Matrices

A 2×2 *matrix* (pronounced "two-by-two matrix") is a square block of 4 numbers. For example,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is a 2×2 matrix. It's called a 2×2 matrix because it has 2 rows and 2 columns.

Matrices come in different sizes. There are matrices with 5 rows and 2 columns, matrices with a million rows and 17 columns, etc. In this course the only matrices that we'll see are 2×2, matrices, so we'll often refer to 2×2 matrices as simply *matrices*.

Two matrices are equal if the number in any position of the one matrix equals the number in the same position of the other matrix.

Example.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$$

Scalar multiplication for matrices

To take the product of a scalar and a matrix, just as with vectors, multiply every number in the matrix by the scalar. For example,

$$2\begin{pmatrix} 2 & 1 \\ 5 & 9 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 & 2 \cdot 1 \\ 2 \cdot 5 & 2 \cdot 9 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 10 & 18 \end{pmatrix}$$

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Matrices as functions

Suppose M is a matrix. To be more precise, let's say that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$.

The matrix M and a column vector

$$\begin{pmatrix} u \\ w \end{pmatrix} \in \mathbb{R}^2$$

can be combined to produce another vector in $\mathbb{R}^2$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} (a,b)\binom{u}{w} \\ \\ (c,d)\binom{u}{w} \end{pmatrix} = \begin{pmatrix} au + bw \\ cu + dw \end{pmatrix}$$

Since $a, b, c, d, u, w \in \mathbb{R}$, it follows that $au + bw$ and $cu + dw$ are also numbers. Thus,

$$\begin{pmatrix} au + bw \\ cu + dw \end{pmatrix} \in \mathbb{R}^2$$

To recap, any matrix M carries with it a recipe to transform a vector in $\mathbb{R}^2$ into another vector in $\mathbb{R}^2$. This process defines a function

$$M : \mathbb{R}^2 \to \mathbb{R}^2$$

Example. Let

$$N = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

Then N is a function $N : \mathbb{R}^2 \to \mathbb{R}^2$.

The column vector

$$\begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

is in the domain of the matrix function N (as is any vector in the plane). If we put this vector into N, we will get out the vector

$$\begin{pmatrix} 16 \\ 42 \end{pmatrix}$$

since

$$N\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} (2,1)\binom{6}{4} \\ \\ (5,3)\binom{6}{4} \end{pmatrix} = \begin{pmatrix} 2(6) + 1(4) \\ 5(6) + 3(4) \end{pmatrix} = \begin{pmatrix} 16 \\ 42 \end{pmatrix}$$

Identity matrix

Notice that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \cdot x + 0 \cdot y \\ 0 \cdot x + 1 \cdot y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

That means that any vector we put into the matrix function

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gets returned to us unaltered. This is the identity function whose domain is the set $\mathbb{R}^2$, so we call

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the *identity matrix*. We should probably name this function id, since it is an identity function, but in linear algebra it's more commonly called I.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Again, $I : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity function for the plane.

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Flipping the plane over

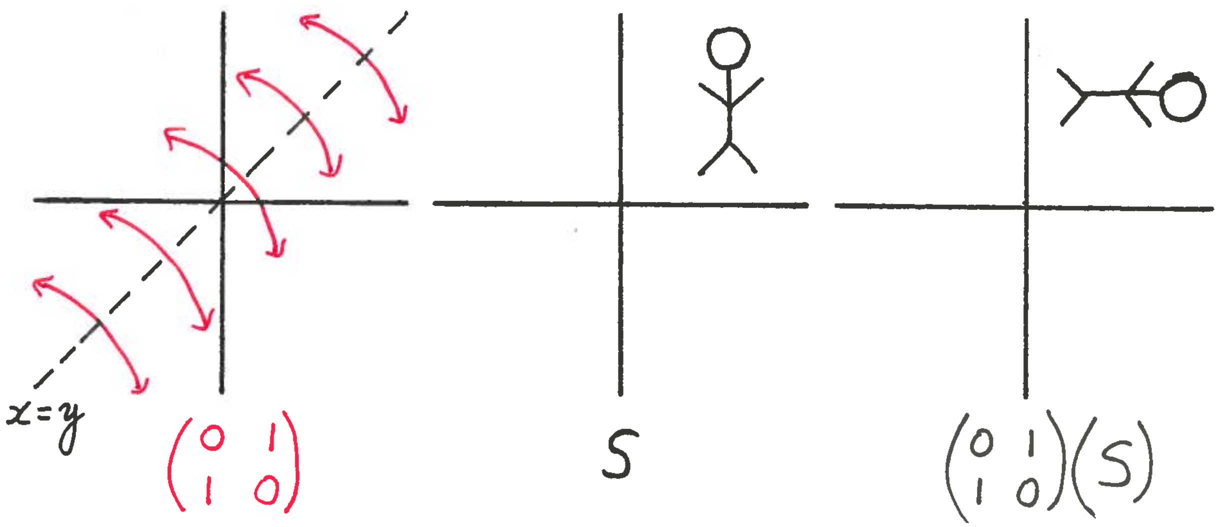
Notice that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \cdot x + 1 \cdot y \\ 1 \cdot x + 0 \cdot y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

Thus, the matrix function

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has the effect of interchanging the x-coordinate with the y-coordinate of every vector in $\mathbb{R}^2$. Geometrically, this matrix function flips the plane over the line $y = x$.



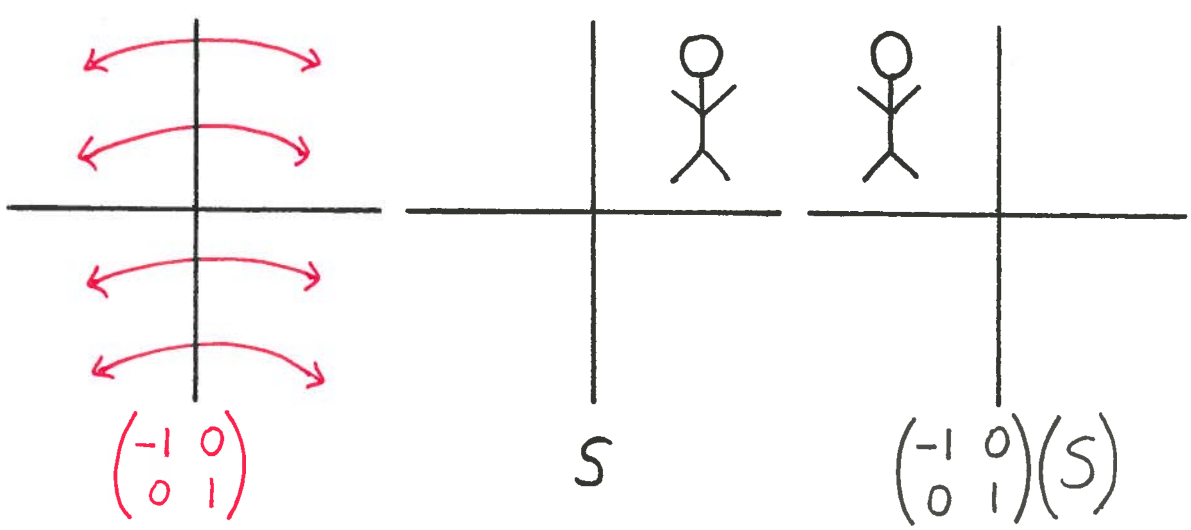
There are other ways to flip the plane over using matrices. For example,

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \cdot x + 0 \cdot y \\ 0 \cdot x + 1 \cdot y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

so we see that the matrix function

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

has the effect of replacing the x-coordinates of vectors with their negatives. Geometrically, this flips the plane over the y-axis.

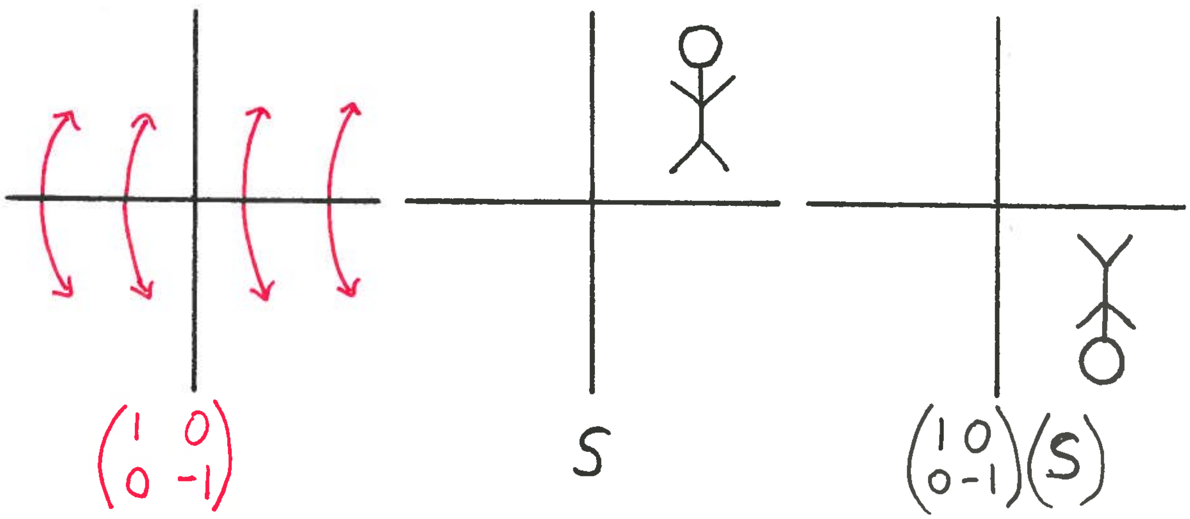


One more flip is worth noting here. The matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

flips the plane over the x-axis, since

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$



Diagonal matrices

Any matrix that has the number 0 as both its upper right and lower left entry is called a *diagonal matrix*.

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

To see how diagonal matrices affect the plane, notice that

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ dy \end{pmatrix}$$

The diagonal matrix above scales the x-coordinate of $\mathbb{R}^2$ by the number a, and it scales the y-coordinate by the number d.

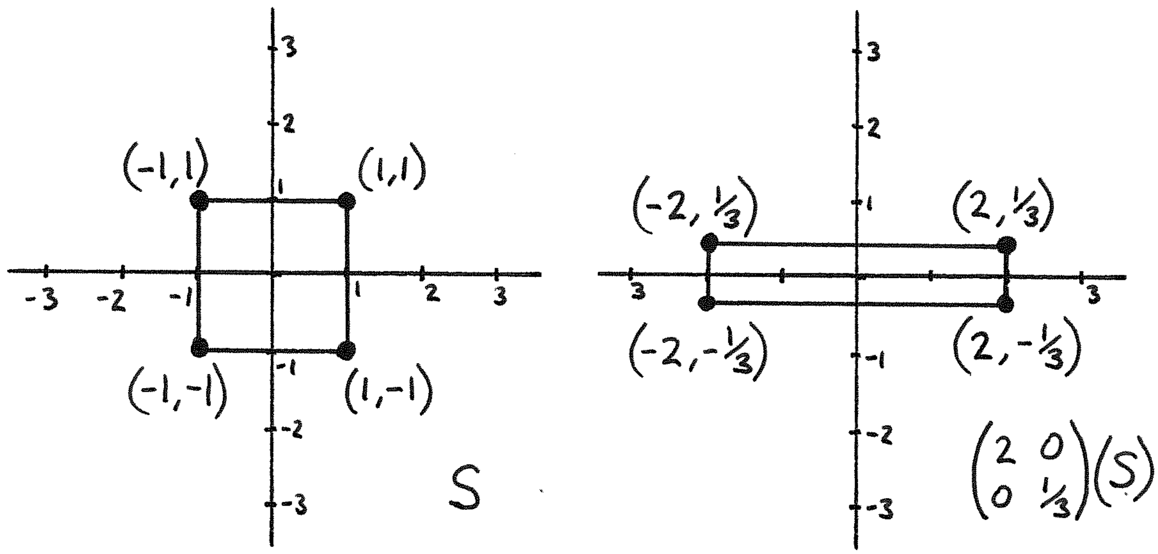
Examples. The diagonal matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

scales the x-coordinates of vectors by 2, and the y-coordinates of vectors by $\frac{1}{3}$ because

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{1}{3}y \end{pmatrix}$$

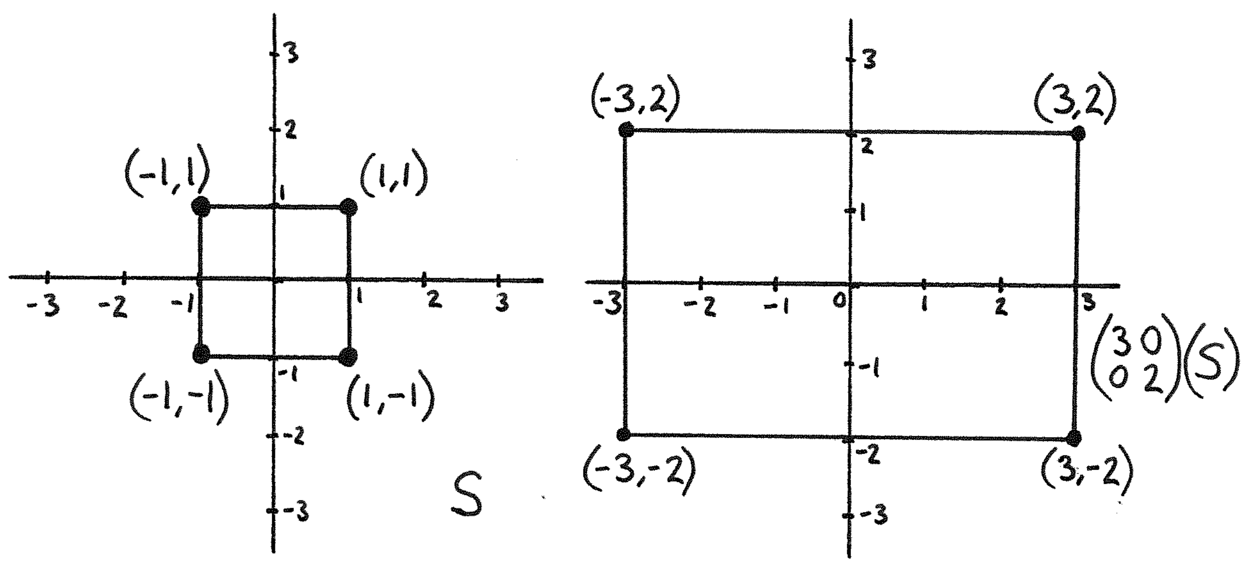
In the picture below, notice that the horizontal dimension is being stretched to be twice as long as it was. The vertical dimension is being shrunk, to be one-third its original size.



The diagonal matrix

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

stretches the x-direction by 3, and the y-direction by 2.



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Matrix multiplication

We can "multiply" two matrices to obtain another matrix. The formula for matrix multiplication is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} (a,b)\binom{u}{w} & (a,b)\binom{v}{z} \\ \\ (c,d)\binom{u}{w} & (c,d)\binom{v}{z} \end{pmatrix} = \begin{pmatrix} au+bw & av+bz \\ \\ cu+dw & cv+dz \end{pmatrix}$$

Example.

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} (1,2)\binom{6}{7} & (1,2)\binom{4}{5} \\ \\ (0,3)\binom{6}{7} & (0,3)\binom{4}{5} \end{pmatrix} = \begin{pmatrix} 20 & 14 \\ 21 & 15 \end{pmatrix}$$

Matrix multiplication is function composition

The formula above explains how to take two different matrices, say M and N, and combine them to get another matrix, M times N, or MN for short.

Matrices are functions, so we have another way of combining two matrices to get another, and that's function composition: $M \circ N$. The claim below says that matrix multiplication and function composition of matrices are the same thing.

Claim: If M and N are matrices then $MN = M \circ N$.

Proof: To check that the claim is true, let's write

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$$

What follows in this proof is perhaps a bit tedious, but it doesn't involve anything more than the definition of a matrix as a function, the definition of matrix multiplication, the definition of function composition, and the distributive law.

The definition of matrix multiplication is that

$$MN = \begin{pmatrix} au+bw & av+bz \\ \\ cu+dw & cv+dz \end{pmatrix}$$

Therefore, using our definition of the matrix function, $MN : \mathbb{R}^2 \to \mathbb{R}^2$ is the matrix function defined by the equation

$$MN\binom{x}{y} = \begin{pmatrix} (au+bw)x + (av+bz)y \\ (cu+dw)x + (cv+dz)y \end{pmatrix}$$

Our claim is that the matrix function MN should be the same function as $M \circ N$. To check that they are the same function, notice that

$$N\binom{x}{y} = \begin{pmatrix} ux+vy \\ wx+zy \end{pmatrix}$$

so that

$$M \circ N\binom{x}{y} = M\begin{pmatrix} ux+vy \\ wx+zy \end{pmatrix} = \begin{pmatrix} a(ux+vy)+b(wx+zy) \\ c(ux+vy)+d(wx+zy) \end{pmatrix}$$

We can now check that $MN\binom{x}{y}$ equals $M \circ N\binom{x}{y}$. We can use the distributive law for each and check that they are both equal to

$$\begin{pmatrix} aux+bwx+avy+bzy \\ cux+dwx+cvy+dzy \end{pmatrix}$$

Because $MN\binom{x}{y} = M \circ N\binom{x}{y}$ for any choice of vector $\binom{x}{y}$, the functions MN and $M \circ N$ are the same. That is, $MN = M \circ N$ ■

Because function composition isn't commutative, neither is matrix multiplication. Try this yourself: write down two matrices, M and N. Probably $MN \neq NM$.

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Inverse matrices

If M is a matrix and the function $M : \mathbb{R}^2 \to \mathbb{R}^2$ is both one-to-one and onto, then M has an inverse function $M^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$. This inverse function M^{-1} will also be a matrix, as we will see shortly.

The definition for inverse functions that we saw earlier in this course was that

$$f \circ f^{-1} = id \quad \text{and} \quad f^{-1} \circ f = id$$

For matrices though, composition is multiplication, and we call the identity function I rather than id. So inverse matrix functions satisfy the rule that

$$MM^{-1} = I \quad \text{and} \quad M^{-1}M = I$$

which means the same thing as

$$MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M^{-1}M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Examples.

- We can check that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

by observing that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Recall that the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

flips the plane over the line $y = x$.

If we flip the plane over the $y = x$ line, and then we flip the plane over the $y = x$ line again, then we'll be back to where we started. It would be as if we hadn't done anything at all. To write that algebraically,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

That means that the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is its own inverse. That is,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

• Flipping the plane over the y-axis is its own inverse.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

• Flipping the plane over the x-axis is its own inverse.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

• If you scale the x-axis in the plane by the number a, and you scale the y-axis by the number d, we can put everything back to where it started by scaling the x-axis by $1/a$ and the y-axis by $1/d$. That is

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$$

You should check this algebraically too. That is, check that

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Determinants

The *determinant* of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the number $ad - cb$.

The above sentence is abbreviated as

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

Example.

$$\det\begin{pmatrix}4 & -2\\ 1 & -3\end{pmatrix} = 4(-3) - 1(-2) = -12 + 2 = -10$$

Claim: If M is a matrix and $\det(M) = 0$, then the function M is not one-to-one.

Proof: Let's suppose that

$$M = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

Then to say that $\det(M) = 0$ means that $ad - cb = 0$. That implies that $ad = cb$, and if neither c nor d equals 0, then $\frac{a}{c} = \frac{b}{d}$.

(It's a bit of a cheat to assume that neither c nor d equals 0. One of them might. But it makes this explanation a little bit shorter if we assume this, and we're just trying to get the idea of this explanation here.)

Observe that

$$\begin{pmatrix}\frac{1}{c}\\ 0\end{pmatrix} \neq \begin{pmatrix}0\\ \frac{1}{d}\end{pmatrix}$$

and yet because $\frac{a}{c} = \frac{b}{d}$ we have

$$M\begin{pmatrix}\frac{1}{c}\\ 0\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\frac{1}{c}\\ 0\end{pmatrix} = \begin{pmatrix}\frac{a}{c}\\ 1\end{pmatrix} = \begin{pmatrix}\frac{b}{d}\\ 1\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\ \frac{1}{d}\end{pmatrix} = M\begin{pmatrix}0\\ \frac{1}{d}\end{pmatrix}$$

Two different inputs into the matrix function M gave us the same output. That means that the function is not one-to-one. ■

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Determinants and inverses

If a matrix has a determinant that equals 0, then it is not one-to-one, so it does not have an inverse. All invertible functions are one-to-one.

If the determinant of the matrix

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

does not equal 0. Then the matrix has an inverse, and it can be found using the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Notice that in the above formula we are allowed to divide by the determinant since we are assuming that it's not 0.

Example. To find

$$\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}^{-1}$$

first check that

$$\det \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 3 \cdot 2 - 1 \cdot 4 = 2$$

Then

$$\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ \frac{-1}{2} & \frac{3}{2} \end{pmatrix}$$

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Planar transformations

In the rest of this text, we'll use the term *planar transformation* to refer interchangeably to any addition function $A_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}^2$ or to any invertible matrix function $M : \mathbb{R}^2 \to \mathbb{R}^2$.

Exercises

For #1-4, find the product of the scalar and the matrix.

1.) $3\begin{pmatrix} 2 & 1 \\ 5 & 9 \end{pmatrix}$

2.) $-4\begin{pmatrix} 0 & 7 \\ -2 & -4 \end{pmatrix}$

3.) $\frac{1}{5}\begin{pmatrix} 10 & 7 \\ -20 & -3 \end{pmatrix}$

4.) $-\frac{1}{2}\begin{pmatrix} \frac{2}{3} & -3 \\ -\frac{3}{5} & 10 \end{pmatrix}$

Find the resulting column vectors in #5-10

5.) $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

6.) $\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \end{pmatrix}$

7.) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ -\frac{2}{3} \end{pmatrix}$

8.) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix}$

9.) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

10.) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 10 \\ 10 \end{pmatrix}$

Multiply the matrices in #11-16

11.) $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$

12.) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 10 \\ 7 & 8 \end{pmatrix}$

13.) $\begin{pmatrix} -2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -4 & 5 \\ 0 & 3 \end{pmatrix}$

14.) $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix}$

15.) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}$

16.) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Use that the inverse of a diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ is $\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$ to find the inverses of the matrices given in #17-20.

17.) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

18.) $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$

19.) $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

20.) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{4}{5} \end{pmatrix}$

What are the determinants of the matrices given in #21-26?

21.) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

22.) $\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$

23.) $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$

24.) $\begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$

25.) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

26.) $\begin{pmatrix} -3 & 8 \\ 2 & 7 \end{pmatrix}$

Each of the matrices #27-32 has a determinant that does not equal 0, so they all have inverse matrices. What are their inverse matrices?

27.) $\begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix}$

28.) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

29.) $\begin{pmatrix} 5 & -3 \\ 2 & 10 \end{pmatrix}$

30.) $\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$

31.) $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 5 \end{pmatrix}$

32.) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Find the solutions of the equations given in #33-36.

33.) $e^{x^2-2} = 5$

34.) $x^7 - 3x^6 = 0$ with $x \neq 0$

35.) $\log_e(2x-5) = 1$ where $x < 3$

36.) $\sqrt[3]{x^3 - 2} = x - 1$