

Polynomial Equations

A *polynomial equation in two variables* is an equation of the form

$$p(x, y) = q(x, y)$$

where both $p(x, y)$ and $q(x, y)$ are polynomials in two variables.

Examples.

- $xy + 2 = y^2 - 3x - 4$

($xy + 2$ is a quadratic polynomial. So is $y^2 - 3x - 4$.)

- $y - x = 2$

($y - x$ is a linear polynomial. 2 is a constant polynomial.)

- $x^2 - 5x + y - 2 = -7xy - y + 2$

- $3x^2 - xy + 4y^2 - 5x + 6 - 7 = 0$

Domains of polynomial equations

Because every polynomial in two variables has a domain of $\mathbb{R}^2$, the implied domain of any polynomial equation in two variables is $\mathbb{R}^2$, the entire plane.

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Solutions of equations

If $p(x, y) = q(x, y)$ is a polynomial equation in two variables, then a point in the plane $(\alpha, \beta) \in \mathbb{R}^2$ is a *solution of the equation* if the number $p(\alpha, \beta)$ equals the number $q(\alpha, \beta)$. That is, if $p(\alpha, \beta) = q(\alpha, \beta)$.

Examples.

- The point in the plane $(1, 2)$ is a solution of the equation $x^2 + y^2 = x + y + 2$ because $1^2 + 2^2 = 5 = 1 + 2 + 2$.

- The point $(0, -1)$ is a solution of the same equation $x^2 + y^2 = x + y + 2$ because $0^2 + (-1)^2 = 1 = 0 + (-1) + 2$.

- The point $(3, 1)$ is not a solution of the same equation $x^2 + y^2 = x + y + 2$ because $3^2 + 1^2 = 10 \neq 6 = 3 + 1 + 2$.

The set of every solution of an equation $p(x, y) = q(x, y)$ is the set

$$S = \{ (\alpha, \beta) \in \mathbb{R}^2 \mid p(\alpha, \beta) = q(\alpha, \beta) \}$$

Thus, if S is the set of solutions of the equation from the previous examples, $x^2 + y^2 = x + y + 2$, then $(1, 2) \in S$ and $(0, -1) \in S$, while $(3, 1) \notin S$.

Solutions as geometric objects

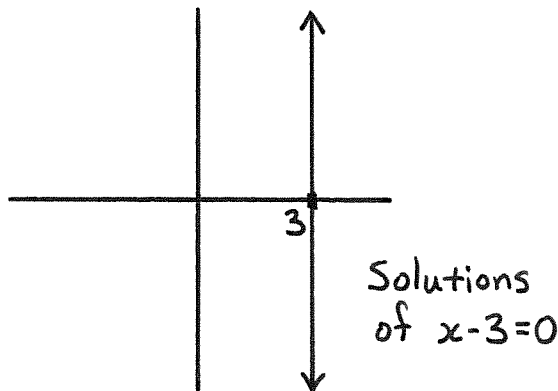
The set of solutions of a polynomial equation in one variable is always finite, so we can just write out a list of the solutions. In contrast, a polynomial equation in two variables can have infinitely many solutions. It would be impossible to write a list of infinitely many points in the plane, but it often is possible to make a drawing of every point in the plane that is a solution of a particular equation.

Examples.

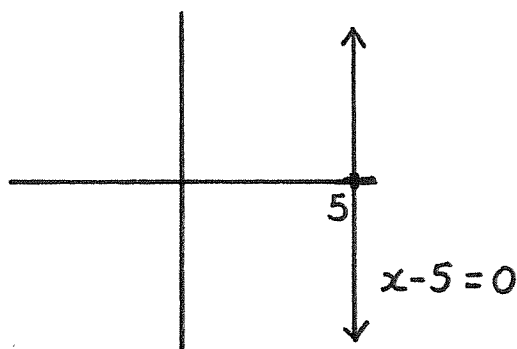
- Let $p(x, y)$ be the linear polynomial x and let $q(x, y)$ be the constant polynomial 3. The solutions of the equation $p(x, y) = q(x, y)$, the equation $x = 3$, are the points (α, β) in the plane with the property that

$$\alpha = p(\alpha, \beta) = q(\alpha, \beta) = 3$$

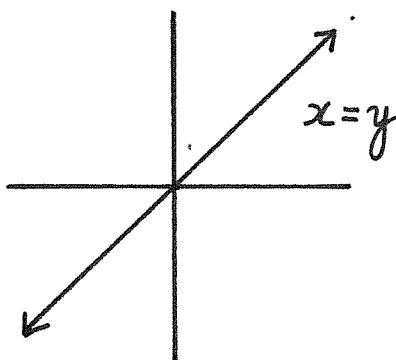
Thus, $(3, 7)$, and $(3, -2)$, and $(3, 100)$ are solutions of the equation $x = 3$. The set of all solutions, of all pairs of numbers whose x -coordinates equal 3, forms a vertical line in the plane.



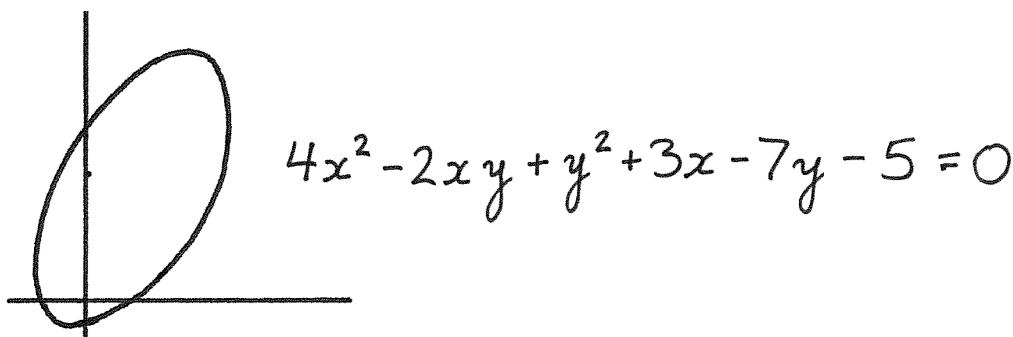
- Similar to the previous example, the set of solutions of the equation $x = 5$ also forms a vertical line in the plane. It's a vertical line comprised of points whose x -coordinates equal 5.



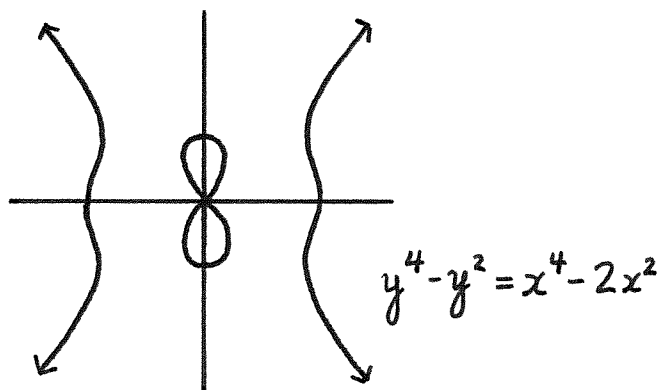
- $x = y$ is an equation in two variables. The solutions of this equation are all of the points (α, β) such that $\alpha = \beta$. In other words, the solutions of this equation are all of the points of the form (α, α) — all of the points whose x -coordinates equal their y -coordinates. There are infinitely many of these solutions, including $(4, 4)$, $(7, 7)$ and $(-10, -10)$. If we placed a tiny dot in the plane for each of these solutions, they'd collectively form a line, a line that's often called the “ $x = y$ line”.



- The set of solutions of the equation $4x^2 - 2xy + y^2 + 3x - 7y - 5 = 0$ form a geometric object called an *ellipse*. We'll have more to say about ellipses later. They're examples of conics.



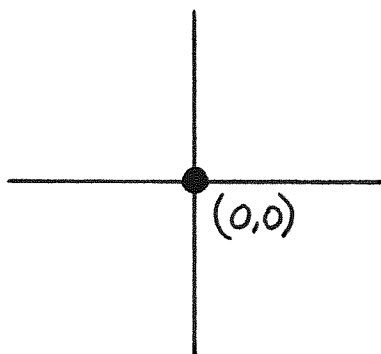
- The set of solutions of the equation $y^4 - y^2 = x^4 - 2x^2$ is called the “devil’s curve”.



• The three previous examples were of equations that had infinitely many solutions. Sometimes a polynomial equation will have no solutions. If you square a number, the result cannot be negative. If you add two nonnegative numbers together, the result is still nonnegative. This is to say that the equation $x^2 + y^2 = -1$ has no solution. There are no pairs of numbers that you can square and then add together to get the negative number -1 .

• Some equations in two variables have a finite number of solutions. The claim below displays one such equation.

Claim: The only solution of the equation $x^2 + y^2 = 0$ is the point $(0, 0)$.



Proof: Suppose that (α, β) is a solution of the equation $x^2 + y^2 = 0$. We’ll show that $(\alpha, \beta) = (0, 0)$. That means that $(0, 0)$ is the only solution.

If (α, β) is a solution of the equation $x^2 + y^2 = 0$, then $\alpha^2 + \beta^2 = 0$. The square of a number can’t be negative. Thus $\alpha^2 \geq 0$ and $\beta^2 \geq 0$. Since neither α^2 nor β^2 are negative, the only way they could sum to 0 is if they both equal zero. That is, $\alpha^2 + \beta^2 = 0$ implies that $\alpha^2 = 0$ and $\beta^2 = 0$. The only number

that you can square to get 0, is 0 itself. In other words, since $\alpha^2 = 0$, we must have that $\alpha = 0$. Since $\beta^2 = 0$, we must have that $\beta = 0$.

We've now shown what we wanted to. If (α, β) is a solution of the equation $x^2 + y^2 = 0$, then $\alpha^2 + \beta^2 = 0$, which implies that $(\alpha, \beta) = (0, 0)$. Thus, $(0, 0)$ is the only solution of the equation $x^2 + y^2 = 0$. ■

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Equivalent equations

Similar to equations in one variable, there are rules for when equations in two variables are equivalent, and equivalent equations have the same solutions. When it comes to polynomial equations in two variables, there are only two rules for equivalent equations that we'll need.

Equivalent by addition:

The equation $p(x, y) + h(x, y) = q(x, y)$
is equivalent to $p(x, y) = q(x, y) - h(x, y)$.

Equivalent by multiplication:

If $c \neq 0$ is a number, then the equation $cp(x, y) = q(x, y)$
is equivalent to $p(x, y) = \frac{q(x, y)}{c}$.

Consequence of equivalent equations:

Equivalent equations have the same set of solutions.

Example.

- The equation

$$8x^2 + 2y^2 + 6x = 4xy + 14y + 10$$

is equivalent by addition to the equation

$$8x^2 + 2y^2 + 6x - (4xy + 14y + 10) = 0$$

We can rearrange the terms above to write this equation as

$$8x^2 - 4xy + 2y^2 + 6x - 14y - 10 = 0$$

We can factor out the constant 2 and rewrite this equation as

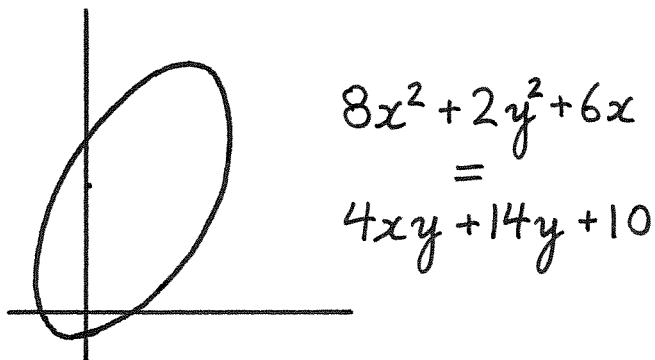
$$2(4x^2 - 2xy + y^2 + 3x - 7y - 5) = 0$$

This is equivalent by multiplication to the equation

$$4x^2 - 2xy + y^2 + 3x - 7y - 5 = \frac{0}{2} = 0$$

We saw the solutions of this equation earlier in the chapter. The solutions form an ellipse. Because equivalent equations have the same set of solutions, the ellipse is also the set of solutions of our original equation from this example

$$8x^2 + 2y^2 + 6x = 4xy + 14y + 10$$



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Planar transformations of solutions

Solutions of polynomial equations are geometric objects, and at times, we'll be interested in how planar transformations affect these geometric objects. The general principal is described in the following claim.

Claim: Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a planar transformation — either an addition function or an invertible matrix — and that $S \subseteq \mathbb{R}^2$ is the set of solutions of the polynomial equation $p(x, y) = q(x, y)$. Then $T(S)$ is the set of solutions of the equation $p \circ T^{-1}(x, y) = q \circ T^{-1}(x, y)$.

Proof: Let's take a point in the set $T(S)$. That is, a point of the form $T(\alpha, \beta)$ where $(\alpha, \beta) \in S$, which means that (α, β) is a solution of the equation $p(x, y) = q(x, y)$, or in other words, that

$$p(\alpha, \beta) = q(\alpha, \beta)$$

We want to show that this point, $T(\alpha, \beta)$, in the set $T(S)$ is a solution of the equation

$$p \circ T^{-1}(x, y) = q \circ T^{-1}(x, y)$$

To see this, we need to check that $p \circ T^{-1}(T(\alpha, \beta))$ equals $q \circ T^{-1}(T(\alpha, \beta))$. We'll do this in a moment, but before we do, remember two things: that the definition of inverse functions is that $T^{-1}(T(x, y)) = (x, y)$, and that $p(\alpha, \beta) = q(\alpha, \beta)$. Now we check that $p \circ T^{-1}(T(\alpha, \beta))$ equals $q \circ T^{-1}(T(\alpha, \beta))$:

$$\begin{aligned} p \circ T^{-1}(T(\alpha, \beta)) &= p(T^{-1}(T(\alpha, \beta))) \\ &= p(\alpha, \beta) \\ &= q(\alpha, \beta) \\ &= q(T^{-1}(T(\alpha, \beta))) \\ &= q \circ T^{-1}(T(\alpha, \beta)) \end{aligned}$$

Thus, points in the set $T(S)$, such as $T(\alpha, \beta)$, are solutions of the equation $p \circ T^{-1}(x, y) = q \circ T^{-1}(x, y)$. ■

The previous claim is important to stress. We'll call it the Principle of Transforming Solutions (**POTS**). If S is the set of solutions of $p(x, y) = q(x, y)$, then $T(S)$ is the set of solutions of $p \circ T^{-1}(x, y) = q \circ T^{-1}(x, y)$.

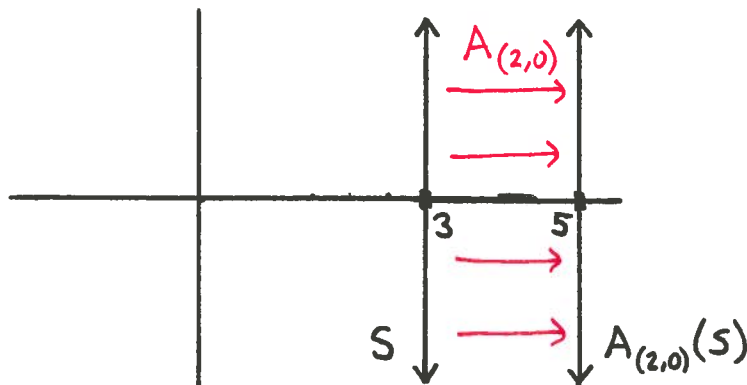
solutions: $S \xrightarrow{T} T(S)$

equations: $p(x, y) = q(x, y) \xrightarrow{T^{-1}} p \circ T^{-1}(x, y) = q \circ T^{-1}(x, y)$

Example.

• Let $p(x, y) = x$ and $q(x, y) = 3$. If S is the set of solutions of the equation $p(x, y) = q(x, y)$, which is the equation $x = 3$, then we've seen that S is a vertical line.

The addition function $A_{(2,0)} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ moves points in the plane right by 2 units, so the set $A_{(2,0)}(S)$ is the line of solutions of $x = 3$ shifted right by 2 units. It's the vertical line of points whose x -coordinates equal 5.



POTS tells us that the line $A_{(2,0)}(S)$ is the set of solutions of the equation

$$p \circ A_{(2,0)}^{-1}(x, y) = q \circ A_{(2,0)}^{-1}(x, y)$$

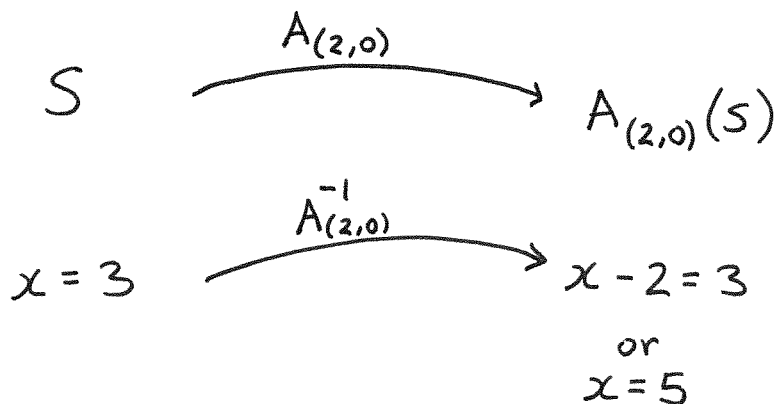
Because $A_{(2,0)}^{-1}(x, y) = (x - 2, y)$, you can check that the equation

$$p \circ A_{(2,0)}^{-1}(x, y) = q \circ A_{(2,0)}^{-1}(x, y)$$

is the same as the equation

$$x - 2 = 3$$

In summary, POTS tells us that the line $A_{(2,0)}(S)$ is the set of solutions of $x - 2 = 3$, or equivalently, of $x = 5$.



POTS

This is very important. It's worth repeating.

If S is the set of solutions of $p(x, y) = q(x, y)$,
then $T(S)$ is the set of solutions of $p \circ T^{-1}(x, y) = q \circ T^{-1}(x, y)$.

$$S \xrightarrow{T} T(S)$$

$$p(x, y) = q(x, y) \xrightarrow{T^{-1}} p \circ T^{-1}(x, y) = q \circ T^{-1}(x, y)$$

Exercises

Let's look at the equation $5x + y - 2 = 3x^2 - xy$. Determine whether the points given in #1-4 are solutions of this equation.

- 1.) $(2, 3)$ 2.) $(1, 0)$ 3.) $(-1, -2)$ 4.) $(3, 4)$

5.) Are the equations $x^2 + 2 = x$ and $x^2 - x + 2 = 0$ equivalent by addition?

6.) Are the equations $xy = y^2$ and $xy - x = y^2 - y$ equivalent by addition?

7.) Are $xy + 3 = y^2 - x^2$ and $3 = y^2 - x^2 - xy$ equivalent by addition?

8.) Are the equations $2x - 3 = x^2$ and $2x = x^2$ equivalent by addition?

9.) Are the equations $2xy - 3x^2 = y^2 - x - y$ and $xy - 3x^2 = y^2 - x - y$ equivalent by multiplication?

10.) Are $5x^2 - 5x + 10 = 20y^2$ and $x^2 - x + 2 = 4y^2$ equivalent by multiplication?

We saw in this chapter that $x^2 + y^2 = 0$ has a single solution, the point $(0, 0)$. In #11-14, let $p(x, y) = x^2 + y^2$ and $q(x, y) = 0$, so that $p(x, y) = q(x, y)$ is an equation whose only solution is $(0, 0)$.

11.) What is $A_{(2,3)}(0, 0)$?

12.) What are $p \circ A_{(2,3)}^{-1}(x, y)$ and $q \circ A_{(2,3)}^{-1}(x, y)$?

13.) What's the only solution of the polynomial equation $p \circ A_{(2,3)}^{-1}(x, y) = q \circ A_{(2,3)}^{-1}(x, y)$? (You can use POTS and #11 to answer this.)

14.) Use POTS and the polynomials $p(x, y) = x^2 + y^2$ and $q(x, y) = 0$ (as in #13) to write a polynomial equation in two variables that has only one solution: the point (a, b) in the plane.

$\log_e(x)$ is the most common logarithm used in math. There are lots of benefits to using logarithms base e , and these benefits will be explained in calculus. Because of these benefits, some call logarithm base e the “natural logarithm”, and they write it as $\ln(x)$. (Scientists often prefer to write $\ln(x)$. Mathematicians often write $\log_e(x)$ as $\log(x)$. To make matters more confusing, if a calculator has a button for $\log(x)$, it probably means $\log_{10}(x)$.) Because plenty of people write $\log_e(x)$ as $\ln(x)$, we should practice seeing and writing the logarithm base e in this way. Find the values asked for in #15-30.

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|-----------------------------------|-----------------------------------|---------------------------|--------------------------------|
| 15.) $\ln^{-1}(2)$ | 16.) $\ln(e)$ | 17.) $\ln(e^2)$ | 18.) $\ln(e^3)$ |
| 19.) $\ln(e^4)$ | 20.) $\ln(\frac{1}{e})$ | 21.) $\ln(\frac{1}{e^2})$ | 22.) $\ln(\frac{1}{e^3})$ |
| 23.) $\ln(\sqrt{e})$ | 24.) $\ln(\sqrt[3]{e})$ | 25.) $\ln(\sqrt[4]{e})$ | 26.) $\ln(\frac{1}{\sqrt{e}})$ |
| 27.) $\ln(\frac{1}{\sqrt[3]{e}})$ | 28.) $\ln(\frac{1}{\sqrt[4]{e}})$ | 29.) $\ln(1)$ | 30.) $\ln^{-1}(e)$ |

Find the solutions of the following equations in one variable.

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|------------------------|------------------------|-----------------------------|
| 31.) $\ln(2x - 3) = 0$ | 32.) $\ln(3x - 5) = 2$ | 33.) $3\ln(7 - x) - 4 = -5$ |
|------------------------|------------------------|-----------------------------|