. In the remainder of this chapter — in Claims 21 and 22 — we'll see why two of the remaining 4 identities are true. You'll check that the other two are true for homework. Also for homework you'll be asked to explain why the "difference formulas" for sine and cosine are true.

Claim (21). If $x \in \mathbb{R}$, then $\tan(\frac{\pi}{2}-x) = \cot(x)$.

Proof: We know from the definition of tangent that

$$\tan\left(\frac{\pi}{2}-x\right) = \frac{\sin(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)}$$

On page 276 in the chapter "Cosecant, Secant, and Cotangent", we had checked that the first two cofunction identities from the top of this page are true. Now we can use those two identities to add to the equation above:

$$\begin{aligned}\tan\left(\frac{\pi}{2}-x\right) &= \frac{\sin(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)}\\ &= \frac{\cos(x)}{\sin(x)}\end{aligned}$$

And then we can add one more line to this string of equations using the definition of cotangent:

$$\begin{aligned}\tan\left(\frac{\pi}{2}-x\right) &= \frac{\sin(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)}\\ &= \frac{\cos(x)}{\sin(x)}\\ &= \cot(x)\end{aligned}$$

This is what we had claimed.

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Claim (22). If $x \in \mathbb{R}$, then $\cot(\frac{\pi}{2}-x) = \tan(x)$.

Proof: We can replace the number x in Claim 21 with the number $[\frac{\pi}{2}-x]$. The result is that

$$\tan\left(\frac{\pi}{2}-\left[\frac{\pi}{2}-x\right]\right) = \cot\left(\left[\frac{\pi}{2}-x\right]\right)$$

Notice that $\frac{\pi}{2}-[\frac{\pi}{2}-x]$ can be written more simply as just x, so simplifying the previous equation yields

$$\tan(x) = \cot\left(\frac{\pi}{2}-x\right)$$

We can flip the right and left sides of the equation to give us

$$\cot\left(\frac{\pi}{2}-x\right) = \tan(x)$$

and that's what we wanted to show. ■

Exercises

Write a proof for each of the following four claims.

Claims A and B are the last of the six cofunction identities listed in this chapter. You might want to use the definitions of sec and csc along with the cofunction identities for sin and cos. The proofs will be somewhat similar to the proofs of Claims 21 and 22.

Claims C and D are called difference formulas. Some books list them as important identities. You don't have to be too familiar with these identities though as they are really just a combination of the more important sum formulas and even/odd identities for cos and sin. (Remember that $x - y = x + (-y)$.)

Claim A. If $x \in \mathbb{R}$, then $\sec(\frac{\pi}{2} - x) = \csc(x)$.

Claim B. If $x \in \mathbb{R}$, then $\csc(\frac{\pi}{2} - x) = \sec(x)$.

Claim C. If $x, y \in \mathbb{R}$, then $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$.

Claim D. If $x, y \in \mathbb{R}$, then $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$.