Precomposing Equations

Let's "precompose" the function $f(x) = x^3 - 2x + 9$ with the function $g(x) = 4 - x$. (Precompose f with g means that we'll look at $f \circ g$. We would call $g \circ f$ "postcomposing" f with g.)

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= g(x)^3 - 2g(x) + 9 \\ &= (4-x)^3 - 2(4-x) + 9 \end{aligned}$$

To get the same answer in perhaps a slightly different way, first we write the formula for $f(x)$.

$$x^3 - 2x + 9$$

Second, we think of g as the function that replaces x with $4 - x$.

$$x \longmapsto 4-x$$

Third, the formula for $f \circ g(x)$ can be obtained be rewriting the formula for $f(x)$, except that we'll replace every x with $4 - x$.

$$(4-x)^3 - 2(4-x) + 9$$

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Suppose $p(x,y) = 3x^2 - xy + 2$. Let's precompose p with the addition function $A_{(-3,4)}$. That is, we'll find $p \circ A_{(-3,4)}(x,y)$.

First, write down the formula for p

$$3x^2 - xy + 2$$

Second, write down what $A_{(-3,4)}$ replaces each of the coordinates of the vector (x,y) with.

$$A_{(-3,4)}(x,y) = (x-3, y+4)$$

$$x \longmapsto x-3$$

$$y \longmapsto y+4$$

Third, the formula for $p \circ A_{(-3,4)}(x,y)$ is found by rewriting the formula for p, except that we'll replace each x with $x-3$, and replace each y with $y+4$.

$$3(x-3)^2 - (x-3)(y+4) + 2$$

This can be simplified.

$$\begin{aligned} p \circ A_{(-3,4)}(x,y) &= 3(x-3)^2 - (x-3)(y+4) + 2 \\ &= 3x^2 - 18x + 27 - xy - 4x + 3y + 12 + 2 \\ &= 3x^2 - xy - 22x + 3y + 41 \end{aligned}$$

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Let's precompose $q(x, y) = x + y - 1$ with the matrix

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

First, write the formula for q.

$$x + y - 1$$

Second, write what M replaces each of the coordinates of the vector (x, y) with.

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+y \end{pmatrix}$$

$$x \longmapsto x+2y$$

$$y \longmapsto 3x+y$$

Third, the formula for $q \circ M(x, y)$ is found by rewriting the formula for q, except that we'll replace each x with $x + 2y$ and each y with $3x + y$.

$$(x+2y) + (3x+y) - 1$$

It can be simplified as

$$q \circ M(x, y) = 4x + 3y - 1$$

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As we saw in the previous chapter, if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a planar transformation, and if S is the set of solutions of $p(x,y) = q(x,y)$, then $T(S)$ is the set of solutions of $p(x,y) \circ T^{-1} = q(x,y) \circ T^{-1}$. We can say this more economically as follows:

The equation for S precomposed with T^{-1}
is an equation for $T(S)$.

To precompose an equation with T^{-1}, there are three steps to be followed.

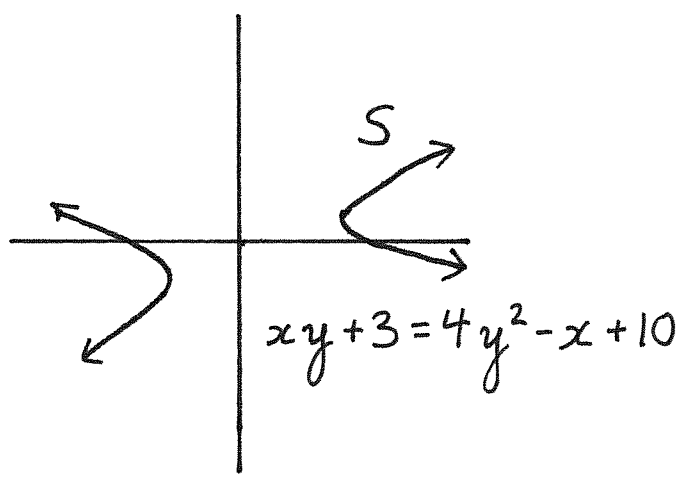
Step 1: Write the original equation.

Step 2: Write what T^{-1} replaces each of the coordinates of the vector (x,y) with.

Step 3: Rewrite the equation from Step 1, except replace every x and every y with the formulas identified in Step 2.

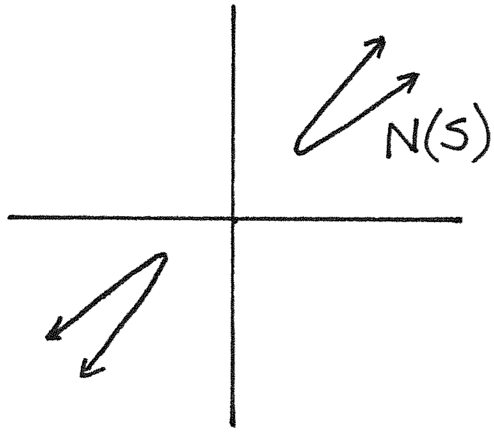
We'll practice these three steps–practice precomposing equations–with the next problem.

Problem: Suppose that S is the subset of the plane that is the set of solutions of the equation $xy + 3 = 4y^2 - x + 10$.



Let N be the invertible matrix $N = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Give an equation that

$N(S)$ is the set of solutions of.



Solution: To find an equation for $N(S)$, we have to precompose the equation for S with N^{-1}.

First, write down the equation for S.

$$xy + 3 = 4y^2 - x + 10$$

Second, write what N^{-1} replaces each of the coordinates of the vector (x, y) with.

$$N^{-1} = \frac{1}{2(1)-1(1)}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ -x + 2y \end{pmatrix}$$

$$x \longmapsto x - y$$

$$y \longmapsto -x + 2y$$

Third, rewrite the equation for S, except replace each x with $x - y$ and each y with $-x + 2y$.

$$(x-y)(-x+2y)+3=4(-x+2y)^2-(x-y)+10$$

Now simplify the equation so that the answer–an equation that has $N(S)$ as its set of solutions–is

$$-x^2 + 3xy - 2y^2 + 3 = 4x^2 - 16xy + 16y^2 - x + y + 10$$

Exercises

For #1-8, write all polynomials in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

1.) Precompose the equation $x - y + 3 = y - 4$ with $A_{(1,2)}$.

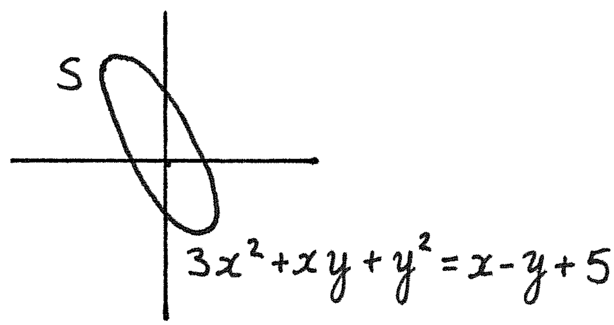
2.) Precompose the equation $2x^2 - 3xy + y - 1 = 2xy + y^2$ with $A_{(-2,3)}$.

3.) Precompose the equation $x + 2 = xy - y - 1$ with $\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$.

4.) Precompose the equation $x^2 + x - 3y = 2y^2 - 5y + 2$ with $\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}$.

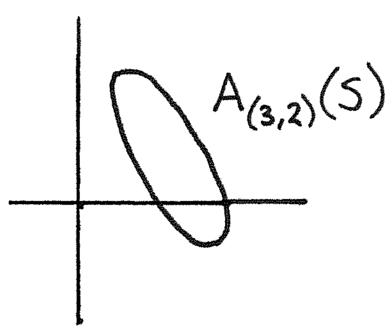
For #5-8, suppose that S is the set of solutions of the equation

$$3x^2 + xy + y^2 = x - y + 5$$

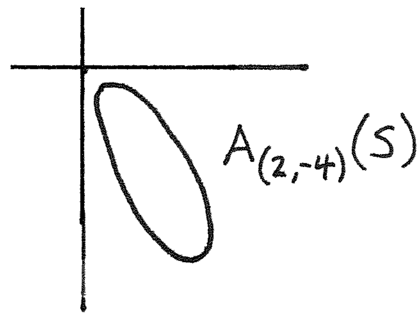


For #5-8, use that the equation for S precomposed with T^{-1} is an equation for $T(S)$.

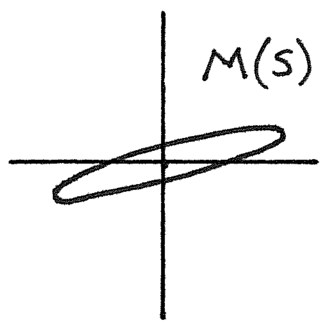
5.) What is $A_{(3,2)}^{-1}$? What is the equation for $A_{(3,2)}(S)$?



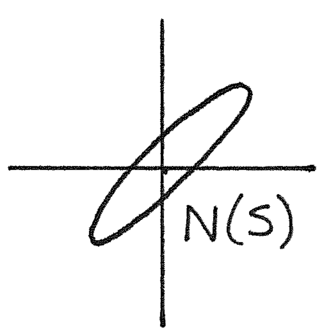
6.) What is $A_{(2,-4)}^{-1}$? What is the equation for $A_{(2,-4)}(S)$?



7.) Suppose $M = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$. What is M^{-1}? What is the equation for $M(S)$?



8.) Suppose $N = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$. What is N^{-1}? What is the equation for $N(S)$?



$$f(x) = \begin{cases} x-1 & \text{if } x \in (-\infty, 0); \\ x^2 & \text{if } x \in [0,4]; \quad \text{and} \\ 57 & \text{if } x \in (4, \infty). \end{cases}$$

Find the following values.

9.) $f(-2)$ 12.) $f(1)$ 15.) $f(4)$

10.) $f(-1)$ 13.) $f(2)$ 16.) $f(5)$

11.) $f(0)$ 14.) $f(3)$ 17.) $f(6)$

Multiply the following matrices.

18.) $\begin{pmatrix} 3 & 4 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 1 & 1 \end{pmatrix}$ 19.) $\begin{pmatrix} 2 & -2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}$

Solve the following equations.

20.) $\log_e(x)^2 - 5\log_e(x) + 6 = 0$ 21.) $(x^3 - 3x^2 + 2x - 3)^2 = -1$