Polar Coordinates and Multiplication

In the last chapter we wrote complex numbers in the form x + iy where x and y are real numbers. We can think of this as writing complex numbers using Cartesian coordinates. Every complex number is the sum of a number on the real axis and a number on the imaginary axis. We saw that writing complex numbers in this way made it simple to add and subtract complex numbers. It was not as simple to multiply and divide complex numbers written in Cartesian coordinates.

In this chapter we'll look at complex numbers using polar coordinates. We'll see that multiplication and division of complex numbers written in polar coordinates has a nice geometric interpretation involving scaling and rotating.

Unit circle

We wrote $C^1 \subseteq \mathbb{R}^2$ to refer to the unit circle in the plane of vectors. This is the circle of all vectors that have norm 1, the circle of all vectors that can be written in the form $(\cos(\theta), \sin(\theta))$.



We now want to look at the unit circle in \mathbb{C} , the plane of complex numbers. The unit circle is the set of all complex numbers whose norms equal 1. Equivalently, and similarly to the plane of vectors, the unit circle in the plane of complex numbers can also be described as the set of complex numbers that can be written in the form $\cos(\theta) + i\sin(\theta)$. Even though the unit circle of complex numbers is not exactly the same thing as the unit circle in the plane of vectors—one is a circle of numbers, the other is a circle of vectors—they are close enough to being the same that we'll just recycle the symbols C^1 and use them in this chapter to refer to the unit circle of complex numbers. That is, for the rest of this text, $C^1 \subseteq \mathbb{C}$. To repeat most of the remarks from the previous paragraph using set notation:

$$C^{1} = \{ x + iy \in \mathbb{C} \mid |x + iy| = 1 \}$$
$$= \{ x + iy \in \mathbb{C} \mid \sqrt{x^{2} + y^{2}} = 1 \}$$
$$= \{ x + iy \in \mathbb{C} \mid x^{2} + y^{2} = 1 \}$$
$$= \{ \cos(\theta) + i\sin(\theta) \mid \theta \in \mathbb{R} \}$$

Examples.

• $1 \in \mathbb{C}$, and |1| = 1, so $1 \in C^1$. Note that $1 = 1 + i0 = \cos(0) + i\sin(0)$.

• $-1 \in \mathbb{C}$, and $-1 = -1 + i0 = \cos(\pi) + i\sin(\pi)$, so -1 is also a point in the unit circle. Note that |-1| = 1.

• $i \in \mathbb{C}$ is a point in the unit circle because

$$|i| = |0 + i1| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$$

and because

$$i = 0 + i1 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$

• $-i \in \mathbb{C}$ is a point on the unit circle because

$$i = 0 - i1 = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)$$

and because

$$|-i| = |0-i1| = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$$

• $\frac{\sqrt{3}}{2} + i\frac{1}{2} \in \mathbb{C}$ is a point on the unit circle since $\frac{\sqrt{3}}{2} + i\frac{1}{2} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$

The 5 examples above are illustrated on the next page.



Polar coordinates

Any number, $z \in \mathbb{C}$, in the plane of complex numbers can be identified by the direction in which you'd have to travel in a straight line from 0 to reach the number—this direction is a point in the unit circle—and by its distance from 0—its norm. Thus, each complex number is identified by a pair of coordinates: a number in the unit circle, and a norm.



A point on the unit circle is a number of the form $\cos(\theta) + i\sin(\theta)$ and a norm is a real number that is greater than or equal 0. Thus, the above paragraph states that any complex number can be written in the form

$$r(\cos(\theta) + i\sin(\theta))$$

for some $r \ge 0$ and some $\theta \in \mathbb{R}$. The real number r is the norm of the complex number $r(\cos(\theta) + i\sin(\theta))$. Writing complex numbers in the form $r(\cos(\theta) + i\sin(\theta))$ is what is referred to as *polar coordinates* for the plane of complex numbers.



Examples.

• Complex numbers written in polar coordinates include $3(\cos(\frac{\pi}{7}) + i\sin(\frac{\pi}{7})), \frac{1}{2}(\cos(4) + i\sin(4)), \text{ and } \cos(\frac{5}{6}) + i\sin(\frac{5}{6}).$ In the last of these three examples, r = 1. The norms of these three complex numbers are $3, \frac{1}{2}$, and 1, respectively.

You can write a complex number $x + iy \in \mathbb{C}$ in polar coordinates in much the same way that a vector in the plane can be written in polar coordinates. Use the formula

$$x + iy = |x + iy| \left(\frac{x}{|x + iy|} + i\frac{y}{|x + iy|}\right)$$

Example.

• To write 3 - i2 in polar coordinates, first find the norm of 3 - i2: $|3 - i2| = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}$

Then



Multiplication by a positive real number scales

If s > 0 is a real number, and if $r(\cos(\theta) + i\sin(\theta))$ is a complex number written in polar coordinates, then the product of s and $r(\cos(\theta) + i\sin(\theta))$ is the number

$$sr(\cos(\theta) + i\sin(\theta))$$



Both $r(\cos(\theta) + i\sin(\theta))$ and $sr(\cos(\theta) + i\sin(\theta))$ have the same "unit circle coordinate" of $\cos(\theta) + i\sin(\theta)$. Where they are different is in their "norm coordinates", r and sr, respectively. That is, multiplying by the positive real number s doesn't change the direction of complex numbers, it only scales their distance from 0.

Multiplying complex numbers by 2 makes them twice as far from 0. Multiplying by $\frac{1}{3}$ makes complex numbers $\frac{1}{3}$ as far from 0.



Multiplication in the unit circle

In the next theorem we'll multiply two different complex numbers in the unit circle. Remember that a number in the unit circle is exactly a number of the form $\cos(\theta) + i\sin(\theta)$ where $\theta \in \mathbb{R}$, or equivalently, a number of the form $\cos(\alpha) + i\sin(\alpha)$ where $\alpha \in \mathbb{R}$. The next theorem shows that multiplying two complex numbers in the unit circle is as easy as adding two real numbers.



Proof: For this proof, we'll multiply the two complex numbers

 $\cos(\alpha) + i\sin(\alpha)$ and $\cos(\theta) + i\sin(\theta)$ in the manner described in the previous chapter. In the last line of the proof, we'll use the angle sum formulas for cosine and sine. Recall that those formulas are

$$\cos(\alpha + \theta) = \cos(\alpha)\cos(\theta) - \sin(\alpha)\sin(\theta)$$
$$\sin(\alpha + \theta) = \sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta)$$

Now to check that the theorem is true:

$$\left(\cos(\alpha) + i\sin(\alpha) \right) \left(\cos(\theta) + i\sin(\theta) \right)$$

= $\cos(\alpha) \cos(\theta) + \cos(\alpha)i\sin(\theta) + i\sin(\alpha)\cos(\theta) + i\sin(\alpha)i\sin(\theta)$
= $\cos(\alpha)\cos(\theta) + i\left(\cos(\alpha)\sin(\theta) + \sin(\alpha)\cos(\theta)\right) + i^2\sin(\alpha)\sin(\theta)$
= $\left(\cos(\alpha)\cos(\theta) - \sin(\alpha)\sin(\theta)\right) + i\left(\sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta)\right)$
= $\cos(\alpha + \theta) + i\sin(\alpha + \theta)$

Example.

•
$$\left(\cos(2) + i\sin(2)\right)\left(\cos(3) + i\sin(3)\right) = \cos(5) + i\sin(5)$$

For this next corollary, which follows from the previous theorem, recall that \mathbb{N} is the set of natural numbers, numbers of the form 1, 2, 3, 4, ...

Corollary (24). (De Moivre's Formula) If $n \in \mathbb{N}$, then

$$\left(\cos(\theta) + i\sin(\theta)\right)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof: If n=1, this corollary says

$$(\cos(\theta) + i\sin(\theta))^1 = \cos(\theta) + i\sin(\theta)$$

which is clearly true. Any number to the first power is itself.

If n = 2, then Theorem 23 tells us that

$$\left(\cos(\theta) + i\sin(\theta)\right)^2 = \left(\cos(\theta) + i\sin(\theta)\right) \left(\cos(\theta) + i\sin(\theta)\right)$$
$$= \cos(\theta + \theta) + i\sin(\theta + \theta)$$
$$= \cos(2\theta) + i\sin(2\theta)$$

If n = 3, then we can use our equation for when n = 2, along with Theorem 23, to see that

$$\left(\cos(\theta) + i\sin(\theta)\right)^3 = \left(\cos(\theta) + i\sin(\theta)\right)^2 \left(\cos(\theta) + i\sin(\theta)\right)$$
$$= \left(\cos(2\theta) + i\sin(2\theta)\right) \left(\cos(\theta) + i\sin(\theta)\right)$$
$$= \cos(2\theta + \theta) + i\sin(2\theta + \theta)$$
$$= \cos(3\theta) + i\sin(3\theta)$$

We've now checked that the corollary is true if n = 1 or n = 2 or n = 3. This process can be continued indefinitely to show that the theorem is true when n = 4, n = 5, n = 6, etc. You might want to check the next step of the process yourself by checking that the corollary is true when n = 4.

What De Moivre's Formula tells us is that we can find the value of multiplying $\cos(\theta) + i \sin(\theta)$ *n*-times by just adding θ *n*-times, that is by finding $n\theta$. De Moivre's Formula turns the problem of finding powers of certain complex numbers into a problem of multiplying a natural number and a real number, and that's a task that we're very comfortable with.

Example.

• Let's find $\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^{12}$. We could perform this task by multiplying $\frac{\sqrt{3}}{2} + i\frac{1}{2}$ twelve times, in the manner shown in the previous chapter, but that would take a long time. If we recall that $\frac{\sqrt{3}}{2} + i\frac{1}{2} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$ then we can use De Moivre's Formula to more easily conclude that $\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^{12} = 1$. Here's how:

$$\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^{12} = \left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)^{12}$$
$$= \cos\left(12 \cdot \frac{\pi}{6}\right) + i\sin\left(12 \cdot \frac{\pi}{6}\right)$$
$$= \cos(2\pi) + i\sin(2\pi)$$
$$= 1 + i0$$
$$= 1$$



The complex number $\frac{\sqrt{3}}{2} + i\frac{1}{2}$ is $\frac{1}{12}$ of the unit circle away from the number 1, in the counterclockwise direction. Each time we multiply by $\frac{\sqrt{3}}{2} + i\frac{1}{2}$, we move counterclockwise another $\frac{1}{12}$ of the total circumference of the unit circle. Thus, $\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^{12}$ is the number in the unit circle that we arrive at by beginning at 1, and then moving $\frac{1}{12}$ of the way around the unit circle counterclockwise, 12 times. To move $\frac{1}{12}$ of the way around a circle 12 times is to make a complete revolution around the circle, and to end up where you began, in this case, at the number 1. That is, $\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^{12} = 1$.

The geometry of this example is a lot like the geometry of the hour hand of a clock, only instead of running clockwise and starting at the top of a circle, as the hour hand of a clock does, multiplication by $\frac{\sqrt{3}}{2} + i\frac{1}{2}$ runs counterclockwise and begins at the rightmost point of the unit circle, at the number 1. Thus, the algebra of multiplication of complex numbers can encode the arithmetic and geometry of clocks.

• Let's find
$$\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)^3$$
. Notice that $-\frac{\sqrt{3}}{2}+i\frac{1}{2}=\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)$.

Thus, De Moivre's Formula tells us that $\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)^3$ is the complex number that is obtained by beginning at 1, and rotating by an angle of $\frac{5\pi}{6}$ three times, which is an angle of $3\left(\frac{5\pi}{6}\right) = \frac{15\pi}{6}$. Notice that $\frac{15\pi}{6} = \frac{12\pi}{6} + \frac{3\pi}{6} = 2\pi + \frac{\pi}{2}$. That is, to rotate by an angle of $\frac{15\pi}{6}$ is to rotate first by a complete revolution around the unit circle, and then to rotate by an angle of $\frac{\pi}{2}$. Therefore, $\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)^3 = i$.



• The complex number $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ equals $\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$. It is the number in the unit circle that is arrived at by beginning at 1, and rotating by an angle of $\frac{\pi}{4}$, which is $\frac{1}{8}$ of the way around the unit circle in the counterclockwise direction.

counterclockwise direction. To find $\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^5$, we just have to rotate the number 1 around the unit circle by an angle of $\frac{\pi}{4}$, 5 times. That makes an angle of $\frac{5\pi}{4}$. The number in the unit circle obtained by rotating the number 1 by an angle of $\frac{5\pi}{4}$ is the number $-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. Therefore, $\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^5 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$.

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Roots of unity

Let $n \in \mathbb{N}$. Any complex number $z \in \mathbb{C}$ that is a solution to the equation $z^n = 1$ is called an n^{th} root of unity. For example, we saw on page 370 that $\frac{\sqrt{3}}{2} + i\frac{1}{2}$ is a twelfth root of unity, or equivalently, that $\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$ is a twelfth root of unity.

What we'll do here is describe all of the n^{th} roots of unity, for any $n \in \mathbb{N}$. To begin with, notice that the solutions of the equation $z^n = 1$ are the solutions of the equation $z^n - 1 = 0$, or equivalently, they are the roots of the polynomial $z^n - 1$. This is a polynomial of degree n.

You may have learned in Math 1050 that a degree n polynomial with real coefficients has at most n real numbers as roots. The algebra of real numbers and complex numbers is similar enough that the same explanation would apply essentially word-for-word to explain that $z^n - 1$ has at most n complex numbers as roots. In fact it has exactly n roots. That is, there are exactly n different n^{th} roots of unity. They're described in the following theorem.

Theorem (25). Let $n \in \mathbb{N}$. Let k be any one of the n numbers in the set $\{0, 1, 2, \ldots, n-1\}$. Then the n^{th} roots of unity are the n complex numbers of the form

$$\cos\left(k\cdot\frac{2\pi}{n}\right) + i\sin\left(k\cdot\frac{2\pi}{n}\right)$$

Before looking at a proof of this theorem, let's see some examples.

Examples.

• If n = 2, then the 2 second roots of unity—the 2 solutions of $z^2 = 1$ —are

$$\cos(0 \cdot \frac{2\pi}{2}) + i\sin(0 \cdot \frac{2\pi}{2}) = \cos(0) + i\sin(0) = 1 + i0 = 1, \text{ and}$$
$$\cos(1 \cdot \frac{2\pi}{2}) + i\sin(1 \cdot \frac{2\pi}{2}) = \cos(\pi) + i\sin(\pi) = 0 + i(-1) = -1.$$

• If n = 3, then the 3 third roots of unity—the 3 solutions of $z^3 = 1$ —are

$$\cos(0 \cdot \frac{2\pi}{3}) + i\sin(0 \cdot \frac{2\pi}{3}) = \cos(0) + i\sin(0) = 1 + i0 = 1,$$

$$\cos(1 \cdot \frac{2\pi}{3}) + i\sin(1 \cdot \frac{2\pi}{3}) = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \text{ and}$$

$$\cos(2 \cdot \frac{2\pi}{3}) + i\sin(2 \cdot \frac{2\pi}{3}) = \cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3}) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$



Now let's return to the proof of the theorem that $\cos\left(k \cdot \frac{2\pi}{n}\right) + i \sin\left(k \cdot \frac{2\pi}{n}\right)$ is a solution of the equation $z^n = 1$.

Proof: We want to show that $\left(\cos\left(k \cdot \frac{2\pi}{n}\right) + i\sin\left(k \cdot \frac{2\pi}{n}\right)\right)^n = 1$ if $k \in \{0, 1, 2, \dots, n-1\}.$

First note that an angle of $k \cdot 2\pi$ is exactly k full rotations around the unit circle. Therefore, $\cos(k \cdot 2\pi) + i \sin(k \cdot 2\pi)$ is the complex number obtained by beginning at 1, and completing k full rotations around the unit circle, thus ending where we began, at the number 1. That is, $\cos(k \cdot 2\pi) + i \sin(k \cdot 2\pi) = 1$. We'll use this below.

Now applying De Moivre's Formula, we have

$$\left(\cos\left(k\cdot\frac{2\pi}{n}\right)+i\sin\left(k\cdot\frac{2\pi}{n}\right)\right)^n = \cos\left(nk\cdot\frac{2\pi}{n}\right)+i\sin\left(nk\cdot\frac{2\pi}{n}\right)$$
$$= \cos(k\cdot2\pi)+i\sin(k\cdot2\pi)$$
$$= 1$$

which shows that $\cos\left(k \cdot \frac{2\pi}{n}\right) + i \sin\left(k \cdot \frac{2\pi}{n}\right)$ is a solution of the equation $z^n = 1$ regardless of whether k equals $0, 1, 2, \ldots$, or n - 1. That is, $\cos\left(k \cdot \frac{2\pi}{n}\right) + i \sin\left(k \cdot \frac{2\pi}{n}\right)$ is an n^{th} root of unity.

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Multiplication by a number in the unit circle rotates

Suppose that α is a real number, so that $\cos(\alpha) + i\sin(\alpha)$ is a complex number in the unit circle.

Let $r(\cos(\theta)+i\sin(\theta))$ be a complex number written in polar coordinates, so that $r \ge 0$ and $\theta \in \mathbb{R}$. To see the effect that multiplication by $\cos(\alpha)+i\sin(\alpha)$ has on $r(\cos(\theta)+i\sin(\theta))$, remember that multiplication of complex numbers is commutative. That is, we can rearrange the order of numbers when we multiply by them. Keeping this in mind, and applying Theorem 23 which explained how to multiply two numbers in the unit circle, we have

$$\left(\cos(\alpha) + i\sin(\alpha)\right) \left(r(\cos(\theta) + i\sin(\theta))\right)$$

= $r\left(\cos(\alpha) + i\sin(\alpha)\right) \left(\cos(\theta) + i\sin(\theta)\right)$
= $r\left(\cos(\alpha + \theta) + i\sin(\alpha + \theta)\right)$

Notice that multiplication by $\cos(\alpha) + i\sin(\alpha)$ doesn't change the "norm coordinate" (the number r), it just affects the "unit circle coordinate". It affects the unit circle coordinate by adding the number α to the number θ .



Multiplication by cos(x)+isin(x)

You may recall that this is essentially the process by which we defined rotation of the plane of vectors in the chapter "Rotation Matrices". That is, multiplication by $\cos(\alpha) + i\sin(\alpha)$ is a rotation of the plane of complex numbers by angle α .

Examples.

• $i = 0 + i1 = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})$, so multiplication by *i* rotates the plane of complex numbers counterclockwise by an angle of $\frac{\pi}{2}$.



• $-1 = -1 + i0 = \cos(\pi) + i\sin(\pi)$ so multiplication by -1 rotates the plane of complex numbers counterclockwise by an angle of π .



Notice that if you rotate the plane of complex numbers by an angle of $\frac{\pi}{2}$, and then you rotate by an angle of $\frac{\pi}{2}$ again, the net effect is to have rotated the plane of complex numbers by an angle of π . Using the two examples above, the geometry of the previous sentence is encoded in the algebraic formula that $i^2 = -1$.

• For a numeric example of the formula on the previous page, the product of $\cos(3) + i\sin(3)$ and $4(\cos(2) + i\sin(2))$ equals

$$4(\cos(3+2) + i\sin(3+2)) = 4(\cos(5) + i\sin(5))$$

Multiplication scales and rotates

Suppose that α is a real number, and that $s \ge 0$. Then $s(\cos(\alpha) + i\sin(\alpha))$ is a complex number. Any complex number can be written in this way.

Let $r(\cos(\theta) + i\sin(\theta))$ be a complex number written in polar coordinates, so that $r \ge 0$ and $\theta \in \mathbb{R}$. To see the effect that multiplication by $s(\cos(\alpha) + i\sin(\alpha))$ has on $r(\cos(\theta) + i\sin(\theta))$, we'll use again that multiplication of complex numbers is commutative.

$$\left(s \left(\cos(\alpha) + i \sin(\alpha) \right) \right) \left(r \left(\cos(\theta) + i \sin(\theta) \right) \right)$$

= $sr \left(\cos(\alpha) + i \sin(\alpha) \right) \left(\cos(\theta) + i \sin(\theta) \right)$
= $sr \left(\cos(\alpha + \theta) + i \sin(\alpha + \theta) \right)$

What the above equation shows us is that multiplication by $s(\cos(\alpha) + i\sin(\alpha))$ changes the norm of the number $r(\cos(\theta) + i\sin(\theta))$ from r to sr. It scales the number by s. It also adds α to the unit circle coordinate, changing that coordinate from $\cos(\theta) + i\sin(\theta)$ to $\cos(\alpha + \theta) + i\sin(\alpha + \theta)$. This is a rotation by angle α .

In summary we have:

Multiplication by $s(\cos(\alpha) + i\sin(\alpha))$ scales the complex numbers in the plane by sand it rotates the plane by angle α .



Examples.

• Multiplying by the number $2\left(\cos\left(\frac{\pi}{7}\right) + i\sin\left(\frac{\pi}{7}\right)\right)$ scales the numbers in the plane by 2, and rotates the plane counterclockwise by an angle of $\frac{\pi}{7}$.

• To find the product of $2\left(\cos\left(\frac{\pi}{7}\right)+i\sin\left(\frac{\pi}{7}\right)\right)$ and $8\left(\cos\left(\frac{4\pi}{7}\right)+i\sin\left(\frac{4\pi}{7}\right)\right)$ just multiply their norm coordinates $(2 \cdot 8 = 16)$ and add the angles of their circle coordinates $\left(\frac{\pi}{7}+\frac{4\pi}{7}=\frac{5\pi}{7}\right)$ to see that

$$2\left(\cos\left(\frac{\pi}{7}\right) + i\sin\left(\frac{\pi}{7}\right)\right) \cdot 8\left(\cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)\right)$$
$$= 16\left(\cos\left(\frac{5\pi}{7}\right) + i\sin\left(\frac{5\pi}{7}\right)\right)$$

Division

The formula for dividing two complex numbers written in polar coordinates is also straightforward. Instead of multiplying their norms and adding the angles of their circle coordinates, divide their norms and subtract their circle coordinates

$$\frac{r(\cos(\theta) + i\sin(\theta))}{s(\cos(\alpha) + i\sin(\alpha))} = \frac{r}{s} \Big(\cos(\theta - \alpha) + i\sin(\theta - \alpha)\Big)$$

Example.

• To find the quotient of the numbers $8\left(\cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)\right)$ and $2\left(\cos\left(\frac{\pi}{7}\right) + i\sin\left(\frac{\pi}{7}\right)\right)$ we need to divide the norm coordinates $\left(\frac{8}{2} = 4\right)$ and subtract the angles of the circle coordinates $\left(\frac{4\pi}{7} - \frac{\pi}{7} = \frac{3\pi}{7}\right)$:

$$\frac{8\left(\cos\left(\frac{4\pi}{7}\right)+i\sin\left(\frac{4\pi}{7}\right)\right)}{2\left(\cos\left(\frac{\pi}{7}\right)+i\sin\left(\frac{\pi}{7}\right)\right)} = 4\left(\cos\left(\frac{3\pi}{7}\right)+i\sin\left(\frac{3\pi}{7}\right)\right)$$

Polar coordinates and addition

It's easier to multiply and divide complex numbers when they are written in polar coordinates—such as $r(\cos(\theta) + i\sin(\theta))$ —than it is when they are written in Cartesian coordinates—such as x + iy. However, adding two complex numbers written in Cartesian coordinates is easy to do, and while there is a method for adding numbers written in polar coordinates, it's a bit complicated, and we won't talk about it here.

Exercises

In #1-6, match the complex numbers written in polar coordinates with the complex numbers drawn to the right. Remember that the norm, or distance from 0, of the number $r(\cos(\theta) + i\sin(\theta))$ is r. The number $\cos(\theta) + i\sin(\theta)$ is the point in the unit circle that represents the direction of the number $r(\cos(\theta) + i\sin(\theta))$.



7.) What's the norm of the complex number $5(\cos(16) + i\sin(16))$?

- 8.) What's the norm of the complex number $\frac{1}{3}(\cos(\pi) + i\sin(\pi))?$
- 9.) Write the complex number 3 + i7 in polar coordinates.

10.) Write the complex number 6 + i2 in polar coordinates.

- 11.) Find the product of 3 and $4(\cos(5) + i\sin(5))$.
- 12.) Find the product of 6 and $2(\cos(-6) + i\sin(-6))$.
- 13.) Find $\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^7$.
- 14.) Find $\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^3$.
- 15.) Find $\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^4$.
- 16.) Find $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^4$.

- 17.) Find $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^6$.
- 18.) Which number is $\left(\cos\left(k \cdot \frac{2\pi}{n}\right) + i\sin\left(k \cdot \frac{2\pi}{n}\right)\right)^n$, where $n \in \mathbb{N}$ and $k \in \{1, 2, \dots, n-1\}$?
- 19.) How many different third roots of unity are there?
- 20.) How many different seventh roots of unity are there?

21.) How many different 34th roots of unity are there?

- 22.) Find the product of $\cos(2) + i\sin(2)$ and $3(\cos(8) + i\sin(8))$.
- 23.) Find the product of $\cos(7) + i\sin(7)$ and $2(\cos(5) + i\sin(5))$.
- 24.) Find the product of $9(\cos(3) + i\sin(3))$ and $2(\cos(4) + i\sin(4))$.
- 25.) Find the product of $8(\cos(5) + i\sin(5))$ and $6(\cos(1) + i\sin(1))$.
- 26.) Find the product of $3(\cos(2) + i\sin(2))$ and $4(\cos(6) + i\sin(6))$.
- 27.) Find the quotient

$$\frac{6(\cos(5) + i\sin(5))}{3(\cos(1) + i\sin(1))}$$

28.) Find the quotient

$$\frac{15(\cos(8) + i\sin(8))}{5(\cos(2) + i\sin(2))}$$

All further exercises in this chapter have nothing to do with complex numbers. Match the functions with their graphs.

29.) $\sin(x)$	30.) $\frac{1}{2}\sin(x)$	31.) $\sin(x + \frac{\pi}{2})$
32.) $2\sin(x)$	33.) $-\sin(x)$	34.) $\sin(x) + 1$
35.) $\sin(-x)$	36.) $\sin(x - \frac{\pi}{2})$	37.) $\sin(2x)$
38.) $\sin(\frac{x}{2})$		



Match the functions with their graphs.

39.)
$$\sin(x)$$
 40.) $\csc(x)$
41.) $f(x) = \begin{cases} \sin(x) & \text{if } x \le 0; \\ \csc(x) & \text{if } x > 0. \end{cases}$ 42.) $g(x) = \begin{cases} \csc(x) & \text{if } x < 0; \\ \sin(x) & \text{if } x \ge 0. \end{cases}$









In the exercises from the previous chapter we reviewed when some equations have no solutions. The list below describes the solutions (if there are solutions) of some common equations.

•
$$x^2 = c$$
 implies $x = \sqrt{c}$ or $x = -\sqrt{c}$
• $ax^2 + bx + c = 0$ implies $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$
• $\sqrt{x} = c$ implies $x = c^2$
• $e^x = c$ implies $x = \log_e(c)$
• $\log_e(x) = c$ implies $x = e^c$

- $\tan(x) = c$ implies $x = \arctan(c) + n\pi$ where $n \in \mathbb{Z}$
- $\cos(x) = c$ implies $x = \arccos(c) + n2\pi$ or $-\arccos(c) + n2\pi$ where $n \in \mathbb{Z}$
- $\sin(x) = c$ implies $x = \arcsin(c) + n2\pi$ or $-\arcsin(c) + (n2+1)\pi$ where $n \in \mathbb{Z}$

For #43-58, use the rules above to find the solutions of the given equations.

43.) $e^x = 5$	51.) $x^2 + 6x + 1 = 0$
44.) $\sqrt{x} = 6$	52.) $\log_e(x) = -37$
45.) $\cos(x) = \frac{2}{3}$	53.) $x^2 = 16$
46.) $x^2 = 4$	54.) $\sqrt{x} = \frac{1}{2}$
47.) $\tan(x) = 43$	55.) $\sin(x) = \frac{4}{5}$
48.) $\log_e(x) = -2$	56.) $e^x = 27$
$49.) \ 6x^2 - x - 2 = 0$	57.) $\cos(x) = -\frac{1}{3}$
50.) $\sin(x) = -\frac{1}{2}$	58.) $\tan(x) = -1$

Use that $e^x e^y = e^{x+y}$ to simplify the following expressions.

59.) $e^{x}e^{2x}$ 60.) $e^{x+1}e^{3-2x}$

61.)
$$e^{2x+1}e^{-4x-5}$$

Use your answers from the problems above to solve the following equations.

62.)
$$e^{x}e^{2x} = e^{-5}$$

63.) $e^{x+1}e^{3-2x} = 5$
64.) $e^{2x+1}e^{-4x-5} = e^{-13}$
Use that $\log_e(x) + \log_e(y) = \log_e(xy)$ to simplify the following expressions.
65.) $\log_e(x^2) + \log_e(x^{-1})$
66.) $\log_e(x^2) + \log_e(x^3)$
67.) $\log_e(x) + \log_e(x^6)$

Use your answers from the problems above to solve the following equations.

- 68.) $\log_e(x^2) + \log_e(x^{-1}) = -11$
- 69.) $\log_e(x^2) + \log_e(x^3) = 5$
- 70.) $\log_e(x) + \log_e(x^6) = -6$

Use that $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$ to simplify the following expressions. 71.) $x + \frac{1}{x}$ 72.) $\frac{x+1}{2x} + \frac{3}{4}$ 73.) $\frac{2x+8}{3} + \frac{4x-7}{5}$

Use your answers from the problems above to solve the following equations.

74.)
$$x + \frac{1}{x} = 5$$

75.) $\frac{x+1}{2x} + \frac{3}{4} = -8$
76.) $\frac{2x+8}{3} + \frac{4x-7}{5} = -3$