# Partial Inverse Trigonometric Functions

Although the three basic trigonometric functions—sin, cos, and tan—do not have inverse functions, they do have what can be called "partial inverses". These partial inverses are called arcsine, arccosine, and arctangent.

We'll begin this chapter with a review of the definition of the most famous example of a partial inverse function, the square-root function. Then we'll see that arcsine, arccosine, and arctangent can be defined in a similar manner.

#### Square-root

The graph of  $x^2$  is shown below. It does not pass the horizontal line test. That is, there are horizontal lines that intersect the graph of  $x^2$  in more than one point. This tells us that the function  $x^2$  is not one-to-one, meaning that there are two different numbers that we can use as inputs for the function  $x^2$ that will result in the same output. For example,  $(-3)^2 = 9 = (3)^2$ .



A function has an inverse only if it is both one-to-one and onto. (Remember that a function is onto if the target of the function is the range of the function.) Since  $x^2 : \mathbb{R} \to \mathbb{R}$  isn't one-to-one, it has no inverse. However, we can restrict the domain of  $x^2$  to the set of nonnegative numbers, and the resulting function is one-to-one. We can see this below as the graph of  $x^2 : [0, \infty) \to \mathbb{R}$ passes the horizontal line test. There aren't any horizontal lines in the plane that intersect the graph in more than one point.



The range of the function  $x^2 : [0, \infty) \to \mathbb{R}$  is the set of numbers that are outputs of the function. It's the set of numbers that appear as the *y*coordinates of the points in the graph. It's the set  $[0, \infty)$ .

We'll replace the target of  $x^2 : [0, \infty) \to \mathbb{R}$  with its range,  $[0, \infty)$ . The result is a function that is onto as well as one-to-one

$$x^2: [0,\infty) \to [0,\infty)$$



Any function that is one-to-one and onto has an inverse. Therefore, the function  $x^2 : [0, \infty) \to [0, \infty)$  has an inverse. We named this inverse function  $\sqrt{x} : [0, \infty) \to [0, \infty)$ . To see the graph of the inverse function, we just have to flip the graph of our original function over the y = x line.



Because the function  $x^2 : [0, \infty) \to [0, \infty)$  is only part of the function  $x^2$  with its implied domain of  $\mathbb{R}$ , and because  $\sqrt{x}$  is the inverse of this smaller part, we call  $\sqrt{x}$  a partial inverse of  $x^2$ .

What it means for  $x^2 : [0, \infty) \to [0, \infty)$  and  $\sqrt{x} : [0, \infty) \to [0, \infty)$  to be inverses is that they reverse each other's assignments. For example, because  $5^2 = 25$ , we know that  $\sqrt{25} = 5$ . Because,  $\sqrt{9} = 3$  we know that  $3^2 = 9$ . The chart below shows more examples of the dual relationship between these two inverse functions.

$x^2$	$\sqrt{x}$
$0^2 = 0$	$\sqrt{0} = 0$
$1^2 = 1$	$\sqrt{1} = 1$
$2^2 = 4$	$\sqrt{4} = 2$
$3^2 = 9$	$\sqrt{9} = 3$
$4^2 = 16$	$\sqrt{16} = 4$
$5^2 = 25$	$\sqrt{25} = 5$

## Arcsine

Sine is not one-to-one. Its graph fails the horizontal line test.



There is however a segment of the graph of sin that satisfies the horizontal line test.



Notice that the segment of the graph highlighted above stretches between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$  on the *x*-axis, so it is the graph of sin with its domain restricted to  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ . It's the graph of sin :  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ . Because this graph passes the horizontal line test, sin :  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$  is one-to-one.



The range of  $\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$  is the set of numbers that appear as the y-coordinates of points in its graph. This set is [-1, 1]. Replacing the target of our function with its range gives us a function that is onto, in addition to being one-to-one. That is,

$$\sin:\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \to \left[-1,1\right]$$

is one-to-one and onto, and thus has an inverse. We could write this inverse as  $[\pi, \pi]$ 

$$\sin^{-1}: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

but because this partial inverse of sin is an important function in math, it has been given a special name. It's called the *arcsine* function, and it's written as  $\begin{bmatrix}
x & x \\ y & z
\end{bmatrix}$ 

$$\operatorname{arcsin}: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

To find the graph of arcsin, we flip the graph of  $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$  over the y = x line.





The chart on the next page demonstrates some values of the arcsine function. Because arcsine is the inverse of  $\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \left[-1, 1\right]$ , it reverses the assignments of  $\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \left[-1, 1\right]$ .

$\sin(x)$	$\arcsin(x)$
$\sin(-\frac{\pi}{2}) = -1$	$\operatorname{arcsin}(-1) = -\frac{\pi}{2}$
$\sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$	$\operatorname{arcsin}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
$\sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$	$\operatorname{arcsin}(-\frac{1}{\sqrt{2}}) = -\frac{\pi}{4}$
$\sin(-\frac{\pi}{6}) = -\frac{1}{2}$	$\operatorname{arcsin}(-\frac{1}{2}) = -\frac{\pi}{6}$
$\sin(0) = 0$	$\arcsin(0) = 0$
$\sin(\frac{\pi}{6}) = \frac{1}{2}$	$\operatorname{arcsin}(\frac{1}{2}) = \frac{\pi}{6}$
$\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$	$\operatorname{arcsin}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$
$\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$	$\operatorname{arcsin}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$
$\sin(\frac{\pi}{2}) = 1$	$\operatorname{arcsin}(1) = \frac{\pi}{2}$

#### Arccosine

Cosine is also not one-to-one. Its graph fails the horizontal line test.  $^{\rm 299}$ 



The below segment of the cosine graph does satisfy the horizontal line test.



The segment stretches between 0 and  $\pi$  on the *x*-axis. It's the graph of cosine restricted to the domain  $[0, \pi]$ , the graph of  $\cos : [0, \pi] \to \mathbb{R}$ . Because its graph passes the horizontal line test,  $\cos : [0, \pi] \to \mathbb{R}$  is one-to-one.



To make  $\cos : [0, \pi] \to \mathbb{R}$  into an onto function, we can replace its target with its range. To do this, we just need to identify the range. It's the set of numbers that appear as *y*-coordinates of points in the graph. It's the set [-1, 1]. Thus,  $\cos : [0, \pi] \to [-1, 1]$  is onto as well as one-to-one. Therefore, it has an inverse function that's called the *arccosine* function:

$$\operatorname{arccos}: [-1,1] \to [0,\pi]$$

It's graph is obtained by flipping the graph of  $\cos : [0, \pi] \to [-1, 1]$  over the y = x line.



The arccosine function is a partial inverse of the cosine function. The chart on the next page shows some values of arccosine.

$\cos(x)$	$\arccos(x)$
$\cos(0) = 1$	$\arccos(1) = 0$
$\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$	$\operatorname{arccos}(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$
$\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$	$\operatorname{arccos}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$
$\cos(\frac{\pi}{3}) = \frac{1}{2}$	$\operatorname{arccos}(\frac{1}{2}) = \frac{\pi}{3}$
$\cos(\frac{\pi}{2}) = 0$	$\arccos(0) = \frac{\pi}{2}$
$\cos(\frac{2\pi}{3}) = -\frac{1}{2}$	$\operatorname{arccos}(-\frac{1}{2}) = \frac{2\pi}{3}$
$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$	$\operatorname{arccos}(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
$\cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$	$\operatorname{arccos}(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
$\cos(\pi) = -1$	$\arccos(-1) = \pi$

### Arctangent

Tangent is not one-to-one, but the segment of its graph between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$  on the x-axis does pass the horizontal line test.



It's also onto, because any real number is the *y*-coordinate of some point of its graph. Therefore,  $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$  has an inverse function, the *arctangent* function

 $\arctan: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ 

Its graph is the graph of  $\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$  flipped over the y = x line.



The chart below shows the dual relationship between tangent and its partial inverse, arctangent.

$\tan(x)$	$\arctan(x)$
$\tan(-\frac{\pi}{3}) = -\sqrt{3}$	$\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
$\tan(-\frac{\pi}{4}) = -1$	$\arctan(-1) = -\frac{\pi}{4}$
$\tan(-\frac{\pi}{6}) = -\frac{1}{\sqrt{3}}$	$\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
$\tan(0) = 0$	$\arctan(0) = 0$
$\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$	$\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$
$\tan(\frac{\pi}{4}) = 1$	$\arctan(1) = \frac{\pi}{4}$
$\tan(\frac{\pi}{3}) = \sqrt{3}$	$\arctan(\sqrt{3}) = \frac{\pi}{3}$

# Exercises

For #1-3, write the values as numbers that do not involve the letters  $\cos$ ,  $\sin$ , or  $\tan$ .

- 1.)  $\arcsin(\frac{1}{\sqrt{2}})$  2.)  $\arccos(\frac{\sqrt{3}}{2})$  3.)  $\arctan(\sqrt{3})$
- 4.)  $\sin(\frac{\pi}{10}) = \frac{\sqrt{5}-1}{4}$ . What does  $\arcsin\left(\frac{\sqrt{5}-1}{4}\right)$  equal? 5.)  $\cos(\frac{\pi}{12}) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ . What does  $\arccos\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)$  equal?
- 6.)  $\tan(\frac{\pi}{10}) = \sqrt{1 \frac{2}{\sqrt{5}}}$ . What does  $\arctan\left(\sqrt{1 \frac{2}{\sqrt{5}}}\right)$  equal?

For #7-9, match the functions with their graphs.

7.)  $\arcsin(x)$  8.)  $\arccos(x)$  9.)  $\arctan(x)$ 



11.) Find the vector 
$$\begin{pmatrix} 1 & 0 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$
.

<sup>10.)</sup> Write the vector (2, -4) in polar coordinates.

Match the functions with their graphs.

12.)  $\arctan(x)$ 13.)  $\arctan(x + \frac{\pi}{2})$ 14.)  $2\arctan(x)$ 15.)  $\arctan(x - \frac{\pi}{2})$ 16.)  $\arctan(x) + \frac{\pi}{2}$ 17.)  $\frac{1}{2}\arctan(x)$ 18.)  $\arctan(x) - \frac{\pi}{2}$ 19.)  $-\arctan(x)$ 20.)  $\arctan(-x)$ 













G.)



Find the set of solutions of the following equations in one variable.

- 21.)  $x^2 = 25$ 22.)  $(x - 2)^2 = 9$ 23.)  $\log_3(x + 1) = 4$
- 24.)  $e^{x^2} = e^{x+2}$