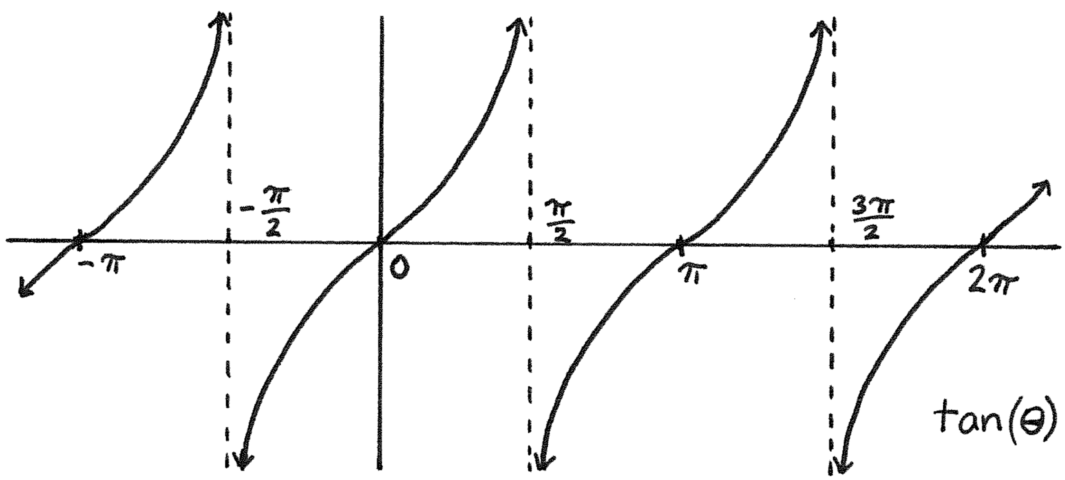
Tangent and Right Triangles

The *tangent* function is the function defined as

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

The implied domain of the tangent function is every number θ except for those which have $\cos(\theta) = 0$: the numbers $\ldots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

Graph of tangent



Period of tangent

Tangent is a periodic function with period π, meaning that

$$\tan(\theta + \pi) = \tan(\theta)$$

This follows from Lemma 10 of the previous chapter which stated that

$$\cos(\theta + \pi) = -\cos(\theta) \quad \text{and} \quad \sin(\theta + \pi) = -\sin(\theta)$$

Therefore,

$$\tan(\theta + \pi) = \frac{\sin(\theta + \pi)}{\cos(\theta + \pi)} = \frac{-\sin(\theta)}{-\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$$

Tangent is an odd function

Recall that an *even* function is a function $f(x)$ that has the property that $f(-x) = f(x)$ for every value of x. Examples of even functions include x^2, x^4, x^6, and $\cos(x)$.

An *odd* function is a function $g(x)$ that has the property $g(-x) = -g(x)$ for every value of x. Examples of odd functions include x^3, x^5, x^7, and $\sin(x)$.

We can add tangent to our list of odd functions. To see why tangent is odd, we'll use that sine is odd and cosine is even:

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

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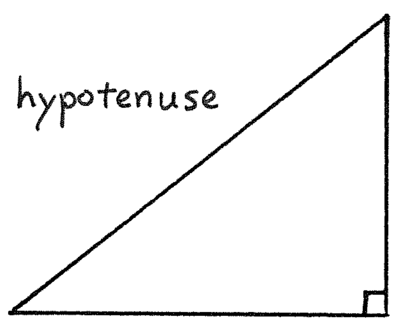
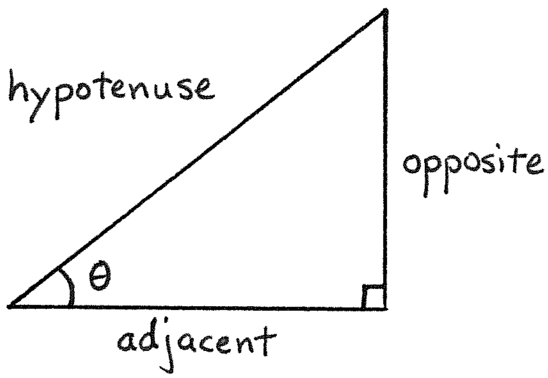
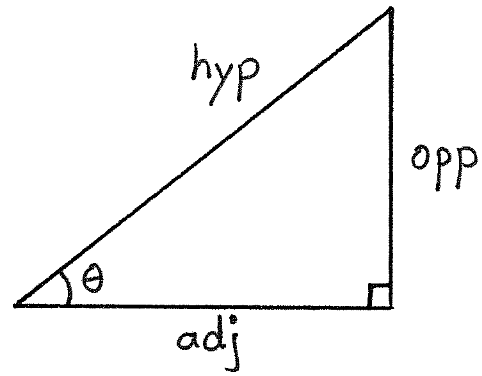
Trigonometry for right triangles

Suppose that we have a right triangle. That is, a triangle one of whose angles equals $\frac{\pi}{2}$. We call the side of the triangle that is opposite the right angle the *hypotenuse* of the triangle.



If we focus our attention on a second angle of the right triangle, an angle that we'll call θ, then we can label the remaining two sides as either being *opposite* from θ, or *adjacent* to θ. We'll call the lengths of the three sides of the triangle *hyp*, *opp*, and *adj*. See the picture on the following page.



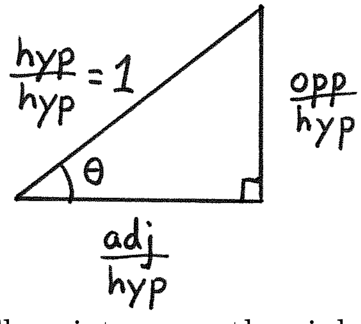


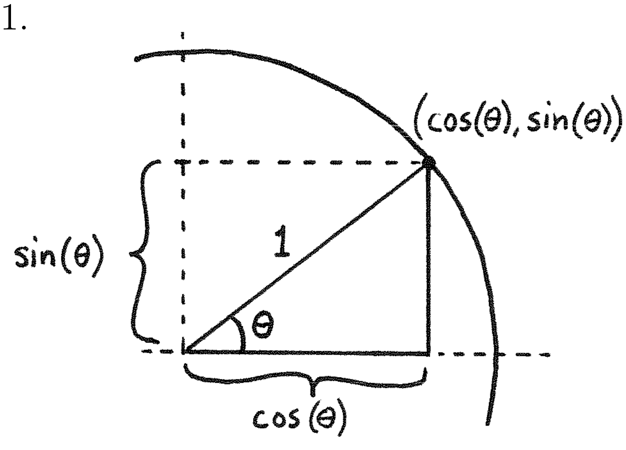
The following proposition relates the lengths of the sides of the right triangle shown above to the measure of the angle θ using the trigonometric functions sine, cosine, and tangent.

Proposition (13). In a right triangle as shown above, the following equations hold:

$$\sin(\theta) = \frac{opp}{hyp} \qquad \cos(\theta) = \frac{adj}{hyp} \qquad \tan(\theta) = \frac{opp}{adj}$$

Proof: We begin with the triangle on the top right of this page, a right triangle with its side lengths labelled as instructed above: *hyp* for the hypotenuse, *opp* for the side opposite θ, and *adj* for the remaining side, which is adjacent to θ. The triangle below on the left is the first triangle scaled by the number $\frac{1}{hyp}$. That is, all lengths have been divided by the number *hyp*, but the angles remain unchanged. What we have then is a new right triangle whose hypotenuse has length $\frac{hyp}{hyp} = 1$.





The picture on the right shows $\sin(\theta)$ and $\cos(\theta)$. Notice that the triangles from the left and right are the same, so $\sin(\theta) = \frac{opp}{hyp}$ and $\cos(\theta) = \frac{adj}{hyp}$.

Last, notice that from the definition of tangent we have

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{opp}{hyp}}{\frac{adj}{hyp}} = \frac{opp}{adj}$$

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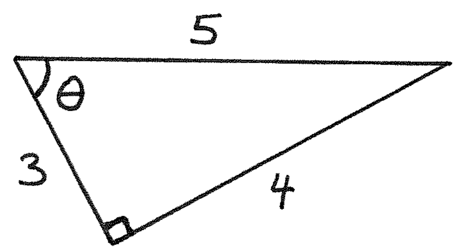
Problem. Find $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for the angle θ shown below.



Solution. The hypotenuse is the side that is opposite the right angle. It has length 5. Of the two remaining sides, the one that is opposite of θ has length 4, and the one that is adjacent to θ has length 3. Therefore,

$$\sin(\theta) = \frac{opp}{hyp} = \frac{4}{5}$$

$$\cos(\theta) = \frac{adj}{hyp} = \frac{3}{5}$$

$$\tan(\theta) = \frac{opp}{adj} = \frac{4}{3}$$

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Exercises

For #1-7, use the definition of tangent, that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, to identify the given value. You can use the chart on page 227 for help.

1.) $\tan\left(\frac{\pi}{3}\right)$

2.) $\tan\left(\frac{\pi}{4}\right)$

3.) $\tan\left(\frac{\pi}{6}\right)$

4.) $\tan(0)$

5.) $\tan\left(-\frac{\pi}{6}\right)$

6.) $\tan\left(-\frac{\pi}{4}\right)$

7.) $\tan\left(-\frac{\pi}{3}\right)$

Suppose that β is a real number, that $0 \leq \beta \leq \frac{\pi}{2}$, and that $\cos(\beta) = \frac{1}{3}$. Use Lemmas 7-12 from the previous chapter, the definition of tangent, and the periods of sine, cosine, and tangent to find the following values.

8.) $\sin(\beta)$

9.) $\tan(\beta)$

10.) $\sin(\beta + \frac{\pi}{2})$

11.) $\cos(\beta - \frac{\pi}{2})$

12.) $\cos(\beta + \pi)$

13.) $\sin(\beta + \pi)$

14.) $\cos(-\beta)$

15.) $\sin(-\beta)$

16.) $\cos(\beta + 2\pi)$

17.) $\sin(\beta + 2\pi)$

18.) $\tan(\beta + \pi)$

Match the numbered piecewise defined functions with their lettered graphs below.

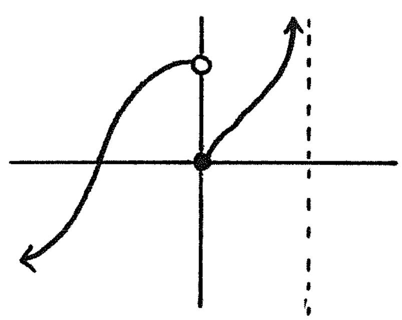
19.) $f(x) = \begin{cases} \sin(x) & \text{if } x \in [0, \infty); \quad \text{and} \\ \tan(x) & \text{if } x \in (-\frac{\pi}{2}, 0). \end{cases}$

20.) $g(x) = \begin{cases} \tan(x) & \text{if } x \in (0, \frac{\pi}{2}); \quad \text{and} \\ \cos(x) & \text{if } x \in (-\infty, 0]. \end{cases}$

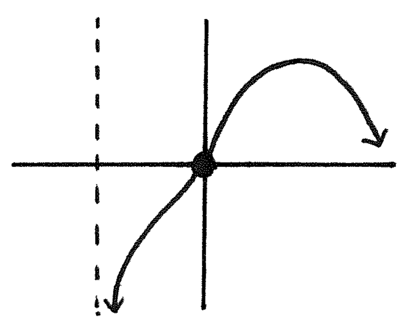
21.) $h(x) = \begin{cases} \tan(x) & \text{if } x \in [0, \frac{\pi}{2}); \quad \text{and} \\ \cos(x) & \text{if } x \in (-\infty, 0). \end{cases}$

22.) $p(x) = \begin{cases} \sin(x) & \text{if } x \in (0, \infty); \quad \text{and} \\ \tan(x) & \text{if } x \in (-\frac{\pi}{2}, 0]. \end{cases}$

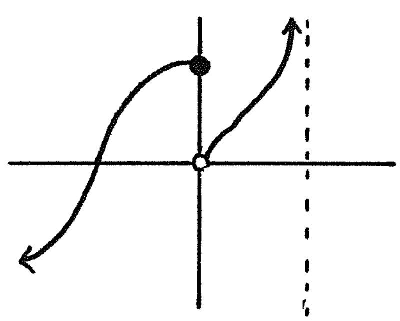
A.)



B.)

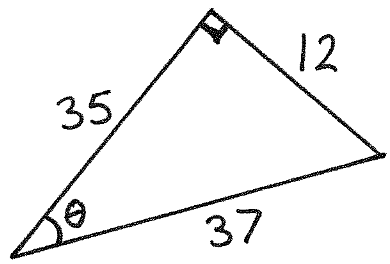


C.)



Use Propostion 13 to find $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for the angles θ given below. You might have to begin by using the Pythagorean Theorem to find the length of a side that is not labelled with a length.

23.)



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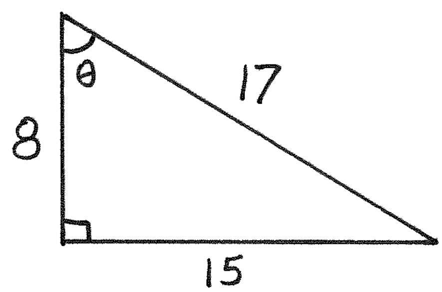

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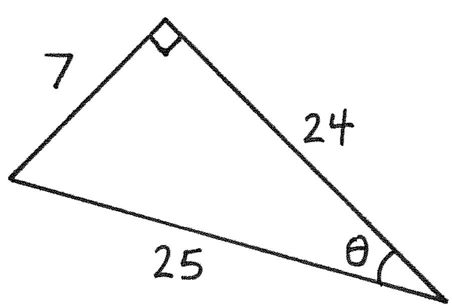

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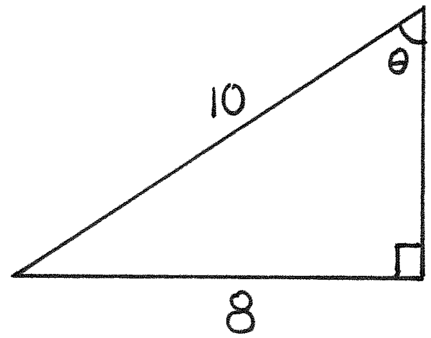
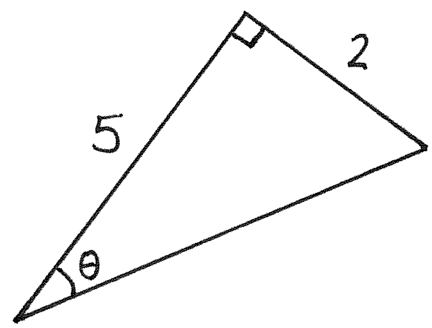


27.)



28.)