Inverse Functions

In this chapter we'll cover what an inverse function is, when a function has an inverse, what inverse functions are useful for, how to graph an inverse function, and how to find the inverse of a function.

One-to-one

Suppose $f : A \to B$ is a function. We call f *one-to-one* if every distinct pair of objects in A is assigned to a distinct pair of objects in B. In other words, each object of the target has at most one object from the domain assigned to it.

There is a way of phrasing the previous definition in a more mathematical language:

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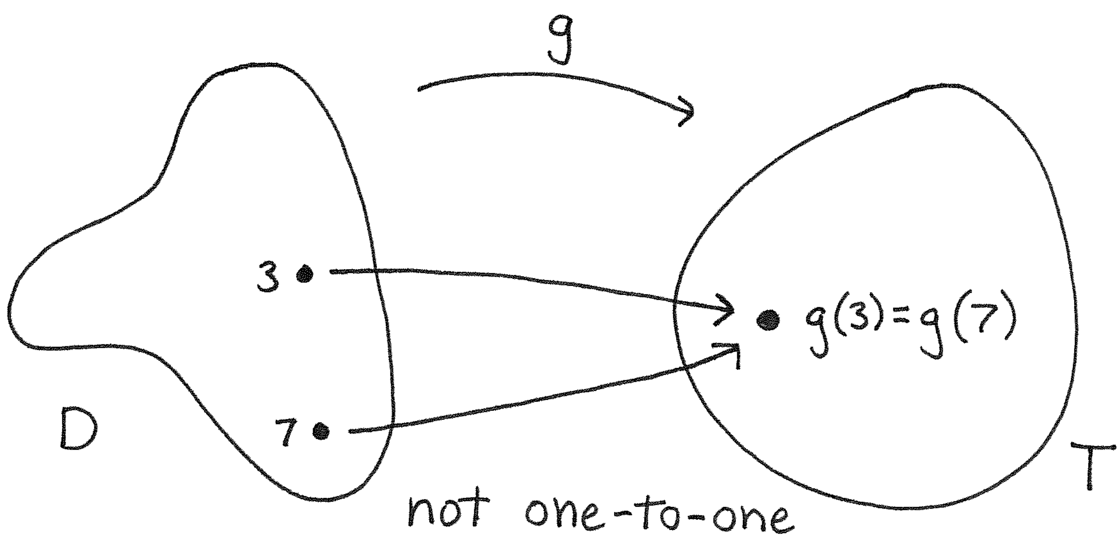
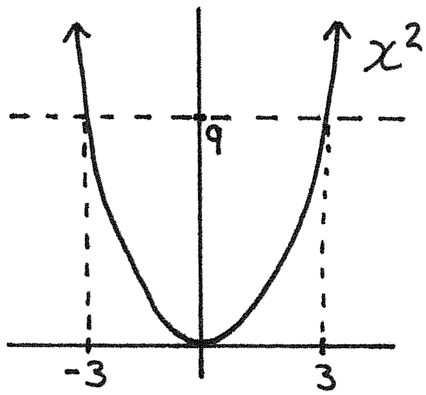
f is *one-to-one* if whenever we have two objects, a and c, in the domain of f with $a \neq c$, then we are guaranteed that $f(a) \neq f(c)$.

The picture below illustrates a function that is not one-to-one, since as you can see in the picture, there are two different objects in the domain of $g : D \to T$ that are assigned by g to the same object in the target. Namely, $3 \neq 7$ but $g(3) = g(7)$.



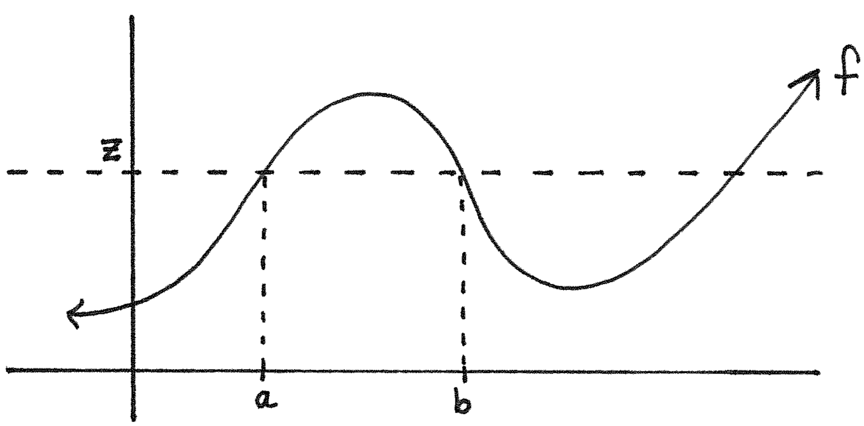
Example. $h : \mathbb{R} \to \mathbb{R}$ where $h(x) = x^2$ is not one-to-one because $3 \neq -3$ and yet $h(3) = h(-3)$ since $h(3)$ and $h(-3)$ both equal 9.



Horizontal line test

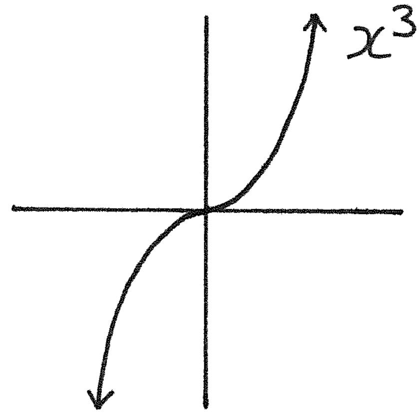
If a horizontal line intersects the graph of $f(x)$ in more than one point, then $f(x)$ is not one-to-one.

The reason $f(x)$ would not be one-to-one is that the graph would contain two points that have the same second coordinate. In the picture below, we have drawn a function f that intersects the horizontal line at height z in more than one point, for example in the points (a, z) and (b, z). That means that $f(a) = z$ and $f(b) = z$, and one-to-one functions can't assign two different objects in the domain to the same object of the target.



If *every* horizontal line in the plane intersects the graph of a function *at most once*, then the function is one-to-one.

Example. Below is the graph of $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x^3$. Any horizontal line that could be drawn would intersect the graph of g in at most one point, so g is one-to-one.



Range

Recall that the target of a function $f : D \to T$ is the set T. That means that for any $d \in D$, we have that $f(d) \in T$, or in other words, whenever we put an object from the domain into the function, the output is an object in the target.

But it might not be that every object in T has an object from D assigned to it by the function f. For example, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ has $\mathbb{R}$ as its target. However, if you square a real number, the result is never a negative number ($2^2 = 4$, $(-3)^2 = 9$, etc.). That is, none of the negative numbers in the target have a number from the domain assigned to it.

The *range* of a function f is the subset of the target consisting of objects that actually "come out of" f.

- Let $h : \{1, 2, 3, 4\} \to \{3, 4, 7, 8, 9\}$ be the function given by

$$h(1) = 9 \qquad h(2) = 4$$

$$h(3) = 4 \qquad h(4) = 8$$

If we put the numbers from the domain "in to" h, the only numbers that "come out" are 9, 4, and 8. That means that the range of h is $\{9, 4, 8\}$.

- If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x - 2$, then $g(3) = 3 - 2 = 1$. We put 3 in to g, and got 1 out, so 1 is an object in the range.

Using our notation for sets, we could write the range of $f : D \to T$ as the set

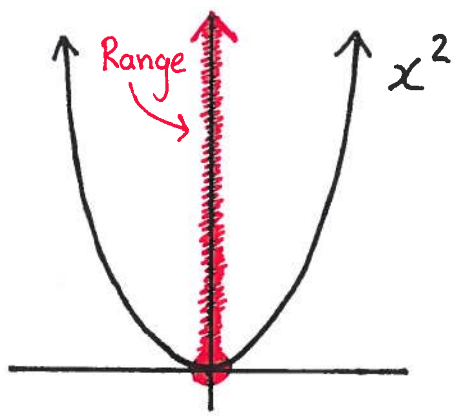
$$\{ y \in T \mid f(x) = y \text{ for some } x \in D \}$$

Meaning that the range of f is the subset of the target consisting of objects that are assigned an object from the domain.

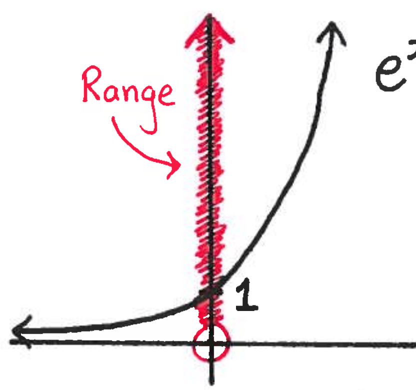
Another way to say what the range is, is to say that it is the smallest set that can serve as the target of the function.

Examples.

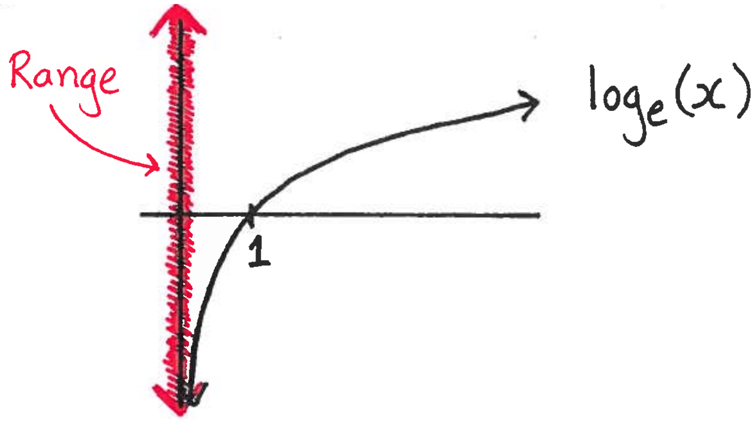
- The range of $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is the set $[0, \infty)$. That's because if you square a real number, the result will never be negative, but it could be 0, and it could be any positive number.



- The range of $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = e^x$ is the set $(0, \infty)$. If $x \in \mathbb{R}$ then we know that e^x could be any positive number. It's never 0, and it's never negative.



• The range of $\log_e : (0,\infty) \to \mathbb{R}$ is $\mathbb{R}$. Any real number is an output of $\log_e$.



Onto

Suppose $f : D \to T$ is a function. We call f *onto* if the range of f equals T.

In other words, f is onto if every object in the target has at least one object from the domain assigned to it by f.

Examples.

• The graph of $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is drawn on the previous page. The range of f is $[0,\infty)$. The target of f is $\mathbb{R}$, and $[0,\infty) \neq \mathbb{R}$, so f is not onto.

• The graph of $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = e^x$ is drawn on the previous page. The function g has the set $(0,\infty)$ for its range while its target is $\mathbb{R}$. Because $(0,\infty) \neq \mathbb{R}$, g is not onto.

• The target of $\log_e : (0,\infty) \to \mathbb{R}$ equals the range of $\log_e : (0,\infty) \to \mathbb{R}$. They are both $\mathbb{R}$. So $\log_e : (0,\infty) \to \mathbb{R}$ is onto.

• Let's compare the first example above — the example $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ — to the function $h : \mathbb{R} \to [0,\infty)$ where $h(x) = x^2$. The two functions are similar, and the only difference between the two functions is what their target is.

For the same reason that the range of f is $[0,\infty)$, the range of h is also $[0,\infty)$. Because the target of h is also $[0,\infty)$, the function h is onto, even though the very similar function f is not onto. What makes a function onto is just whether its target is chosen to be the range.

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Inverse functions

The defining characteristic of the cube root function is that for any $x \in \mathbb{R}$ we have that

$$\sqrt[3]{x^3} = x \qquad \text{and} \qquad (\sqrt[3]{x})^3 = x$$

This is just an example of the definition of inverse functions.

Suppose $f : A \to B$ is a function. A function $g : B \to A$ is called the *inverse function* of f if

$$f \circ g = id \qquad \text{and} \qquad g \circ f = id$$

The line above says that f and g are inverse functions if $f \circ g$ and $g \circ f$ are each the identity function, meaning that for any x we have

$$f \circ g(x) = id(x) \qquad \text{and} \qquad g \circ f(x) = id(x)$$

Remember that $id(x) = x$, so the line above can be written more simply as

$$f \circ g(x) = x \qquad \text{and} \qquad g \circ f(x) = x$$

If g is the inverse function of f, then we often rename g as f^{-1}. Then we can rewrite the equations above replacing g with f^{-1} and we have

$$f \circ f^{-1} = id \quad \text{and} \quad f^{-1} \circ f = id$$

or equivalently, that for any x

$$f \circ f^{-1}(x) = x \quad \text{and} \quad f^{-1} \circ f(x) = x$$

Examples.

- Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 2x + 2$. Then f has an inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ where $f^{-1}(x) = \frac{1}{2}x - 1$. Notice that

$$f \circ f^{-1}(x) = f(f^{-1}(x)) = f\Big(\frac{1}{2}x - 1\Big) = 2\Big(\frac{1}{2}x - 1\Big) + 2 = x$$

Similarly

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(2x+2) = \frac{1}{2}\Big(2x+2\Big) - 1 = x$$

• The inverse of addition is subtraction. If $g(x) = x + 5$, then $g^{-1}(x) = x - 5$.

• The inverse of multiplication is division. If $h(x) = 3x$, then $h^{-1}(x) = \frac{x}{3}$.

• The inverse of an odd power is an odd root. If $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$.

• The inverse of exponential with base $a > 0$ (as long as $a \neq 1$) is logarithm base a. If $g(x) = 2^x$ then $g^{-1}(x) = \log_2(x)$.

• The inverse of $\frac{1}{x}$ is itself. If $h(x) = \frac{1}{x}$, then $h^{-1}(x) = \frac{1}{x}$.

• Even powers do not have inverses exactly. They do have partial inverses though, and those are the even roots. To repeat, the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ does not have an inverse function. However, we can restrict the domain and target of f to obtain the function $f : [0, \infty) \to [0, \infty)$ where $f(x) = x^2$. This function does have an inverse, the square-root function $\sqrt[2]{\ } : [0, \infty) \to [0, \infty)$. Because the domain of the square-root function is $[0, \infty)$, it wouldn't make any sense to try to take the square-root of a negative number.

The Inverse of an inverse is the original

If f^{-1} is the inverse of f, then $f^{-1} \circ f = id$ and $f \circ f^{-1} = id$. We can see from the definition of inverse functions above, that f is the inverse of f^{-1}. That is $(f^{-1})^{-1} = f$.

For example, as the inverse of addition is subtraction, the inverse of subtraction is addition. As the inverse of an exponential function is a logarithm, the inverse of a logarithm is an exponential function.

Inverse functions "reverse the assignment"

The definition of an inverse function is given above, but the essence of an inverse function is that it reverses the assignments of the original function. The first chart below shows inputs and their respective outputs for the function x^2. The second chart shows inputs and their respective outputs for the inverse function $\sqrt[2]{x}$. If you reverse the two columns of numbers for the x^2 chart, you'll get the $\sqrt[2]{x}$ chart.

x	x^2
1	1
2	4
3	9
4	16
5	25

x	$\sqrt[2]{x}$
1	1
4	2
9	3
16	4
25	5

The example above isn't particular to x^2 and $\sqrt[2]{x}$. It applies to any pair of inverse functions as the claim below demonstrates.

Claim: For inverse functions f and f^{-1}, if $f(a) = b$, then $f^{-1}(b) = a$.

Proof: If $f(a) = b$, then

$$\begin{aligned} f^{-1}(b) &= f^{-1}(f(a)) \\ &= f^{-1} \circ f(a) \\ &= a \end{aligned}$$

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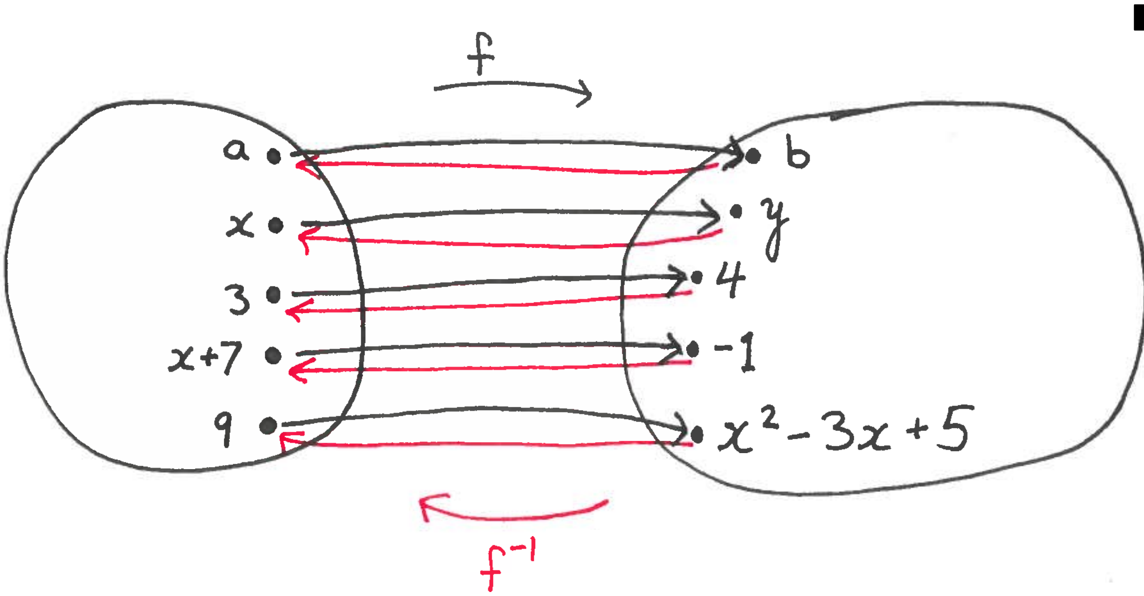

Examples. The claim above states that if $f(a) = b$, then $f^{-1}(b) = a$. The examples below are written in the style of the equivalent statement that if $f(a) = b$, then $a = f^{-1}(b)$.

- If $f(3) = 4$, then $3 = f^{-1}(4)$.
- If $f(-2) = 16$, then $-2 = f^{-1}(16)$.
- If $f(x+7) = -1$, then $x + 7 = f^{-1}(-1)$.
- If $f^{-1}(0) = -4$, then $0 = f(-4)$.
- If $f^{-1}(x^2 - 3x + 5) = 9$, then $x^2 - 3x + 5 = f(9)$.
- Because $5^2 = 25$, we know that $5 = \sqrt[2]{25}$.
- Because $3^4 = 81$, we know that $4 = \log_3(81)$.

In the 7 examples above, we "erased" a function from the left side of the equation by applying its inverse function to the right side of the equation.

When a function has an inverse

A function has an inverse exactly when
it is *both* one-to-one and onto

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The Graph of an inverse

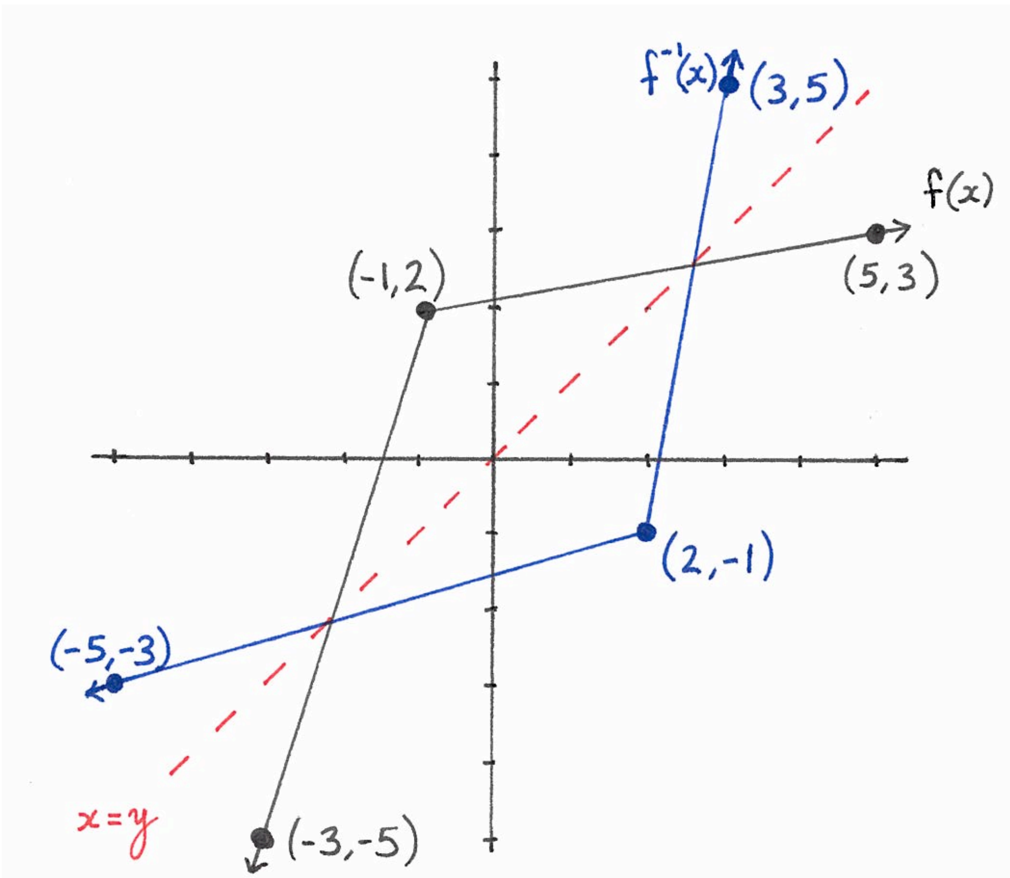
If f is an *invertible* function (that means if f is a function that has an inverse function), and if you know what the graph of f looks like, then you can draw the graph of f^{-1}. The claim below will tell us how.

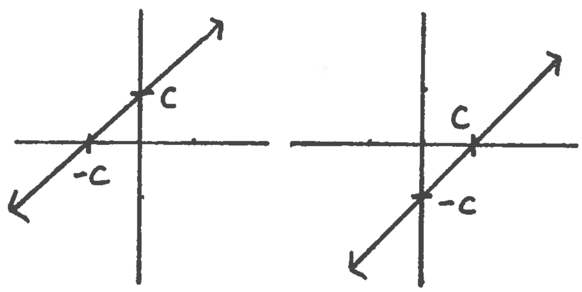
Claim: If (a, b) is a point in the graph of $f(x)$, then (b, a) is a point in the graph of $f^{-1}(x)$.

Proof: If (a, b) is a point in the graph of $f(x)$, then $f(a) = b$, and our claim on the previous page tells us that $f^{-1}(b) = a$. That means f^{-1} assigns b to a, or in other words, that (b, a) is a point in the graph of $f^{-1}(x)$. ■

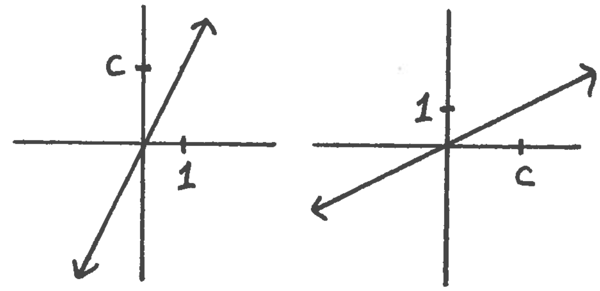
What the above claim tells us is that if we know the graph of an invertible function, we can just switch all of the first and second coordinates of our graph and we'll be left with a graph of the inverse function. Geometrically, when you switch all the first and second coordinates of points in the plane, the result is to flip the plane over the "$x = y$ line".

Examples.

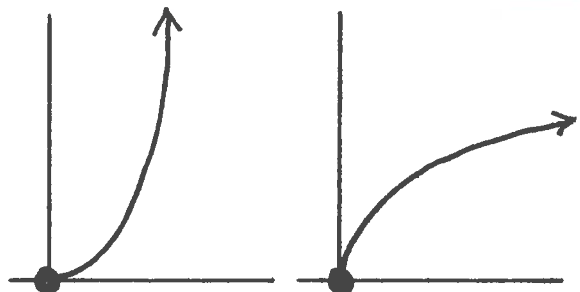




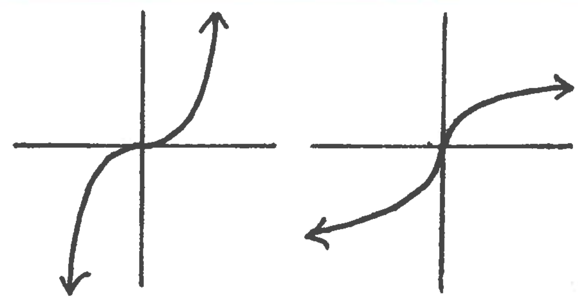
$x + c$ $\quad$ $x - c$



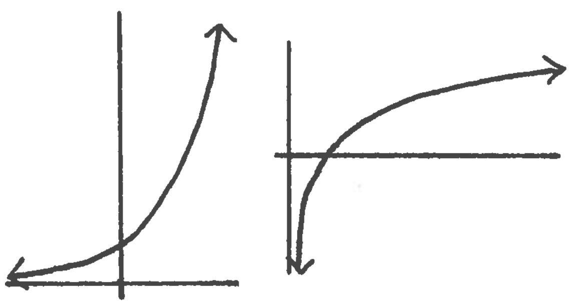
cx $\quad$ $\frac{x}{c}$ $(c \neq 0)$



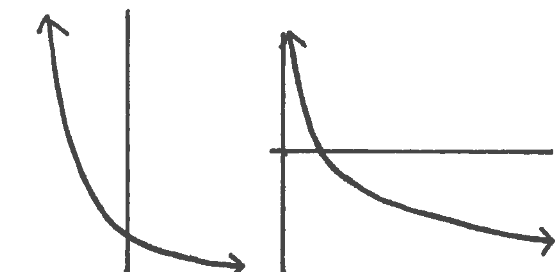
x^n $\quad$ $\sqrt[n]{x}$ $(n \geq 2 \text{ even})$



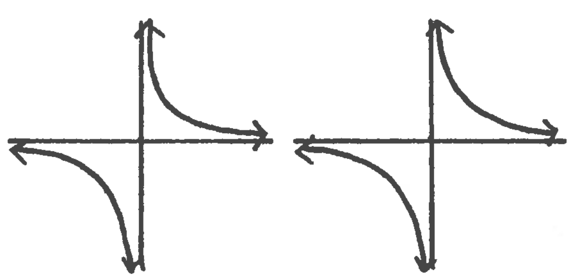
x^n $\quad$ $\sqrt[n]{x}$ $(n \geq 3 \text{ odd})$



a^x $\quad$ $\log_a(x)$ $(a > 1)$



a^x $\quad$ $\log_a(x)$ $(0 < a < 1)$



$1/x$ $\quad$ $1/x$

Exercises

It's easy to solve equations that are formed from inverse functions. For example, if f is a function with an inverse function f^{-1}, and if we are given the equation $f(x+3) = 4$, then we can solve for the variable x by erasing the function f and applying its inverse to the other side of the equation: $x + 3 = f^{-1}(4)$. Next, we can erase addition by 3 by applying its inverse, subtraction by 3, to the other side of the equation: $x = f^{-1}(4) - 3$. We have solved for x.

Just remember,

If	$f(\text{stuff}) = (\text{other stuff})$
then	$(\text{stuff}) = f^{-1}(\text{other stuff})$

For #1-18, solve for x.

1.) $x + 3 = 7$

2.) $3x = 60$

3.) $x^3 = 7$

4.) $\frac{1}{x} = 10$

5.) $e^x = 5$

6.) $x - 2 = 4$

7.) $\sqrt[3]{x} = 2$

8.) $\frac{x}{4} = 20$

9.) $\frac{1}{x} = \frac{1}{3}$

10.) $2x + 5 = 3$

11.) $10^{x-1} = 17$

12.) $\frac{x-1}{3} = 7$

13.) $\log_e(2x) = 4$

14.) $(x+2)^3 = \frac{1}{8}$

15.) $\frac{1}{x+1} = 2$

16.) $\sqrt[3]{7-x} = -4$

17.) $\log_2(3x-1) = 4$

18.) $\frac{1}{2x+3} = 4$

If you know that f is an invertible function, and you have an equation for $f(x)$, then you can find the equation for f^{-1} in three steps.

Step 1 is to replace $f(x)$ with the letter y.

Step 2 is to use algebra to solve for x.

Step 3 is to replace x with $f^{-1}(y)$.

After using these three steps, you'll have an equation for the function $f^{-1}(y)$.

Example: To find the inverse of $f(x) = \log_e(x-5)$, first replace $f(x)$ with y to get $y = \log_e(x-5)$. Second, solve for x as follows: Erase $\log_e$ from the right side of the equation by applying its inverse, exponential base e to the left side of the equation and we have $e^y = x - 5$. Next, we can erase subtraction by 5 on the right by applying its inverse, addition by 5, on the left. That gives us $e^y + 5 = x$. The third step is to simply replace x with $f^{-1}(y)$, and that gives us an equation for the inverse function, $f^{-1}(y) = e^y + 5$.

Some people prefer to use x as the variable for their functions whenever they can. If you do, that's fine, just change the y's on the last line above to x's. That is, $f^{-1}(y) = e^y + 5$ is exactly the same function as $f^{-1}(x) = e^x + 5$, or for that matter, it's also exactly the same function as $f^{-1}(z) = e^z + 5$.

For exercises #19-30, find the inverse function of the given function.

19.) $f(x) = x + 5$

20.) $g(x) = \frac{2}{x-1}$

21.) $h(x) = 2x$

22.) $f(x) = e^{4x}$

23.) $g(x) = \frac{x}{10}$

24.) $h(x) = \sqrt[3]{x-4}$

25.) $f(x) = \frac{3x}{x-2}$

26.) $g(x) = \log_{10}(x+2)$

27.) $h(x) = (2x+1)^3 + 4$

28.) $f(x) = 5^{2x+3} + 1$

29.) $g(x) = x - 4$

30.) $h(x) = \log_e(3x-1)$

For exercises #31-32, find the roots of the quadratic polynomials.

31.) $2x^2 - 3x - 1$

32.) $4x^2 - x - 3$