Quadratic Equations and Conics

A *quadratic equation in two variables* is an equation that's equivalent to an equation of the form

$$p(x, y) = 0$$

where $p(x, y)$ is a quadratic polynomial.

Examples.

- $4x^2 - 3xy - 2y^2 + x - y + 6 = 0$ is a quadratic equation, as are $x^2 - y^2 = 0$ and $x^2 + y^2 = 0$ and $x^2 - 1 = 0$.

- $y = x^2$ is a quadratic equation. It's equivalent to $y - x^2 = 0$, and $y - x^2$ is a quadratic polynomial.

- $xy = 1$ is a quadratic equation. It's equivalent to the quadratic equation $xy - 1 = 0$.

- $x^2+y^2 = -1$ is a quadratic equation. It's equivalent to $x^2+y^2+1 = 0$.

- $x^2 + y = x^2 + 2$ is a not a quadratic equation. It's a linear equation. It's equivalent to $y - 2 = 0$, and $y - 2$ is a linear polynomial.

A *conic* is a set of solutions of a quadratic equation in two variables. In contrast to lines–solutions of linear equations in two variables–it takes a fair amount of work to list all of the possible geometric shapes that can possibly arise as conics. In what remains of this chapter, we'll take a tour of some conics that we already know. After we cover trigonometry in this course, we'll return to conics and explain all of the possible shapes of conics.

No points

We saw in the chapter Polynomial Equations that the quadratic equation

$$x^2 + y^2 = -1$$

has no solution. A sum of squares can never be negative. Thus, $\emptyset$ is an example of conic.

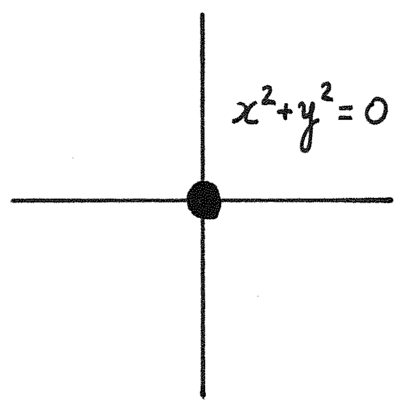
$x^2 = -1$ has no solution too!

One point

We saw in the chapter Polynomial Equations that the quadratic equation

$$x^2 + y^2 = 0$$

has exactly one solution, the point $(0, 0)$. The only way to take two numbers, square them, take the sum of those squares, and get 0, is if the two numbers you start with are 0. Thus, the set containing the single point $(0, 0)$ is a conic.



Parallel lines

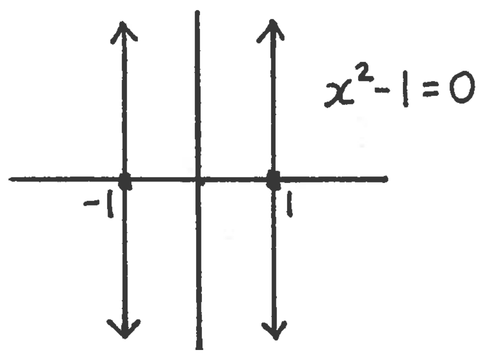
An equation for the vertical line that crosses the x-axis at the number 1 is $x = 1$, or equivalently, $x - 1 = 0$. Similarly, $x + 1 = 0$ is an equation for the vertical line that crosses the x-axis at the number -1. As we saw in the previous chapter, the union of these two vertical lines is the set of solutions of the equation

$$(x - 1)(x + 1) = 0$$

an equation that can be written more simply as

$$x^2 - 1 = 0$$

Thus, two parallel lines can appear as a conic.

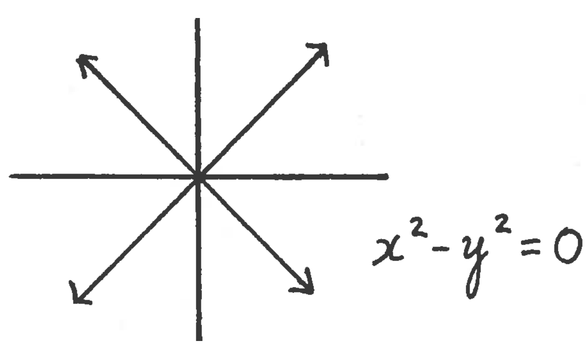


Intersecting lines

It's also possible to have two intersecting lines as a conic. In the previous chapter we saw that the solutions of

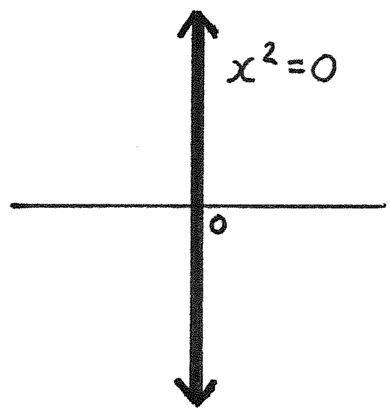
$$x^2 - y^2 = 0$$

are a union of the two lines that have slope 1 and -1, and that each pass through the point $(0, 0)$. Recall that $x - y = 0$ is an equation for the first of these two lines, and that $x + y = 0$ is an equation for the second of these two lines. Therefore, $(x - y)(x + y) = 0$ is an equation for the union of these two lines, and it can be simplified as $x^2 - y^2 = 0$.



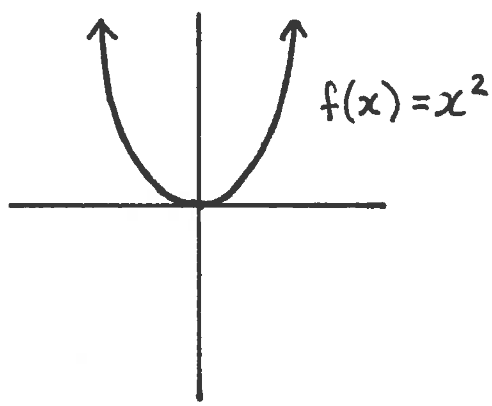
Single line

$x^2 = 0$ is a quadratic equation, so its set of solutions in the plane is a conic. The only number whose square is 0 is 0 itself, so if $x^2 = 0$, then $x = 0$. Thus, the set of solutions of $x^2 = 0$ are those points in the plane whose x-coordinates equal 0. This is a single vertical line, the y-axis. This single line is an example of a conic.



Parabola

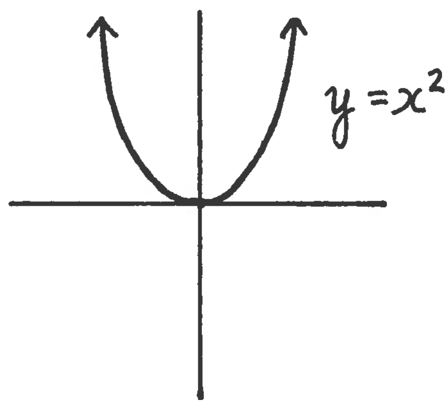
The graph of the function in one variable $f(x) = x^2$ is called a *parabola*.



The points in the graph of $f(x)$ are points of the form $(x, f(x))$. That is, the points in the graph have the form (x, x^2). These are the points in the plane whose second coordinates are the square of their first coordinates, or in other words, points of the form (x, y) where

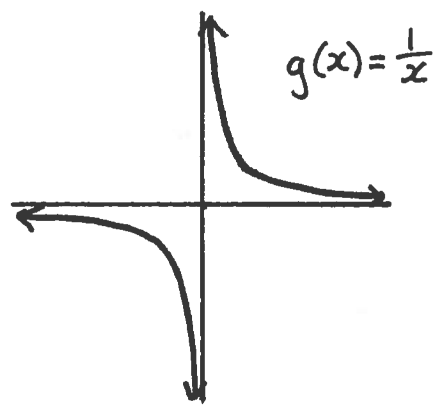
$$y = x^2$$

To recap, the graph of $f(x) = x^2$ in the plane is the same set as the set of solutions of the quadratic equation in two variables given by the equation $y = x^2$. That is, the parabola below is a conic.



Hyperbola

The points in the graph of the function $g(x) = \frac{1}{x}$ are of the form $(x, \frac{1}{x})$.

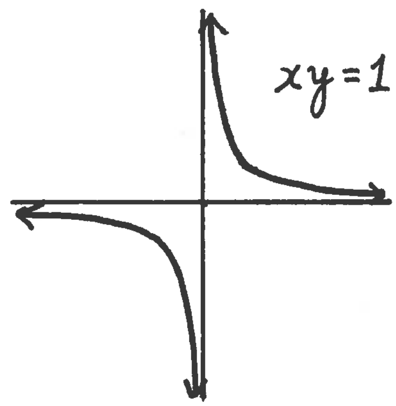


That is, they are points whose y-coordinates are the multiplicative inverse of their x-coordinates. In other words, the graph of $g(x) = \frac{1}{x}$ is exactly the set of points (x, y) in the plane where $y = \frac{1}{x}$.

We know from the domain of the function $g(x) = \frac{1}{x}$ that $x \neq 0$. Thus, we can multiply the equation $y = \frac{1}{x}$ by x to obtain the equivalent equation

$$xy = 1$$

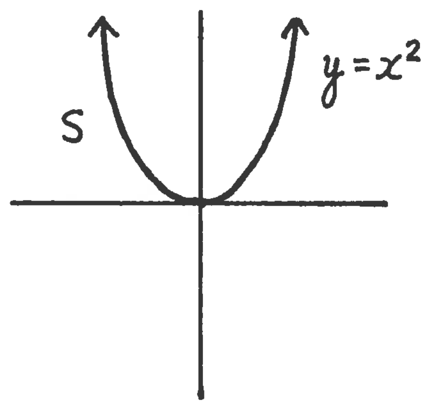
so that we now have that the graph of $\frac{1}{x}$ is exactly the set of solutions of the equation $xy = 1$.



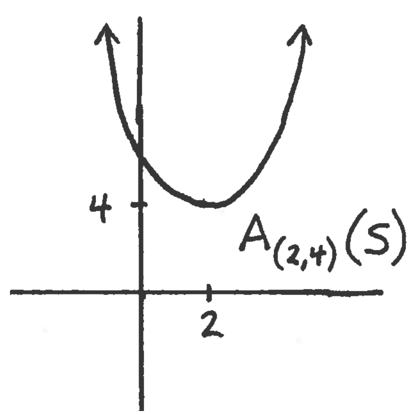
The equation $xy = 1$ is a quadratic equation, so its set of solutions is a conic. It's called a *hyperbola*.

Exercises

#1-5 involve writing equations for planar transformations of the parabola that is the set of solutions of $y = x^2$. We'll call this conic S.

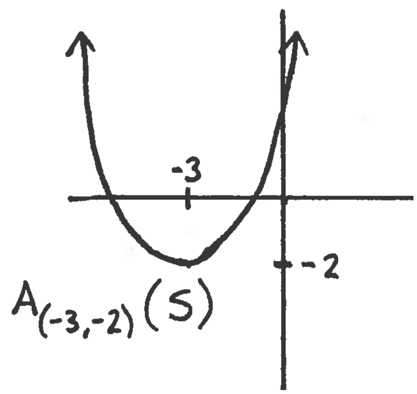


1.) **The parabola shifted right 2 and up 4:**
Recall that the planar transformation $A_{(2,4)} : \mathbb{R}^2 \to \mathbb{R}^2$ moves points in the plane right 2 and up 4. That is, $A_{(2,4)}(x,y) = (x+2, y+4)$. Find an equation for $A_{(2,4)}(S)$. To do this, precompose the equation for S, the equation $y = x^2$, by $A_{(2,4)}^{-1}$. The inverse of $A_{(2,4)}$ is $A_{(-2,-4)}$ and $A_{(-2,-4)}(x,y) = (x-2, y-4)$. (Don't simplify your answer.)



2.) **The parabola shifted left 3 and down 2:**

The planar transformation $A_{(-3,-2)} : \mathbb{R}^2 \to \mathbb{R}^2$ moves points in the plane left 3 and down 2. That is, $A_{(-3,-2)}(x, y) = (x - 3, y - 2)$. Find an equation for $A_{(-3,-2)}(S)$. To do this, precompose the equation for S by $A^{-1}_{(-3,-2)}$. The inverse of $A_{(-3,-2)}$ is $A_{(3,2)}$. (Don't simplify your answer.)

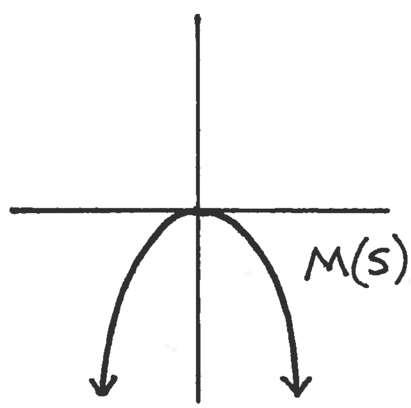


3.) **The parabola flipped over the x-axis:**

The planar transformation $M : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

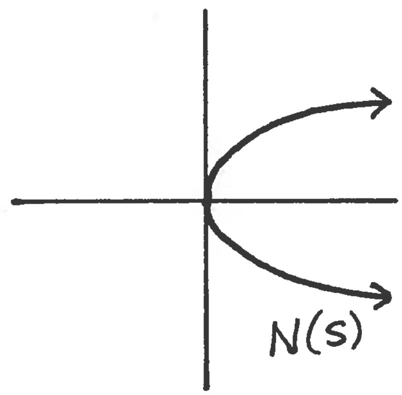
flips the plane over the x-axis. Written using row vectors, $M(x, y) = (x, -y)$. Find an equation for $M(S)$. To do this, precompose the equation for S by M^{-1}. (Remember that $M^{-1} = M$, and don't simplify your answer.)



4.) **The parabola flipped over the $x = y$ line:**
The planar transformation $N : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

flips the plane over the $x = y$ line. Written using row vectors, $N(x, y) = (y, x)$. Find an equation for $N(S)$. (Remember that $N^{-1} = N$, and don't simplify your answer.)

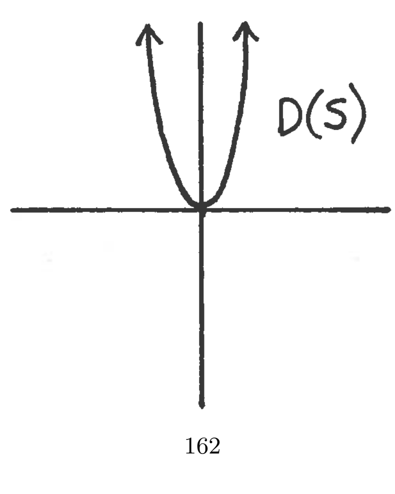


5.) **The parabola scaled by 2 in the y-coordinate:**
The planar transformation $D : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

is a diagonal matrix that scales the y-coordinates of vectors by 2. That is, $D(x, y) = (x, 2y)$. Find an equation for $D(S)$. (Don't simplify your answer, and remember that

$$D^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$



For #6-9, match the numbered function with its lettered graph.

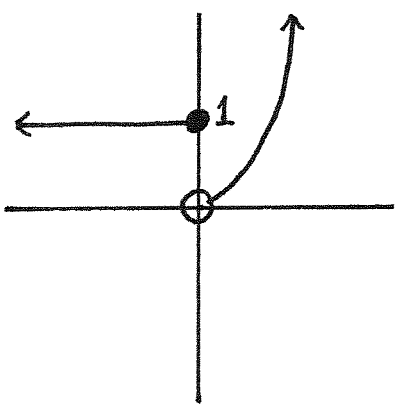
6.) $f(x) = \begin{cases} x^2 & \text{if } x \geq 0; \quad \text{and} \\ 1 & \text{if } x < 0. \end{cases}$

8.) $h(x) = \begin{cases} x^2 & \text{if } x > 0; \quad \text{and} \\ 1 & \text{if } x \leq 0. \end{cases}$

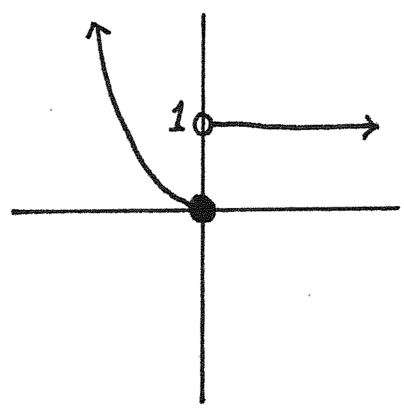
7.) $g(x) = \begin{cases} 1 & \text{if } x \geq 0; \quad \text{and} \\ x^2 & \text{if } x < 0. \end{cases}$

9.) $m(x) = \begin{cases} 1 & \text{if } x > 0; \quad \text{and} \\ x^2 & \text{if } x \leq 0. \end{cases}$

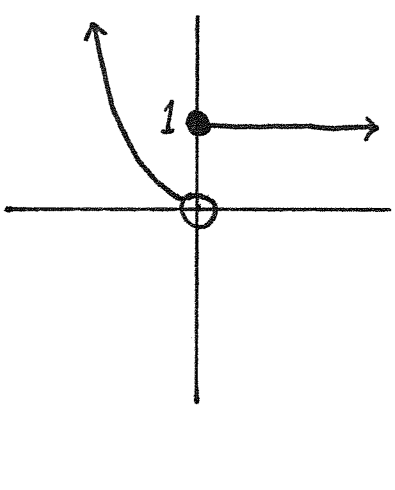
A.)



C.)



B.)



D.)