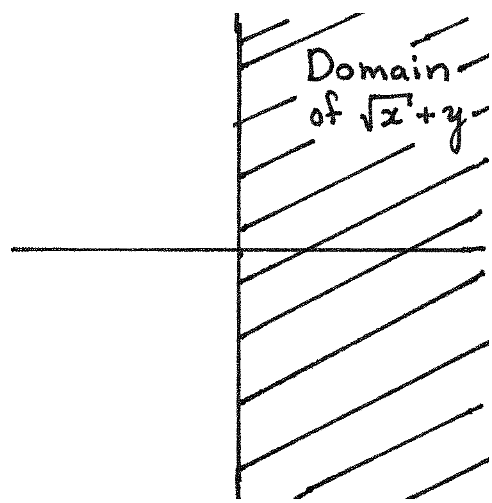
Polynomials in Two Variables

A *function in two variables* is a function $f : D \to \mathbb{R}$ where D is a subset of the plane, $\mathbb{R}^2$.

Examples.

• The function $g : \mathbb{R}^2 \to \mathbb{R}$ where $g(x, y) = xy + 3$ is a function in two variables. Notice that $g(1, 2) = 1 \cdot 2 + 3 = 5$, that $g(0, -7) = 0 \cdot (-7) + 3 = 3$, and that $g(4, 5) = 4 \cdot 5 + 3 = 23$.

• Let D be the set of all vectors in the plane whose x-coordinates are greater than or equal to 0. That is, $D = \{\, (x, y) \in \mathbb{R}^2 \mid x \geq 0 \,\}$. Then $h : D \to \mathbb{R}$ where $h(x, y) = \sqrt{x} + y$ is a function in two variables. Notice that $h(4, 3) = \sqrt{4} + 3 = 2 + 3 = 5$, and that $h(0, -10) = -10$.



• The function $f : \mathbb{R}^2 \to \mathbb{R}$ where $f(x, y) = -x^2 + 2xy + y^2 - 2x + 3y + 10$ is a function in two variables. Check that $f(0, 1) = 0 + 0 + 1 - 0 + 3 + 10 = 14$ and that $f(2, 3) = -4 + 12 + 9 - 4 + 9 + 10 = 32$.

• $g : \mathbb{R}^2 \to \mathbb{R}^2$ where $g(x, y) = 5$ is a constant function. The output of g is always 5, regardless of the input.

Functions in two variables have domains that are subsets of the plane, $\mathbb{R}^2$.

Polynomials

A **degree 0 polynomial in two variables** is a function of the form

$$p(x,y) = a_{0,0}$$

for some constant number $a_{0,0}$.

For example, $p(x,y) = 4$ is a degree 0 polynomial, and so is $q(x,y) = -3$. These are just constant functions, and because of that, degree 0 polynomials are often called *constant polynomials.*

A **degree 1 polynomial in two variables** is a function of the form

$$p(x,y) = a_{1,0}x + a_{0,1}y + a_{0,0}$$

where $a_{1,0}, a_{0,1}, a_{0,0} \in \mathbb{R}$, as long as $a_{1,0}$ and $a_{0,1}$ don't both equal 0.

For example, $p(x,y) = 2x+4y+5$ is a degree 1 polynomial in two variables. So are $q(x,y) = -2x+3$, $f(x,y) = y$, and $g(x,y) = x-y$. Degree 1 polynomials are often called *linear polynomials.*

A **degree 2 polynomial in two variables** is a function of the form

$$p(x,y) = a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0}$$

where $a_{2,0}, a_{1,1}, a_{0,2}, a_{1,0}, a_{0,1}, a_{0,0} \in \mathbb{R}$, as long as $a_{2,0}$, $a_{1,1}$, and $a_{0,2}$ don't all equal 0.

For example, $p(x,y) = 2x^2+4xy+7y^2+3x+2y-8$ is a degree 2 polynomial in two variables. So are $q(x,y) = x^2 - xy$, $f(x,y) = x^2+y^2-1$, and $g(x,y) = xy + x - 3$.

Degree 2 polynomials are often called *quadratic polynomials.*

Domains of polynomials

Polynomials are functions that involve addition and multiplication. You can multiply any collection of numbers, and you can add any collection of numbers. There are no restrictions. The domain of any polynomial in two variables is the entire plane, $\mathbb{R}^2$.

Coefficients and terms

If we have a polynomial such as

$$p(x,y) = a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0}$$

then we call $a_{2,0}x^2$ the x^2-*term* of $p(x,y)$. Similarly, we call $a_{1,1}xy$ the xy-*term*, and $a_{0,1}y$ the y-*term*, etc. We call $a_{0,0}$ the *constant term.*

The number $a_{2,0}$ is called the *coefficient* of the x^2-term of $p(x, y)$. Similarly, we call $a_{1,1}$ the coefficient of the *xy-term*, and $a_{0,1}$ the coefficient of the *y-term*, etc.

Example.

The x-term of the polynomial

$$4x^2 - 3xy + 2y^2 - 5x + 7$$

is $-5x$. The y^2-term is $2y^2$. The constant term is 7. The coefficient of the x^2-term is 4. The coefficient of the xy-term is -3. This polynomial has no y-term, or in other words, the y-term is $0 \cdot y$. The coefficient of the y-term is 0.

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Rows vs. Columns

We had previously written the vector inputs and outputs for an addition function as row vectors, so that, for example, $A_{(2,4)}(3, 8) = (5, 12)$.

We had previously written the vector inputs and outputs of a matrix as column vectors, so that, for example, if

$$M = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

then $M\binom{0}{2} = \binom{6}{2}$. But there really is no difference between a row vector and a column vector. Writing one versus the other is just a matter of preference. And because it's easier to fit row vectors on a line of paper, from now on, we'll also sometimes write row vectors as the inputs and outputs of matrices. Thus, instead of $M\binom{0}{2} = \binom{6}{2}$ we might write $M(0, 2) = (6, 2)$. There's really no difference, but rows fit more easily on lines of paper.

Composing polynomials and planar transformations

Recall that we will use the words *planar transformation* to refer to either an addition function $A_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}^2$ or to an invertible matrix function $M : \mathbb{R}^2 \to \mathbb{R}^2$.

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a planar transformation,
and if $p : \mathbb{R}^2 \to \mathbb{R}$ is a degree n polynomial in two variables,
then $p \circ T : \mathbb{R}^2 \to \mathbb{R}$ is a degree n polynomial in two variables.

Verifying that the above fact is true isn't too difficult, at least for degree 1 and degree 2 polynomials which is most of what we'll apply this fact to. That is, it's not so difficult if you've learned about matrix equations in 3-variables, as is often covered in Math 1050 courses. However, it would take us a little far from where we want to be to review matrix equations, so instead we'll just look at two illustrations of this fact for now. We'll see more examples in the exercises.

Examples.

- Let $A_{(2,1)}(x, y)$ be the addition function of the vector $(2, 1)$. That means that $A_{(2,1)}(x, y) = (x+2, y+1)$. Let $p(x, y)$ be the quadratic polynomial $p(x, y) = x^2 + y^2 + 6$. Then

$$\begin{aligned} p \circ A_{(2,1)}(x, y) &= p(x + 2, y + 1) \\ &= (x + 2)^2 + (y + 1)^2 + 6 \end{aligned}$$

That's a perfectly good answer, and we can usually stop at this step. However, if you're asked to write your answer in the form

$$a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0}$$

as you will be in #33 and #35 in the homework exercises, then you'll have to do a bit more work. You'll have to multiply the expression out and then group like terms. That is,

$$\begin{aligned} p \circ A_{(2,1)}(x, y) &= (x + 2)^2 + (y + 1)^2 + 6 \\ &= (x^2 + 4x + 4) + (y^2 + 2y + 1) + 6 \\ &= x^2 + y^2 + 4x + 2y + 11 \end{aligned}$$

Notice that $x^2 + y^2 + 4x + 2y + 11$ is a quadratic polynomial. We composed a quadratic polynomial, p, with a planar transformation, $A_{(2,1)}$, and we obtained a quadratic polynomial, $p \circ A_{(2,1)}$.

- Let's look at the invertible matrix

$$M = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

Notice that

$$M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ x + 4y \end{pmatrix}$$

Written with row vectors, the line above is $M(x, y) = (2x + 3y, x + 4y)$.

Let $q(x, y) = y^2 - x + 2$. Then

$$\begin{aligned} q \circ M(x, y) &= p(2x + 3y, x + 4y) \\ &= (x + 4y)^2 - (2x + 3y) + 2 \end{aligned}$$

This is usually a good place to stop. Sometimes, including in #34 and #36 of the homework exercises for this chapter, you'll be asked to write your answer in the form

$$a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0}$$

To do that,

$$\begin{aligned} q \circ M(x, y) &= (x + 4y)^2 - (2x + 3y) + 2 \\ &= (x^2 + 8xy + 16y^2) - (2x + 3y) + 2 \\ &= x^2 + 8xy + 16y^2 - 2x - 3y + 2 \end{aligned}$$

We composed a quadratic polynomial, q, with a planar transformation, M, and we obtained a quadratic polynomial, $x^2 + 8xy + 16y^2 - 2x - 3y + 2$.

Exercises

Let $p(x,y) = 2x^2 - xy + x - 1$. Identify the following values.

1.) $p(1,2)$ 2.) $p(0,-7)$ 3.) $p(3,4)$

Let $q(x,y) = x^2 + xy - y^2 + 2x - 3y - 2$. Identify the following values.

4.) $q(2,0)$ 5.) $q(-1,1)$ 6.) $q(2,1)$

Determine whether the polynomials in #7-18 are constant, linear, or quadratic.

7.) 4

8.) $2x^2 - 3xy + 4y^2 + 4x - 3y + 5$

9.) $3x + 4y + 1$

10.) $x^2 + x - y + 2$

11.) $3xy - 2$

12.) $2x + y$

13.) xy

14.) $-x$

15.) $-\frac{17}{5}$

16.) $y^2 + y + 1$

17.) $5x^2$

18.) $y + 3$

For #19-29, give the terms and coefficients asked for of the quadratic polynomial $p(x, y) = 3x^2 - 2xy - y + 4$.

19.) x^2-term

20.) xy-term

21.) y^2-term

22.) x-term

23.) y-term

24.) constant term

25.) coefficient of the x^2-term

26.) coefficient of the xy-term

27.) coefficient of the y^2-term

28.) coefficient of the x-term

29.) coefficient of the y-term

Identify the given row vectors in #30-32 using the matrix

$$M = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

30.) $M(1, 2)$

31.) $M(-3, 4)$

32.) $M(2, -3)$

For the remaining questions, suppose that $p(x, y) = x^2 + y^2$, that $q(x, y) = 3x^2 + xy - y^2 + x - 2$, and that

$$M = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$$

Perform the following compositions of quadratic polynomials in two variables with planar transformations. What is the resulting quadratic polynomial in two variables? Write your answers in the form

$$a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0}$$

33.) $p \circ A_{(1,3)}(x, y)$

34.) $p \circ M(x, y)$

35.) $q \circ A_{(2,-1)}(x, y)$

36.) $q \circ N(x, y)$