Sets

We won't spend much time on the material from this and the next two chapters, Functions and Inverse Functions. That's because these three chapters are mostly a review of some of the math that's a prerequisite for this course. You don't need to know all of it, but you should know much of it. Identify the topics in these three chapters that you are uncomfortable with, and spend the time to catch up on those topics. Ask your classmates, instructor, or perhaps a tutor to help you out.

Sets

A *set* is a collection of objects. For example, the set of countries in North America is a set that contains 3 objects: Canada, Mexico, and the United States.

Set notation. Writing $\{2, 3, 5\}$ is a shorthand for the set that contains the numbers 2, 3, and 5, and no objects other than 2, 3, and 5.

The order in which the objects of a set are written doesn't matter. For example, $\{5, 2, 3\}$ and $\{2, 3, 5\}$ are the same set. Alternatively, the previous sentence could be written as "For example, $\{5, 2, 3\} = \{2, 3, 5\}$."

If B is a set, and x is an object contained in B, we write $x \in B$. If both x and y are objects contained in B then we write $x, y \in B$. If x is not contained in B then we write $x \notin B$.

Examples.

- $5 \in \{2, 3, 5\}$
- $2, 3 \in \{2, 3, 5\}$
- $1 \notin \{2, 3, 5\}$

Subsets. One set is a *subset* of another set if every object in the first set is an object of the second set as well. The set of weekdays is a subset of the set of days of the week, since every weekday is a day of the week.

A more succinct way to express the concept of a subset is as follows:

The set B is a *subset* of the set C if every $b \in B$ is also contained in C.

Writing $B \subseteq C$ is a shorthand for writing "B is a subset of C". Writing $B \nsubseteq C$ is a shorthand for writing "B is *not* a subset of C".

Examples.

- $\{2,3\} \subseteq \{2,3,5\}$
- $\{2,3,5\} \nsubseteq \{3,5,7\}$

Set minus. If A and B are sets, we can create a new set named $A - B$ (spoken as "A minus B") by starting with the set A and removing all of the objects from A that are also contained in the set B.

Examples.

- $\{1,7,8\} - \{7\} = \{1,8\}$
- $\{1,2,3,4,5,6,7,8,9,10\} - \{2,4,6,8,10\} = \{1,3,5,7,9\}$

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Numbers

Among the most common sets appearing in math are sets of numbers. There are many different kinds of numbers. Below is a list of some that are important for this course.

Natural numbers. $\mathbb{N} = \{1,2,3,4,...\}$

Integers. $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, 3, ...\}$

Rational numbers. $\mathbb{Q}$ is the set of fractions of integers. That is, the numbers contained in $\mathbb{Q}$ are exactly those of the form $\frac{n}{m}$ where n and m are integers and $m \neq 0$.

For example, $\frac{1}{3} \in \mathbb{Q}$ and $\frac{-7}{12} \in \mathbb{Q}$.

Real numbers. $\mathbb{R}$ is the set of numbers that can be used to measure a distance or magnitude, or the negative of a number used to measure a distance or magnitude. The set of real numbers can be drawn as a line called "the number line".

$\sqrt{2}$ and π are two of very many real numbers that are not rational numbers.

Numbers as subsets. Any natural number is an integer. Any integer is a rational number, and any rational number is a real number. We can write this more succinctly as

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

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More set notation

Sometimes in math we find it more precise to use a common collection of symbols to express our thoughts. The symbols have less room for interpretation than normal written language, so less confusion arises when we use them. One important example of this is a method we use to describe sets. Let's start with an example, then we'll explain the example:

$$\mathbb{Q} = \left\{ \frac{n}{m} \;\middle|\; n, m \in \mathbb{Z} \text{ and } m \neq 0 \right\}$$

Let's break down this equation one symbol at a time. We know that $\mathbb{Q}$ is the set of rational numbers, and the equal sign means that $\mathbb{Q}$ is the same set as the set on the right of the equal sign.

You know that everything on the right of the equal sign is a set because it's all surrounded by the two curly brackets "{" and "}", so now let's discuss everything in between the curly brackets, moving from left to right. The fraction $\frac{n}{m}$ means that the set on the right side of the equal sign is a set of fractions. The vertical line immediately following $\frac{n}{m}$ means "that satisfy the following rules". Everything to the right of the vertical line is the collection of rules that the fractions of the set satisfy, in this case, n and m are integers, and $m \neq 0$. So the set $\left\{ \frac{n}{m} \mid n, m \in \mathbb{Z} \text{ and } m \neq 0 \right\}$ is the set of all fractions $\frac{n}{m}$ that satisfy the rules that n and m are integers and $m \neq 0$.

Putting it all together, the expression

$$\mathbb{Q} = \left\{ \frac{n}{m} \;\middle|\; n, m \in \mathbb{Z} \text{ and } m \neq 0 \right\}$$

should be read as saying that the set of rational numbers is the set of fractions of integers, as long as the denominator of the fractions do not equal 0. And we already knew that. This is perhaps just a slightly more efficient way to write that. That is, it will be more efficient once you are comfortable with it, and that won't take too long.

Examples.

• $\mathbb{Z} = \left\{ \frac{n}{m} \mid n \in \mathbb{Z} \text{ and } m = 1 \right\}$ says that the integers are the same set as the set of fractions that satisfy the rules that the numerator is an integer and the denominator equals 1, and that's true. Notice that $\frac{3}{1} = 3$ and that $3 \in \mathbb{Z}$. Similarly, $\frac{-17}{1} = -17$ and $-17 \in \mathbb{Z}$, etc.

• $E = \left\{ n \in \mathbb{Z} \mid \frac{n}{2} \in \mathbb{Z} \right\}$ says that E is the set of integers that satisfy the rule that when you divide them by 2, you still have an integer. Notice that $\frac{4}{2} = 2 \in \mathbb{Z}$, that $\frac{-18}{2} = -9 \in \mathbb{Z}$, and that $\frac{3}{2} \notin \mathbb{Z}$. So the above equation of sets is telling us is that E is the set of even numbers.

• $\mathbb{N} = \left\{ n \in \mathbb{Z} \mid n > 0 \right\}$ says that the natural numbers is the set of all integers that are positive. And that's true. The positive numbers in $\{..., -2, -1, 0, 1, 2, 3, ...\}$ are exactly the numbers $\{1, 2, 3, 4, ...\}$.

• $\left\{ x \in \{2, 7, -3\} \mid x < 0 \right\} = \{-3\}$ says that -3 is the only negative number from the collection of 2, 7, and -3.

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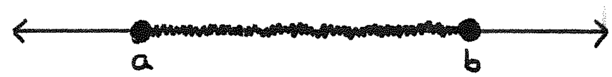
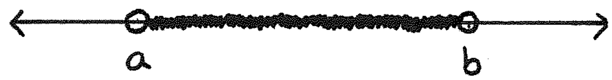
Intervals

Below is a list of 8 important types of subsets of $\mathbb{R}$. In the sets below, a and b are real numbers. That is, $a, b \in \mathbb{R}$. Also, in the sets below $a \leq b$. Any set of one of these types is called an *interval*.

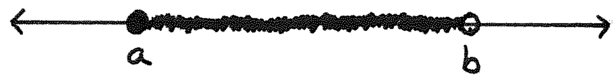
$$[a,b] = \{\, x \in \mathbb{R} \mid a \leq x \leq b \,\}$$



$$(a,b) = \{\, x \in \mathbb{R} \mid a < x < b \,\}$$

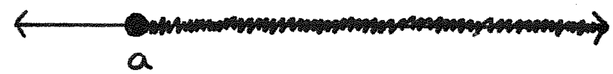


$$[a,b) = \{\, x \in \mathbb{R} \mid a \leq x < b \,\}$$

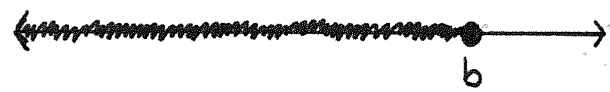


$$(a,b] = \{\, x \in \mathbb{R} \mid a < x \leq b \,\}$$

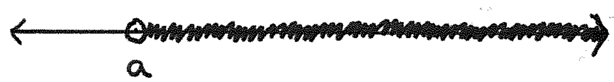
$$[a,\infty) = \{\, x \in \mathbb{R} \mid a \leq x \,\}$$



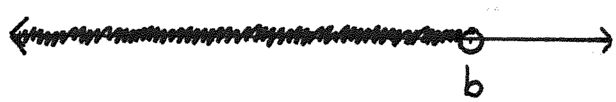
$$(-\infty,b] = \{\, x \in \mathbb{R} \mid x \leq b \,\}$$



$$(a,\infty) = \{\, x \in \mathbb{R} \mid a < x \,\}$$



$$(-\infty,b) = \{\, x \in \mathbb{R} \mid x < b \,\}$$



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The Empty Set

The set $\{4,7,9\}$ is the set that contains the numbers 4, 7, and 9. The set $\{4,7\}$ contains only 4 and 7, the set $\{4\}$ contains only 4, and the set $\{\,\}$ contains, well, nothing.

It probably seems silly to you now, but the set that has nothing in it, that is the set $\{\,\}$, is an important example of a set, and it's a set that comes up frequently in math. It comes up often enough that we've given it a name. We call it *the empty set*, because it's the set that has nothing in it.

In addition to having its own name, we've given it its own symbol. That's because if we wrote the empty set as $\{\,\}$, then readers might just wonder whether that's a typo, that perhaps we intended to type something inside of the curly brackets, but just neglected to. Instead, we use the symbol $\emptyset$ to refer to the empty set.

Example.

If S was the set of all roots of the quadratic $x^2 - 4$, and then $S = \{-2, 2\}$.

If S was the set of all roots of the quadratic x^2, then $S = \{0\}$.

And if S was the set of all real number roots of the quadratic polynomial $x^2 + 1$, then $S = \emptyset$, because $x^2 + 1$ has no real number roots.

The Empty Set is a subset of any other set

In the explanation a few paragraphs before, we began with the set $\{4, 7, 9\}$. We took away an object and were left with the smaller set $\{4, 7\}$. Then we took away another object and were left with the still smaller set $\{4\}$. We ended by taking away the last object to arrive at a still smaller set, namely the empty set, $\emptyset$.

We can think of a set B as being a subset of another set C if we can remove objects from C to obtain B. In that way we can write

$$\emptyset \subseteq \{4\} \subseteq \{4, 7\} \subseteq \{4, 7, 9\}$$

Of course if C is any set, it doesn't matter which one, and if we took away all of the objects from C, then we'd be left with $\emptyset$. That is to say,

$$\emptyset \subseteq C$$

regardless of what the set C is.

Exercises

Determine whether the statements made in #1-15 are true or false.

1.) $3 \in \{3, 6, 1\}$ 2.) $1, 6 \in \{3, 6, 1\}$ 3.) $4 \in \{3, 6, 1\}$

4.) $4 \notin \{3, 6, 1\}$ 5.) $\{2, 3\} \subseteq \{3, 6, 1\}$ 6.) $\{2, 3\} \nsubseteq \{3, 6, 1\}$

7.) $\emptyset \subseteq \{3, 6, 1\}$ 8.) $\{3, 6, 1\} - \{6\} = \{3, 1\}$

9.) $[1, 3) \subseteq [1, 3]$ 10.) $2 \in [1, 3]$ 11.) $1 \in (1, 3]$

12.) $3 \in (1, 3]$ 13.) $[1, 3) \subseteq (1, \infty)$ 14.) $[1, 3) \subseteq (-\infty, 5]$

15.) $\emptyset \subseteq [1, 3)$

Each of the sets in #16-21 is an interval. Name the interval in each of these problems.

16.) $[2, 5) - (3, 5)$ 17.) $[2, 5) - (2, 5)$ 18.) $[2, 5) - [4, 5)$

19.) $[2, 5) - \{2\}$ 20.) $[2, 5) - [2, 3)$ 21.) $[2, 5) - [2, 4]$

Let $T = \{1, 2, 3\}$. Match the numbered sets on the left to the lettered sets on the right that they equal.

22.) $\{\, x \in T \mid \frac{x}{2} \in \mathbb{Z} \,\}$	A.) $\{1, 2, 3\}$
23.) $\{\, x \in T \mid x \in \mathbb{N} \,\}$	B.) $\{1, 2\}$
24.) $\{\, x \in T \mid x > 1 \,\}$	C.) $\{1, 3\}$
25.) $\{\, x \in T \mid \frac{x}{3} \in \mathbb{Z} \,\}$	D.) $\{2, 3\}$
26.) $\{\, x \in T \mid -x \in \mathbb{N} \,\}$	E.) $\{1\}$
27.) $\{\, x \in T \mid \frac{x}{2} \notin \mathbb{Z} \,\}$	F.) $\{2\}$
28.) $\{\, x \in T \mid x \leq 2 \,\}$	G.) $\{3\}$
29.) $\{\, x \in T \mid \frac{1}{x} \in \mathbb{Z} \,\}$	H.) $\emptyset$