Planar Transformations of Graphs

Let $f : D \to \mathbb{R}$ be a function with $D \subset \mathbb{R}$. Then $f(x) = y$ is an equation in two variables

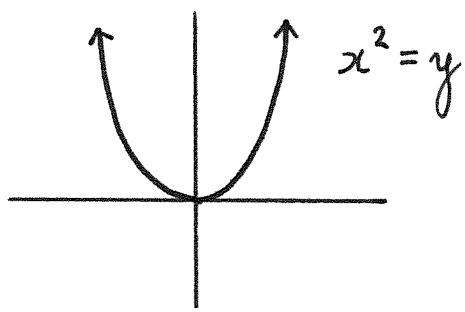
Examples.

- $\log_e(x) = y$ is an equation in two variables.
- $x^2 = y$ is an equation in two variables.
- $4x^3 - 2x + \frac{1}{x} = y$ is an equation in two variables.

Graphs are sets of solutions

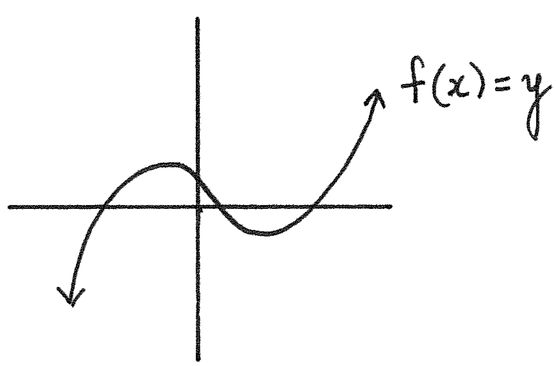
The point $(2, 4)$ is in the graph of the function x^2 because $2^2 = 4$. Similarly, $(3, 9)$, $(5, 25)$, and $(-1, 1)$ are points in the graph of x^2.

Notice that each point listed above, and every point in the graph of x^2, is a solution of the equation $x^2 = y$.



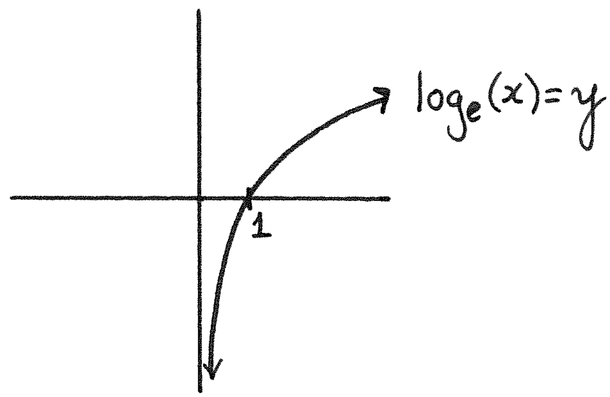
This observation applies to any function $f : D \to \mathbb{R}$ with $D \subset \mathbb{R}$, which is the content of the fact below.

Important Fact: The graph of $f(x)$ is the set of solutions of the equation $f(x) = y$.

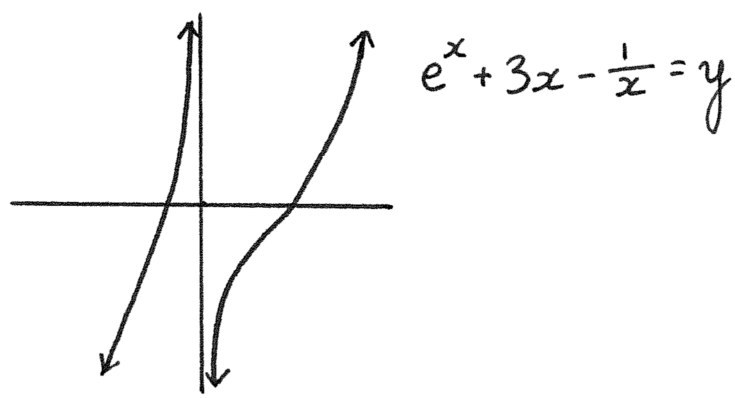


Examples.

• The graph of $\log_e(x)$ is the set of solutions of the equation $\log_e(x) = y$.



• The set of solutions of the equation $e^x + 3x - \frac{1}{x} = y$ is the graph of the function $e^x + 3x - \frac{1}{x}$



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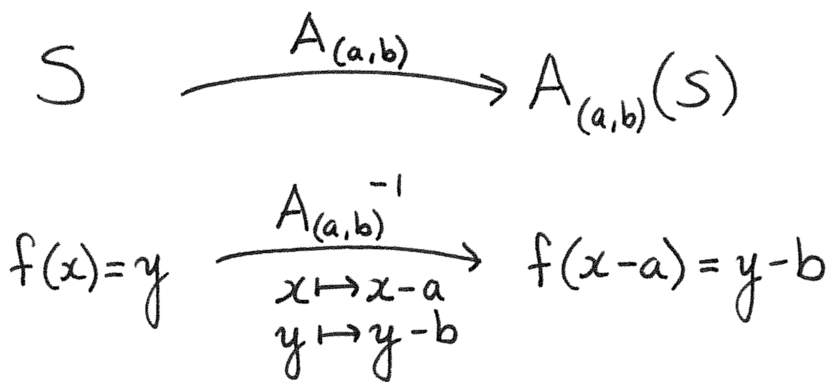
Common transformations of graphs

The next three propositions — 18, 19, and 20 — explain how some common planar transformations can transform the graph of a function $f(x)$ into the graph of a different, but closely related function.

Before reading the three propositions, it might be worth skipping over them to the chart on page 283 that summarizes their content, and to the list of examples that follow. You can come back to the proofs of the propositions at the conclusion of the chapter.

Propostion (18). If $S \subseteq \mathbb{R}^2$ is the graph of $f(x)$, then $A_{(a,b)}(S)$ is the graph of $f(x-a)+b$.

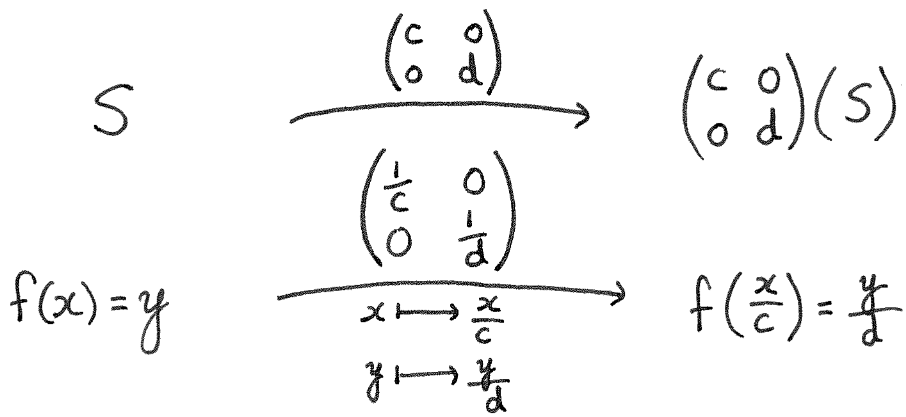
Proof: The graph of $f(x)$, the set S, is the set of solutions of the equation $f(x) = y$.



The equation $f(x-a) = y-b$ is equivalent to the equation $f(x-a)+b = y$, so $A_{(a,b)}(S)$ is the set of solutions of $f(x-a)+b = y$. That is, $A_{(a,b)}(S)$ is the graph of the function $f(x-a)+b$. ■

Propostion (19). Suppose that $c \neq 0$ and that $d \neq 0$. If $S \subseteq \mathbb{R}^2$ is the graph of $f(x)$, then $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}(S)$ is the graph of $df(\frac{x}{c})$.

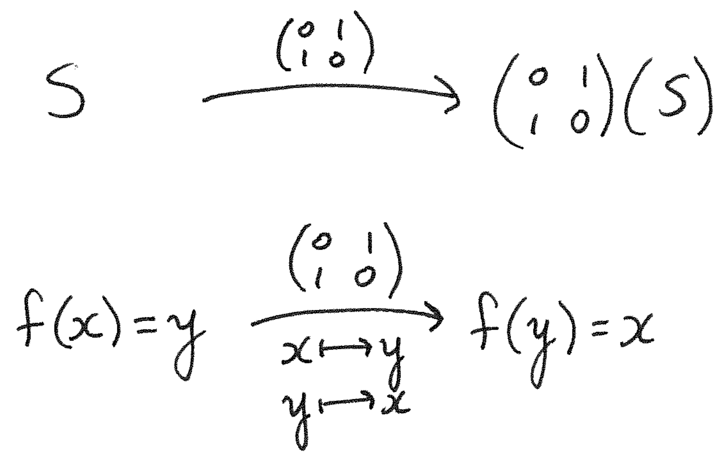
Proof:



The set $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}(S)$ is the set of solutions of the equation $f(\frac{x}{c}) = \frac{y}{d}$, or equivalently, of the equation $df(\frac{x}{c}) = y$. Thus, $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}(S)$ is the graph of $df(\frac{x}{c})$. ■

Propostion (20). Suppose that $f(x)$ is an invertible function. If $S \subseteq \mathbb{R}^2$ is the graph of $f(x)$, then $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(S)$ is the graph of $f^{-1}(x)$.

Proof:



The equation $f(y) = x$ is equivalent to $y = f^{-1}(x)$, so $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(S)$ is the graph of $f^{-1}(x)$. ■

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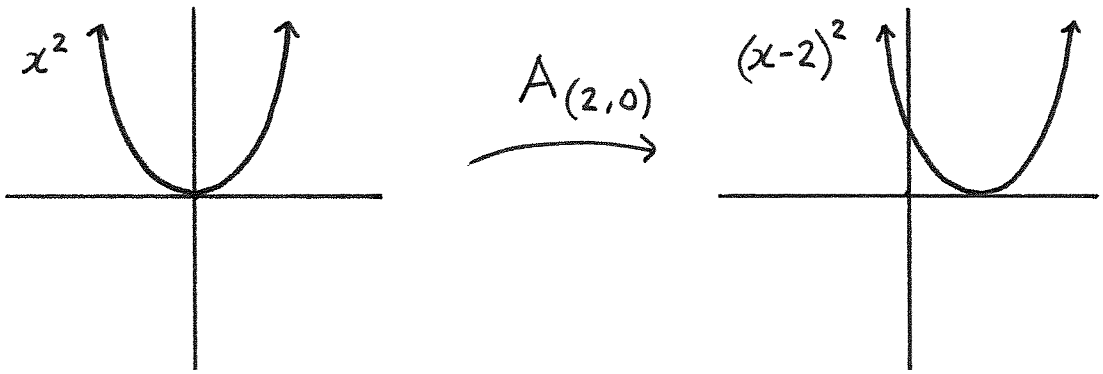
Seven most basic graph transformations

The chart below breaks up Propositions, 18, 19, and 20, into the seven most basic graph transformations. In the chart, $S \subseteq \mathbb{R}^2$ is the graph of a function $f(x)$. The numbers c and d that appear in the third and fourth row are not equal to 0.

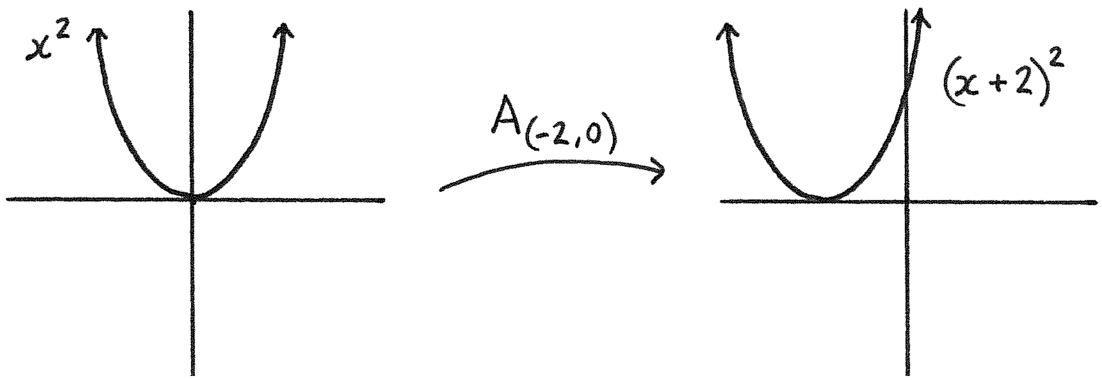
function	function's graph	how to derive the function's graph from the graph of $f(x)$
$f(x-a)$	$A_{(a,0)}(S)$	add a to the x-coordinate
$f(x)+b$	$A_{(0,b)}(S)$	add b to the y-coordinate
$f(\frac{x}{c})$	$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}(S)$	scale the x-coordinate by c
$df(x)$	$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}(S)$	scale the y-coordinate by d
$f(-x)$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}(S)$	flip over the y-axis
$-f(x)$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(S)$	flip over the x-axis
$f^{-1}(x)$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(S)$	flip over the $y=x$ line

Examples.

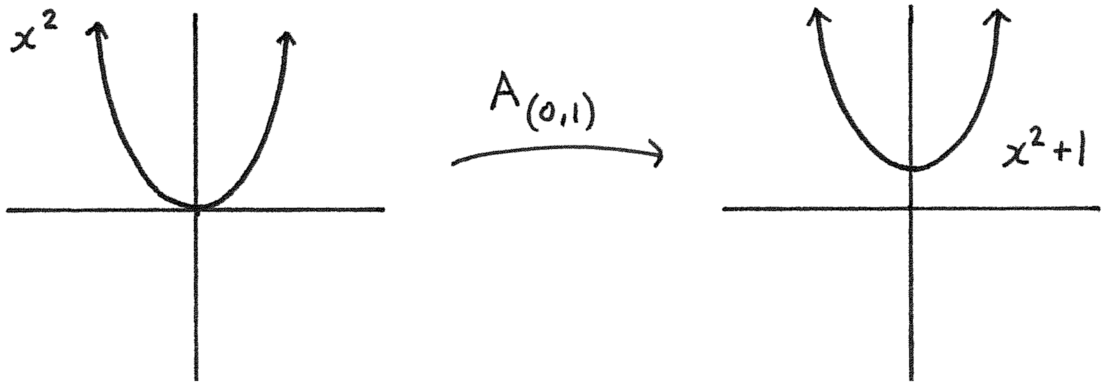
• We know that the graph of x^2 is a parabola. The first row of the chart on the previous page tells us that we can derive the graph of $(x-2)^2$ by adding 2 to the x-coordinate of every point in the graph of x^2.



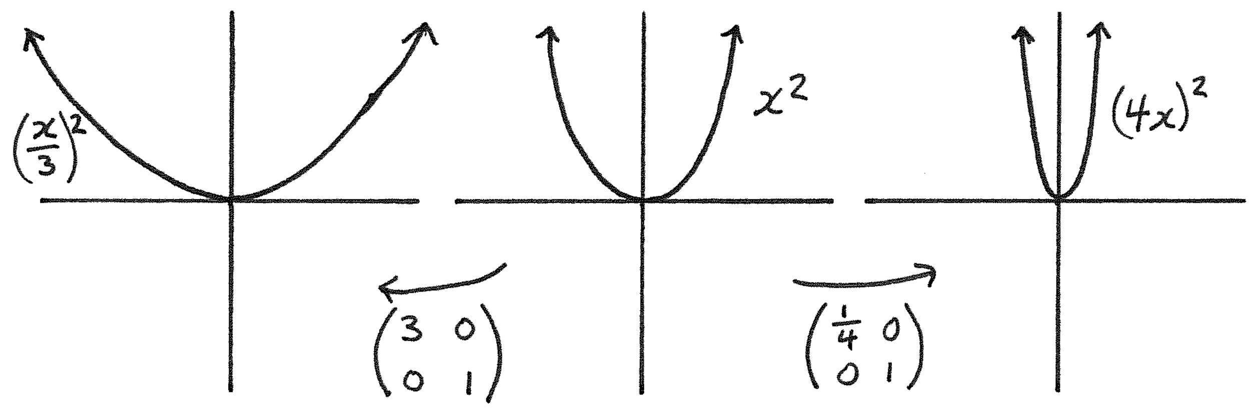
• The graph of $(x+2)^2$ is the graph of x^2 shifted left by 2. That's because $(x+2)^2 = (x-(-2))^2$, and to graph $(x-(-2))^2$ we need to add -2 (or subtract 2) from the x-coordinates of the points in the graph of x^2.



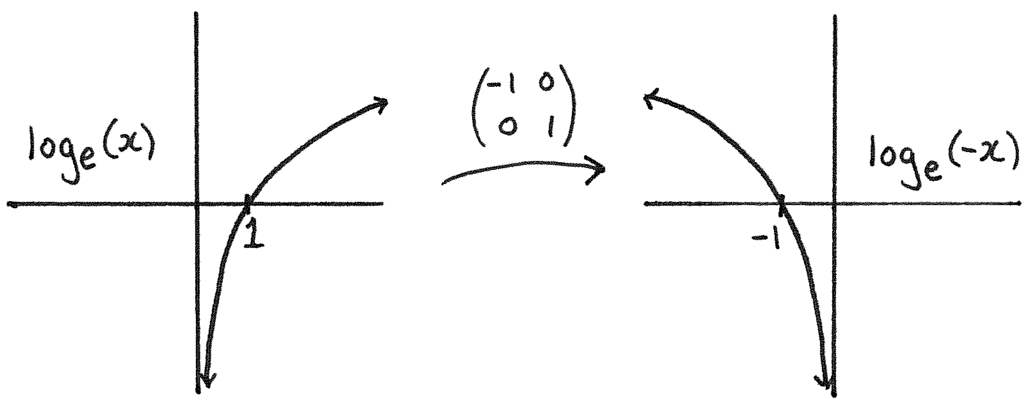
• The graph of x^2+1 is the graph of x^2 with 1 added to the y-coordinate of every point. That is, the graph of x^2+1 is the graph of x^2 shift up by 1 unit.



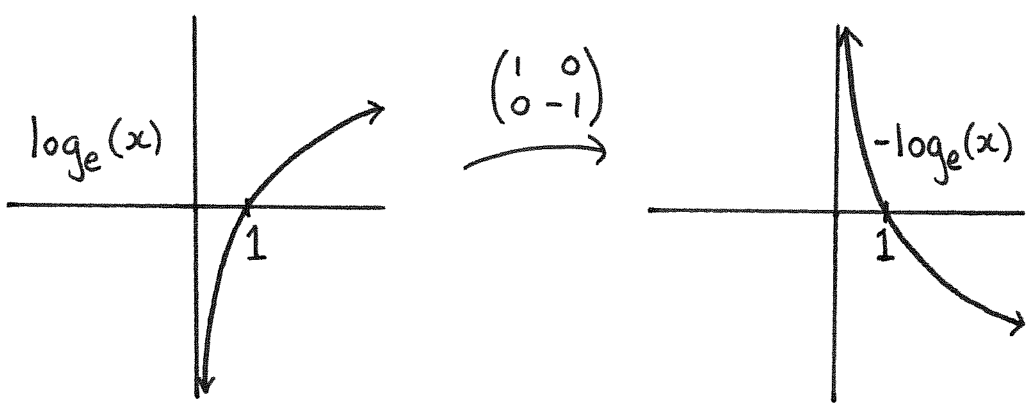
• The graph of $(\frac{x}{3})^2$ is the graph of x^2 scaled by 3 in the x-coordinate. The graph of $(4x)^2 = (\frac{x}{1/4})^2$ is the graph of x^2 scaled by $\frac{1}{4}$ in the x-coordinate.



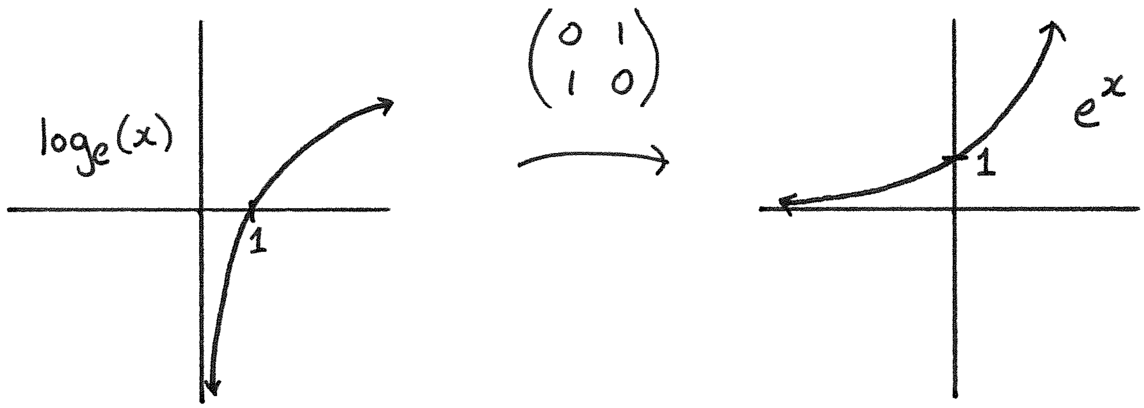
• The graph of $\log_e(-x)$ is the graph of $\log_e(x)$ flipped over the y-axis.



• The graph of $-\log_e(x)$ is the graph of $\log_e(x)$ flipped over the x-axis.



• The graph of the inverse function of $\log_e(x)$, which is the function e^x, is the graph of $\log_e(x)$ flipped over the $y = x$ line.

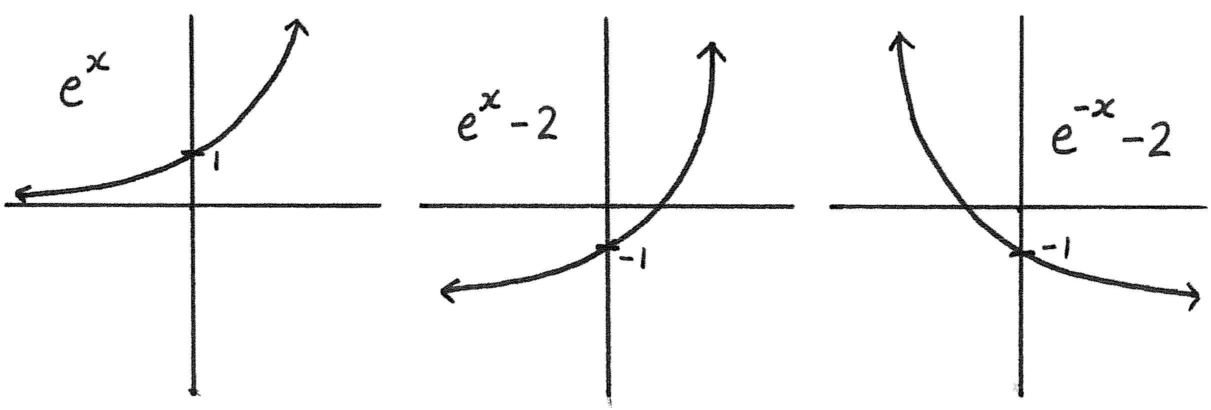


Multiple transformations

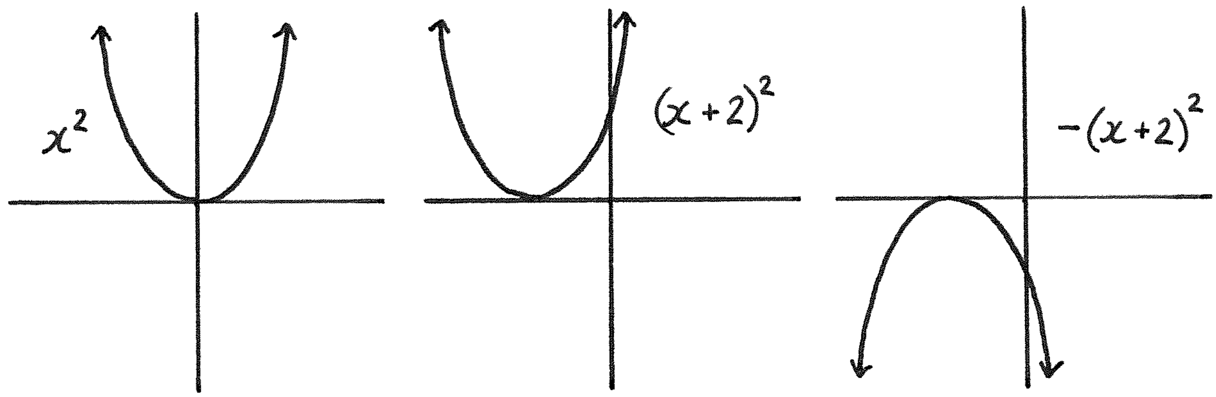
As long as there is only one type of operation involved "inside the function" — either multiplication or addition — and only one type of operation involved "outside the function" — either multiplication or addition — then you can easily combine different rules from the chart of the seven most basic graph transformations. For example, you can easily combine the rules for functions of the form $-2f(x+5)$, $3f(2x)$, $f(x-4)+3$, or $f(-7x)+2$. The next three examples illustrate how this can be done.

Examples.

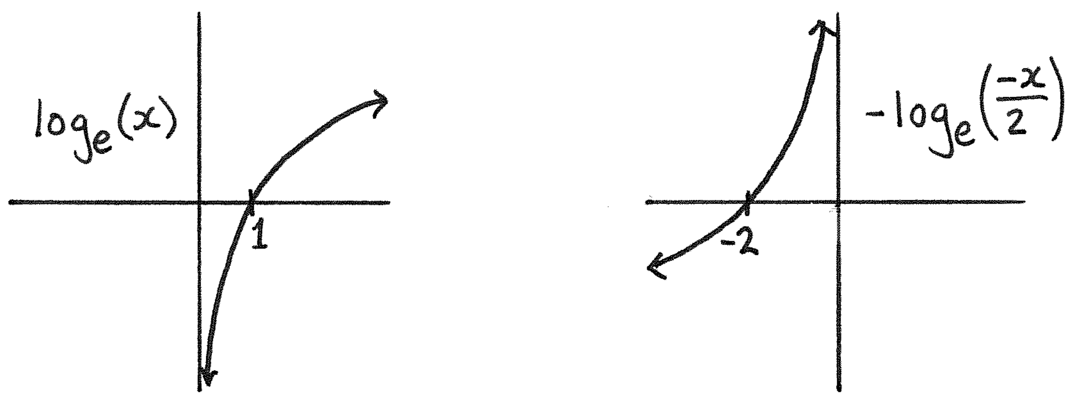
• The graph of $e^{-x} - 2$ is the graph of e^x moved down by 2 units and flipped over the y-axis.



• The graph of $-(x+2)^2$ is the graph of x^2 shifted left by two units and flipped over the x-axis.



• The graph of $-\log_e\left(\frac{-x}{2}\right)$ is the graph of $\log_e(x)$ flipped over the x-axis, flipped over the y-axis, and scaled by 2 in the x-coordinate.



Graph transformations become a little more complicated if we both multiply and add "inside the function" such as with $f(2x+5)$, where we have multiplied by 2 and added 5. It's also a little more complicated if we both multiply and add "outside the function" such as with $3f(x)+6$, where we have multiplied by 3 and added 6. The extra complication with these sorts of functions is that it matters which graph transformation you apply first. For example, scaling the x-coordinate by $\frac{1}{2}$ and then shifting the graph left by 5 results in a different picture then if we shift left by 5 and then scale the x-coordinate by $\frac{1}{2}$.

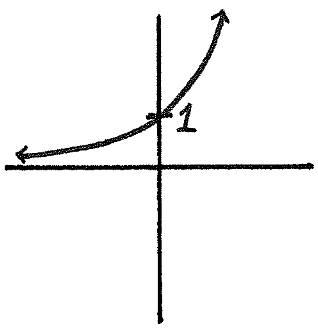
For all of the homework exercises in this text, you'll only be given graph transformation problems where the order of transformations that you need to apply does not affect the final result.

Exercises

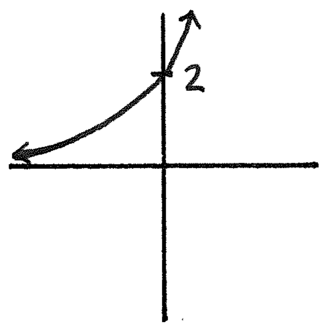
Use the chart on page 283 to help you match the numbered functions with their lettered graphs.

1.) e^x	2.) e^{x+2}	3.) e^{x-2}
4.) $e^x + 2$	5.) $e^x - 2$	6.) $2e^x$
7.) e^{-x}	8.) $-e^x$	9.) $\log_e(x)$

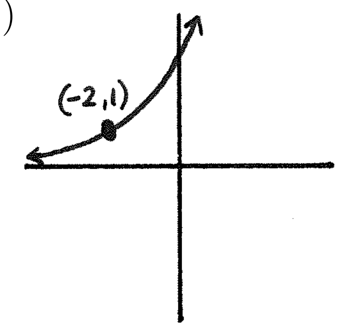
A.)



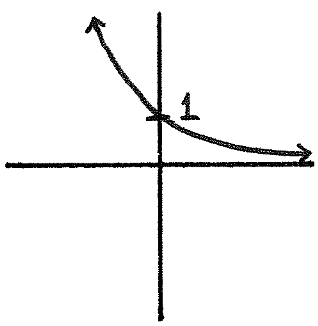
B.)



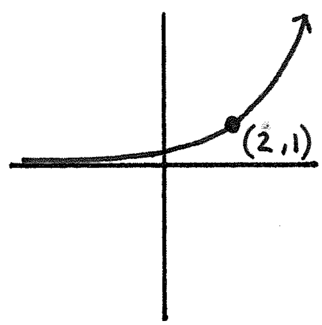
C.)



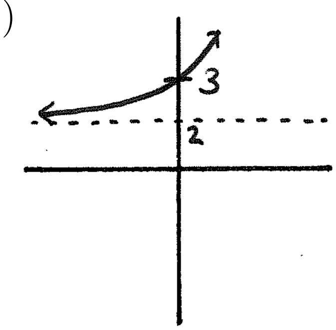
D.)



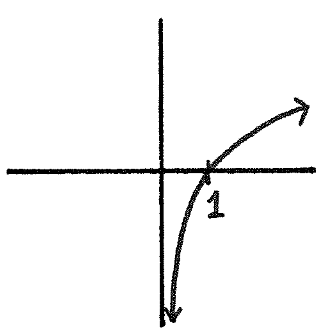
E.)



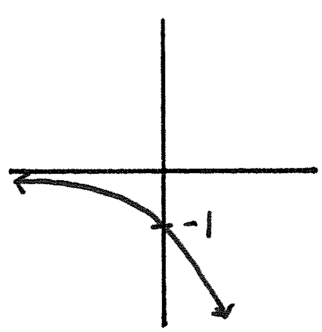
F.)



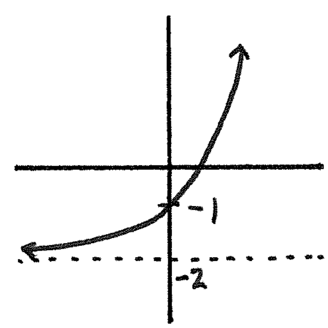
G.)



H.)



I.)



10.) x^2

11.) $(x+1)^2$

12.) $(x-1)^2$

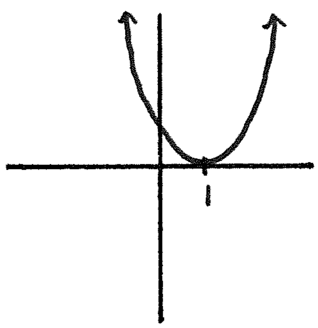
13.) x^2+1

14.) x^2-1

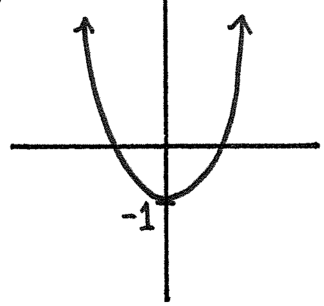
15.) $(-x)^2$

16.) $-x^2$

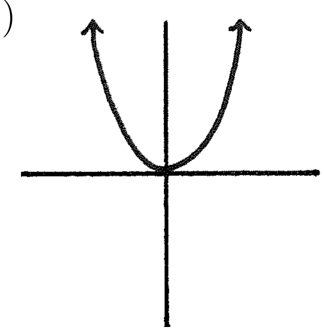
A.)



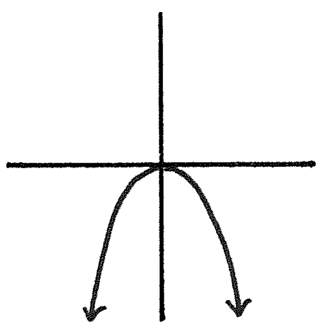
B.)



C.)



D.)



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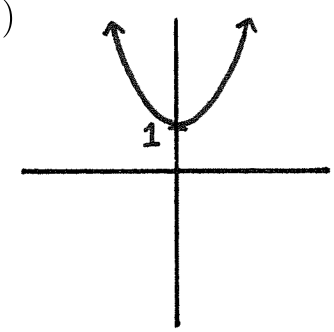
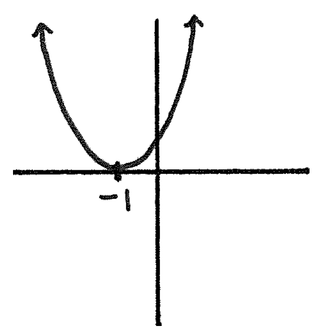


F.)



17.) x^3

18.) $-x^3$

19.) $x^3 - 2$

20.) $(x-2)^3$

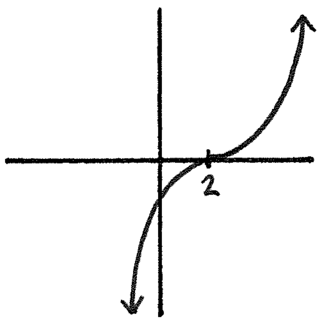
21.) $(x+2)^3$

22.) $(-x)^3$

23.) $\sqrt[3]{x}$

24.) $x^3 + 2$

A.)



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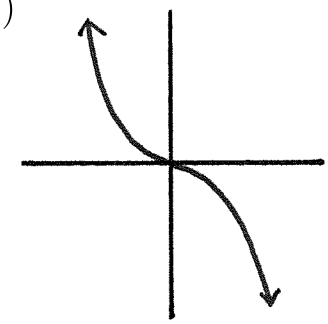

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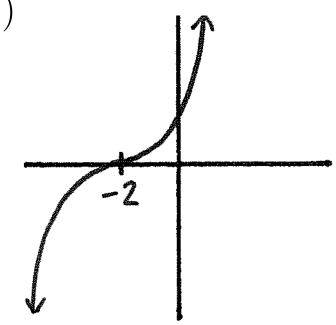

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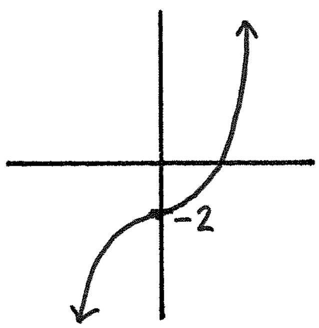

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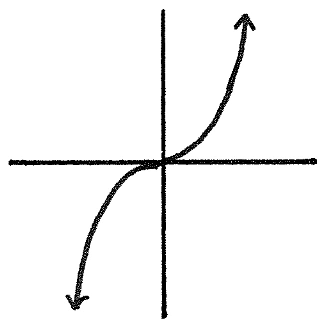

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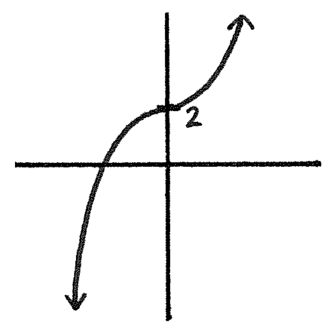
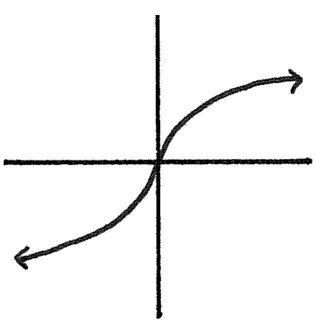


G.)



25.) $\frac{1}{x}$

26.) $\frac{1}{x} + 1$

27.) $-\frac{1}{x}$

28.) $\frac{1}{x} - 1$

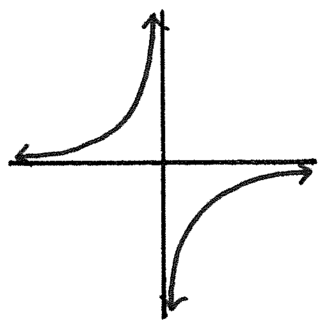
29.) $\frac{1}{x+1}$

30.) $\frac{1}{-x}$

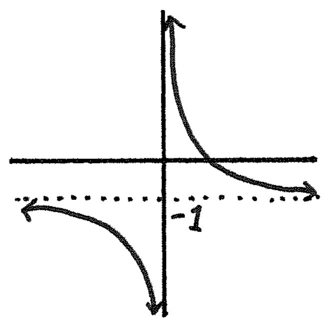
31.) $\frac{1}{x-1}$

32.) The inverse of $\frac{1}{x}$

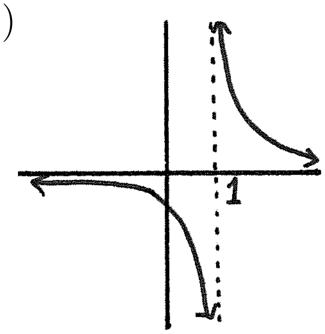
A.)



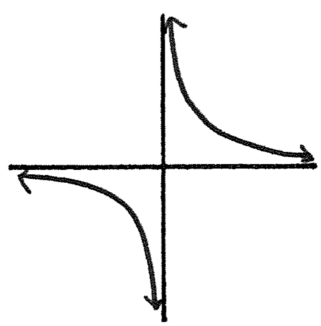
B.)



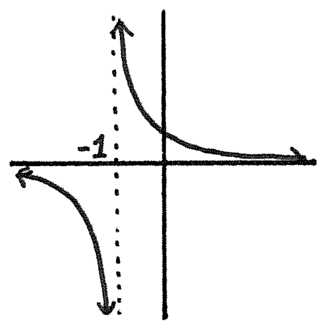
C.)



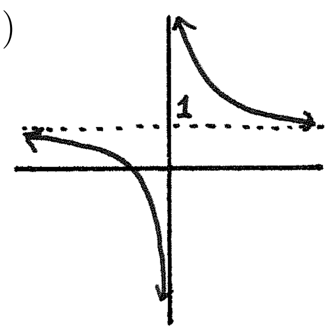
D.)



E.)



F.)

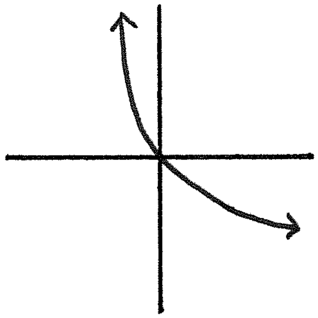


33.) $\log_e(x)$ 34.) $\log_e(x+1)$ 35.) $-\log_e(x+1)$

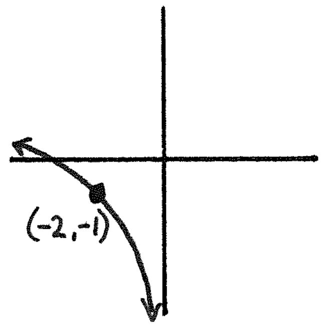
36.) $\log_e(x-1)$ 37.) $\log_e(x-1)+2$

38.) $\log_e\left(\frac{x}{2}\right)$ 39.) $\log_e\left(-\frac{x}{2}\right)$ 40.) $\log_e\left(-\frac{x}{2}\right)-1$

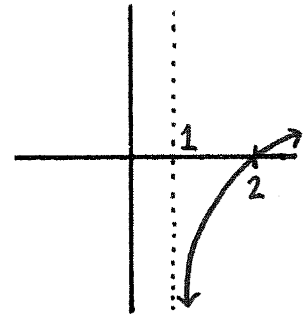
A.)



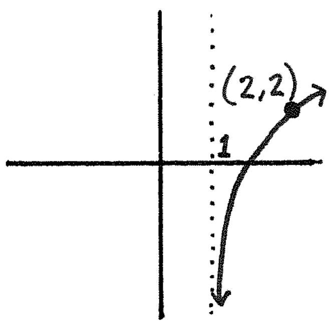
B.)



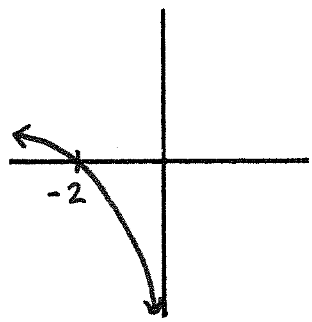
C.)



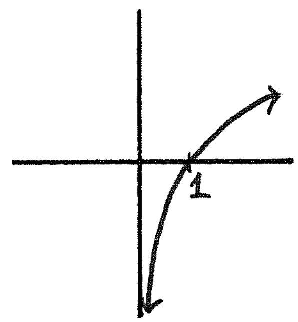
D.)



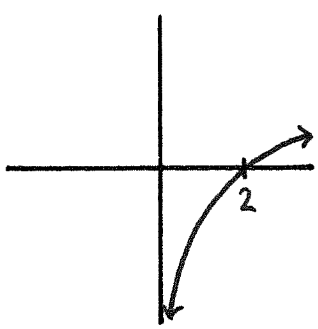
E.)



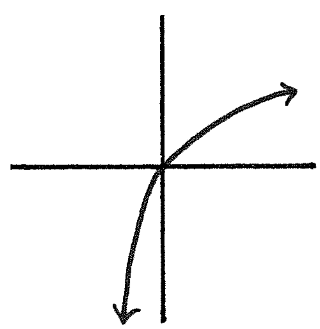
F.)



G.)



H.)



In mathematics, angles are measured by lengths of segments of the unit circle, as we've done in this text. In most contexts though, angles are measured in degrees. In mathematics, a full revolution is an angle that measures 2π, and using degrees, the same angle would be measured as $360°$. That is, when measuring angles, $2\pi = 360°$, or equivalently, $\frac{2\pi}{360°} = 1$, or $\frac{360°}{2\pi} = 1$.

To convert degrees into the mathematical measurement of angles, multiply by $\frac{2\pi}{360°}$. For example, an angle of $180°$ is the same as an angle of

$$180°\Big(\frac{2\pi}{360°}\Big) = 2\pi\Big(\frac{180°}{360°}\Big) = 2\pi\Big(\frac{1}{2}\Big) = \pi$$

Conversely, to change a mathematical measurement of an angle into degrees, multiply by $\frac{360°}{2\pi}$. For example, a right angle of $\frac{\pi}{2}$ can be written in degrees as an angle of

$$\frac{\pi}{2}\Big(\frac{360°}{2\pi}\Big) = \Big(\frac{\frac{\pi}{2}}{2\pi}\Big)360° = \Big(\frac{1}{4}\Big)360° = 90°$$

Convert the following angles in degrees into mathematical angles.

41.) $45°$ 42.) $120°$ 43.) $30°$ 44.) $300°$

Convert the following mathematical angles into degrees.

45.) $\frac{\pi}{6}$ 46.) $\frac{4\pi}{3}$ 47.) $\frac{\pi}{12}$ 48.) $\frac{3\pi}{2}$

Find the solutions of the following equations in one variable.

49.) $5x^2 - 3x = -2$

50.) $e^x = e^{2-x}$

51.) $\log_e\Big((x-3)^{13}\Big) = 26$