Linear Equations and Lines

A *linear equation in two variables* is an equation that's equivalent to an equation of the form

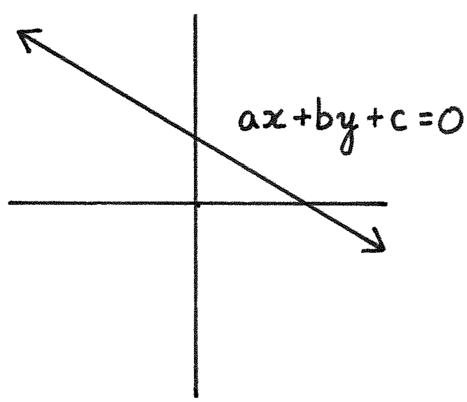
$$ax + by + c = 0$$

where a, b, and c are constant numbers, and where a and b don't both equal 0. (If they were both 0 then the polynomial on the left side of the equal sign wouldn't be linear.)

Examples.

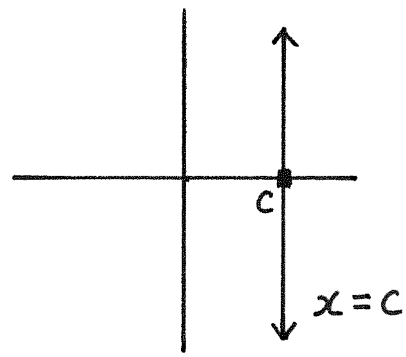
- $3x + 2y - 7 = 0$ is a linear equation.
- $2x = -4y + 3$ is a linear equation. It's equivalent to $2x + 4y - 3 = 0$, and $2x + 4y - 3$ is a linear polynomial.
- $y = 2x + 5$ is a linear equation. It's equivalent to the linear equation $y - 2x - 5 = 0$.
- $x = y$ is a linear equation. It's equivalent to $x - y = 0$.
- $x = 3$ is a linear equation. It's equivalent to $x - 3 = 0$.
- $y = -2$ is a linear equation. It's equivalent to $y + 2 = 0$.

Linear equations are called linear because their sets of solutions form straight lines in the plane. We'll have more to say about these lines in the rest of this chapter.



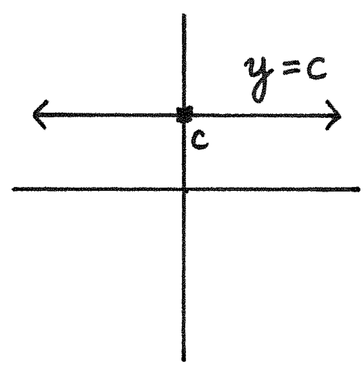
Vertical lines

The solutions of a linear equation $x = c$, where $c \in \mathbb{R}$ is a constant, form a vertical line: the line of all points in the plane whose x-coordinates equal c.



Horizontal lines

If c is a constant number, then the horizontal line of all points in the plane whose y-coordinates equal c is the set of solutions of the equation $y = c$.

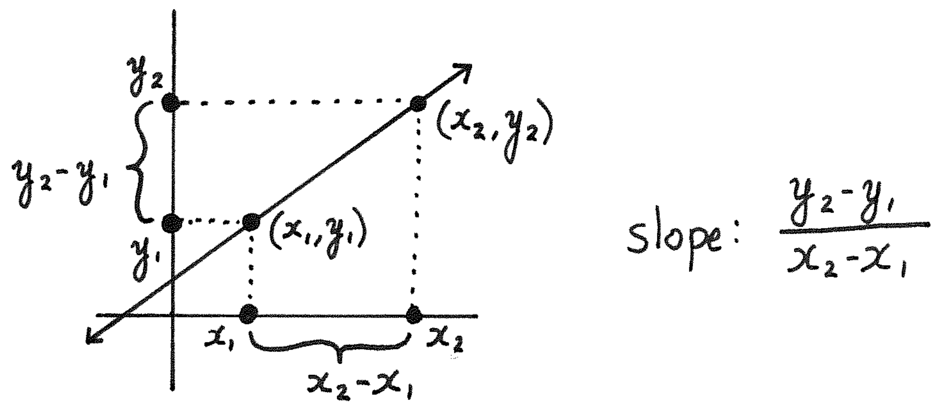


Slope

The slope of a line is the ratio of the change in the second coordinate to the change in the first coordinate. In different words, if a line contains the two points (x_1, y_1) and (x_2, y_2), then the slope is the change in the y-coordinate – which equals $y_2 - y_1$ – divided by the change in the x-coordinate – which equals $x_2 - x_1$.

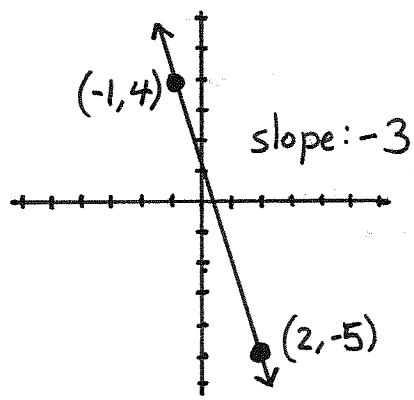
Slope of line containing (x_1, y_1) and (x_2, y_2):

$$\frac{y_2 - y_1}{x_2 - x_1}$$

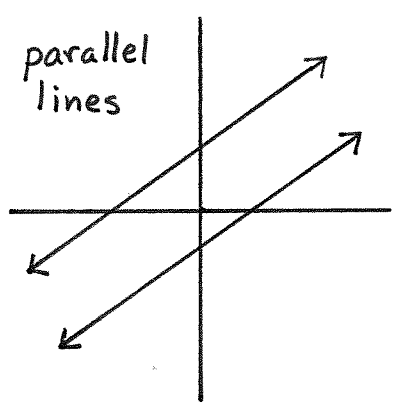


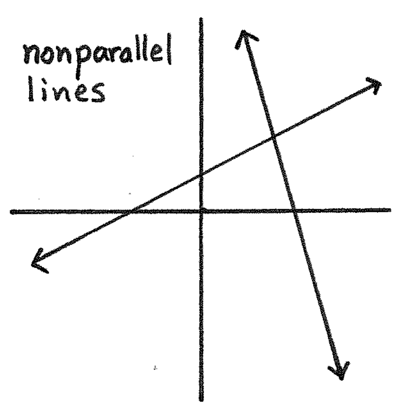
Example: The slope of the line containing the two points $(-1, 4)$ and $(2, -5)$ equals

$$\frac{-5-4}{2-(-1)} = \frac{-9}{3} = -3$$



Lines with the same slope are either equal or parallel, meaning they never intersect.

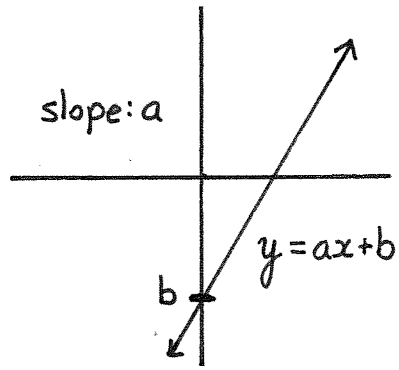




Slope-intercept form for linear equations

In calculus, the most common form of linear equation you'll see is $y = ax+b$, where a and b are constants. For example $y = 2x - 5$ or $y = -2x + 7$. An equation of the form $y = ax + b$ is linear, because it's equivalent to $y - ax - b = 0$. An equation of the form $y = ax + b$ is called a linear equation in *slope-intercept* form.

Claim: The solutions of the equation $y = ax + b$ (where a and b are numbers) form a line of slope a that contains the point $(0, b)$ on the y-axis.



Proof: That $(0, b)$ is a solution of $y = ax + b$ is easy to check. Just replace x with 0 and y with b to see that

$$(b) = b = a(0) + b$$

The point $(1, a + b)$ is also on the line for $y = ax + b$:

$$(a + b) = a + b = a(1) + b$$

Now that we know two points on the line, $(0, b)$ and $(1, a + b)$, we can find the slope of the line. The slope is

$$\frac{(a+b)-b}{1-0} = \frac{a}{1} = a$$

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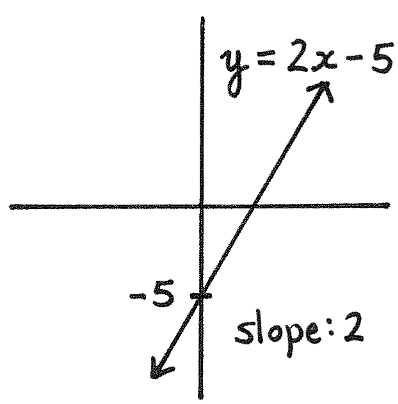
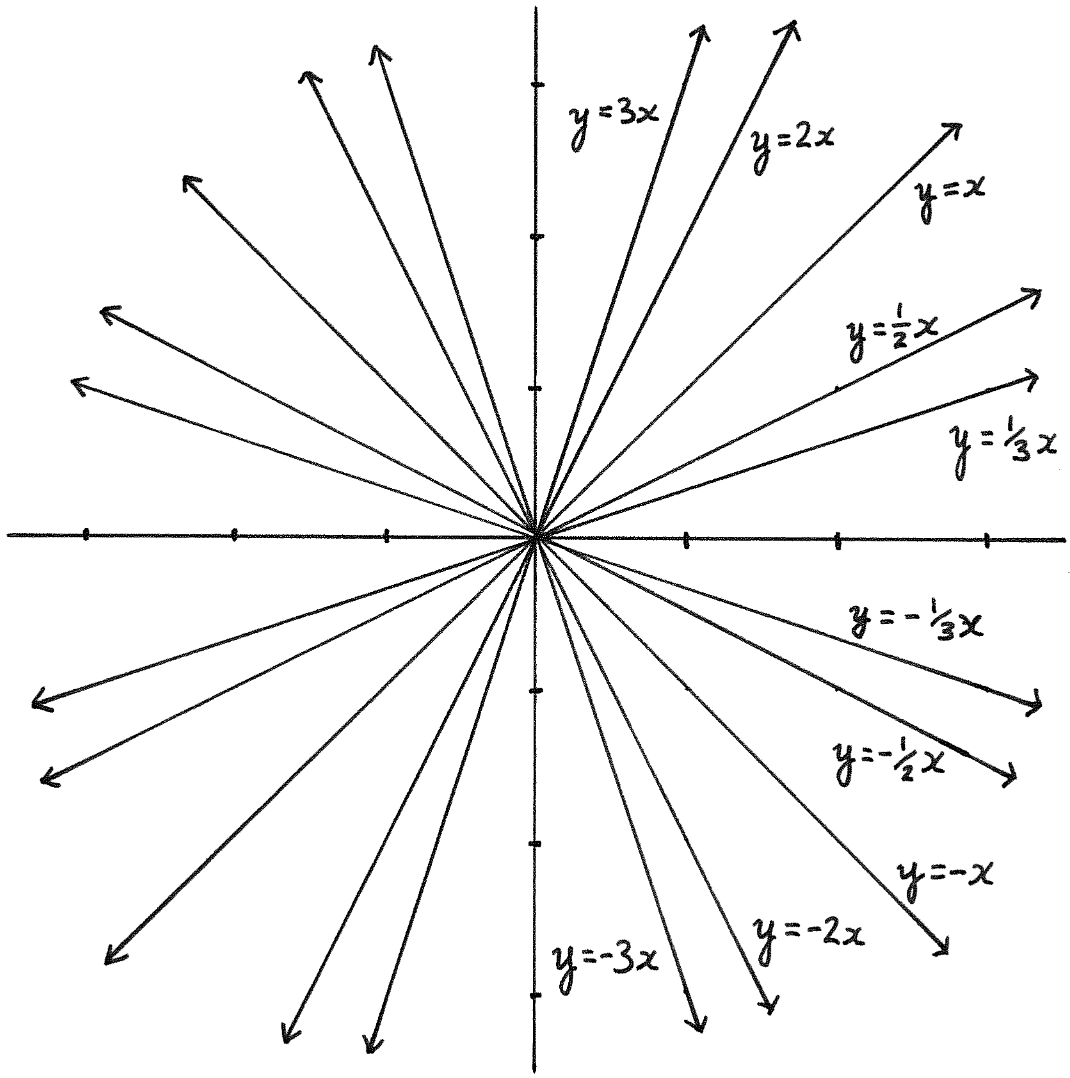


Claim: $y = ax$ is a line of slope a that contains the point $(0, 0)$.



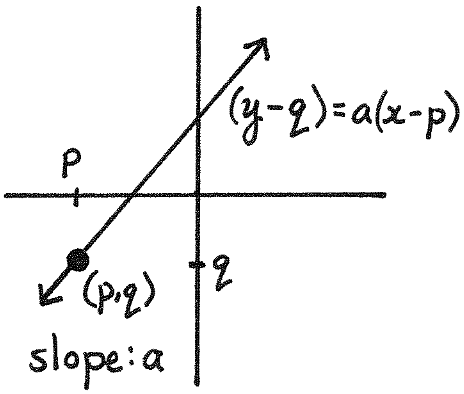
Proof: This claim follows from the previous claim if we write $y = ax$ as $y = ax + 0$. The previous claim tells us that $y = ax + 0$ a line of slope a that contains the point $(0, 0)$. ■

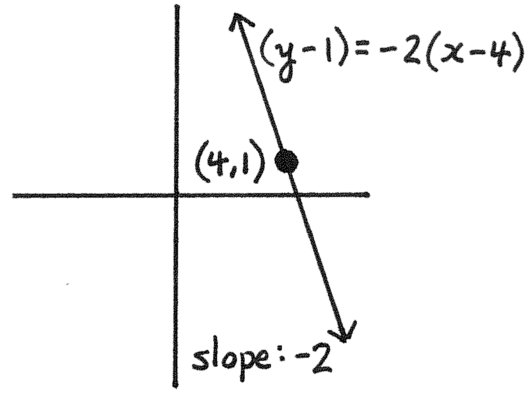
Point-slope form for linear equations

Another common way to write linear equations is as $(y - q) = a(x - p)$, where $a, p, q \in \mathbb{R}$ are constants. This is called the *point-slope* form of a linear equation. Examples include $(y - 2) = 7(x - 3)$ and $(y - 1) = -2(x + 4)$.

Claim: Let $L \subseteq \mathbb{R}^2$ be the line containing the point $(p, q) \in \mathbb{R}^2$ and having slope equal to the number a. Then $(y - q) = a(x - p)$ is an equation for L.

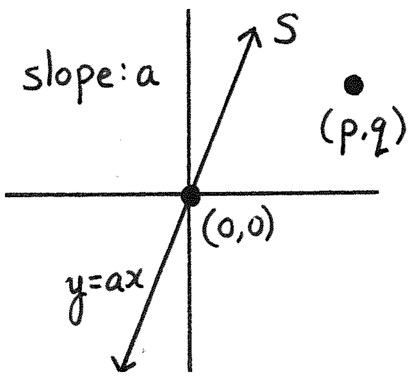
As an example of the claim, $(y - 1) = -2(x - 4)$ is a line of slope -2 that passes through the point in the plane $(4, 1)$.

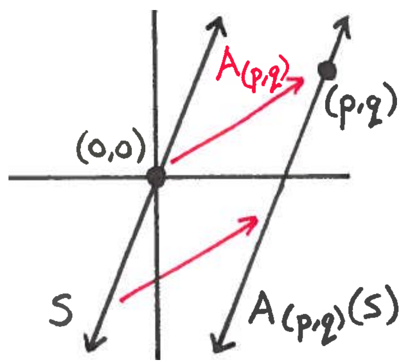


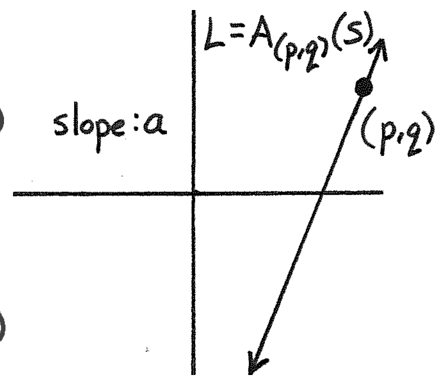


Now let's turn to the proof of the claim.

Proof: Let S be the set of solutions of $y = ax$. As shown in the previous claim, S is a line in the plane whose slope equals a and that contains the point $(0, 0)$.







We can use the addition function $A_{(p,q)}$ to shift the line S horizontally by p, and vertically by q. This new line, $A_{(p,q)}(S)$, is parallel to our original line, so its slope also equals a. The line $A_{(p,q)}(S)$ also contains the point (p, q), because $(0, 0) \in S$ and thus $(p, q) = A_{(p,q)}(0, 0) \in A_{(p,q)}(S)$.

Because $A_{(p,q)}(S)$ has slope a and contains the point (p, q), we see that $A_{(p,q)}(S)$ is the line L from the claim whose equation we would like to identify, and we can find the equation for$A_{(p,q)}(S) = L$ using POTS:

<u>equation for S:</u> $y = ax$

<u>$A^{-1}_{(p,q)} = A_{(-p,-q)}$:</u> $x \longmapsto x - p$

$y \longmapsto y - q$

<u>equation for $L = A_{(p,q)}(S)$:</u> $(y - q) = a(x - p)$

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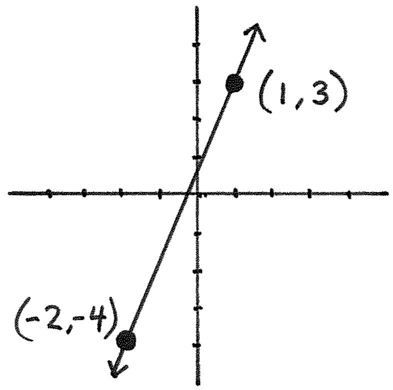
Problem: Find an equation for the line containing the two points $(1, 3)$ and $(-2, -4)$.



Solution: First we'll find the slope of the line. It's

$$\frac{-4-3}{-2-1} = \frac{-7}{-3} = \frac{7}{3}$$

Second, we'll use the previous claim which tells us that a line containing the point $(1, 3)$ and with slope $\frac{7}{3}$ has as an equation

$$(y-3) = \frac{7}{3}(x-1)$$

Exercises

For #1-4, find the slope of the line that contains the two given points in the plane.

1.) $(1, 2)$ and $(3, 4)$

2.) $(5, 10)$ and $(0, 0)$

3.) $(-2, 3)$ and $(2, -1)$

4.) $(3, 2)$ and $(-6, -7)$

For #5-10, identify the slope of the line that is the set of solutions of the given linear equation in slope-intercept form.

5.) $y = 2x + 3$

6.) $y = -4x + 7$

7.) $y = -x - 2$

8.) $y = x + 5$

9.) $y = 17$

10.) $y = 3x - 4$

For #11-16, identify the number b with the property that $(0, b)$ is a point in the line of the solutions of the given linear equation. This number is called the *y-intercept.*

11.) $y = 3x + 8$

12.) $y = 2x + 9$

13.) $y = -3x - 7$

14.) $y = 9$

15.) $y = 5x - 8$

16.) $y = 2x + 5$

For #17-20, write an equation for a line that has the given slope and contains the given point. Write answers in point-slope form $(y-q) = a(x-p)$.

17.) Slope: 4. Point: $(2, 3)$.

18.) Slope: -2. Point: $(-3, 4)$.

19.) Slope: -1. Point: $(1, -5)$.

20.) Slope: 7. Point: $(-4, -3)$.

For #21-24, provide an equation for a line that contains the given pair of points. Write your answers in the slope-intercept form $y = ax + b$.

21) $(8, -2)$ and $(2, 5)$

22.) $(7, 3)$ and $(2, 1)$

23.) $(-3, 5)$ and $(3, 4)$

24.) $(2, 8)$ and $(-2, -5)$

Find the inverse matrices.

25.) $\begin{pmatrix} 3 & 2 \\ -1 & -2 \end{pmatrix}^{-1}$

26.) $\begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}^{-1}$

Find the solutions of the following equations in one variable.

27.) $e^{2x-3} = 0$

28.) $\log_3(x)^2 + 3 = 4\log_3(x)$

Recall that $p_X : \mathbb{R}^2 \to \mathbb{R}$ where $p_X(x, y) = x$ is called the projection onto the x-axis, and that $p_Y : \mathbb{R}^2 \to \mathbb{R}$ where $p_Y(x, y) = y$ is the projection onto the y-axis. In the remaining questions, find the appropriate values.

29.) $p_X(3, 7)$

30.) $p_X(2, 8)$

31.) $p_Y(-2, -5)$

32.) $p_Y(4, 0)$

33.) $p_X(1, 8)$

34.) $p_Y(6, 9)$

35.) $p_Y(-1, 7)$

36.) $p_X(2, 3)$