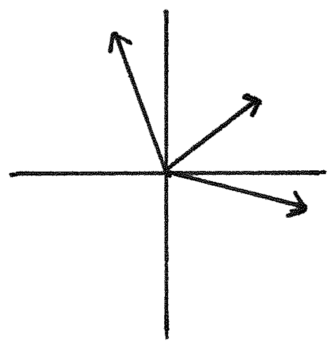
Introduction

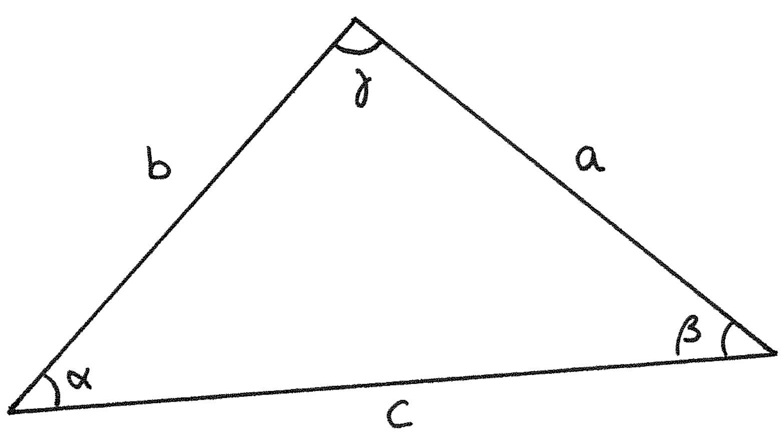
This text will begin with a brief review of some precalculus material that you are assumed to be comfortable with: quadratic polynomials, exponentials, and logarithms, as a few examples. We'll then see a brief introduction to linear algebra in the plane. This is the study of vectors in the plane, of the algebra relating to vectors, and of matrices, which are the most important functions that are used to relate vectors to each other.



vectors

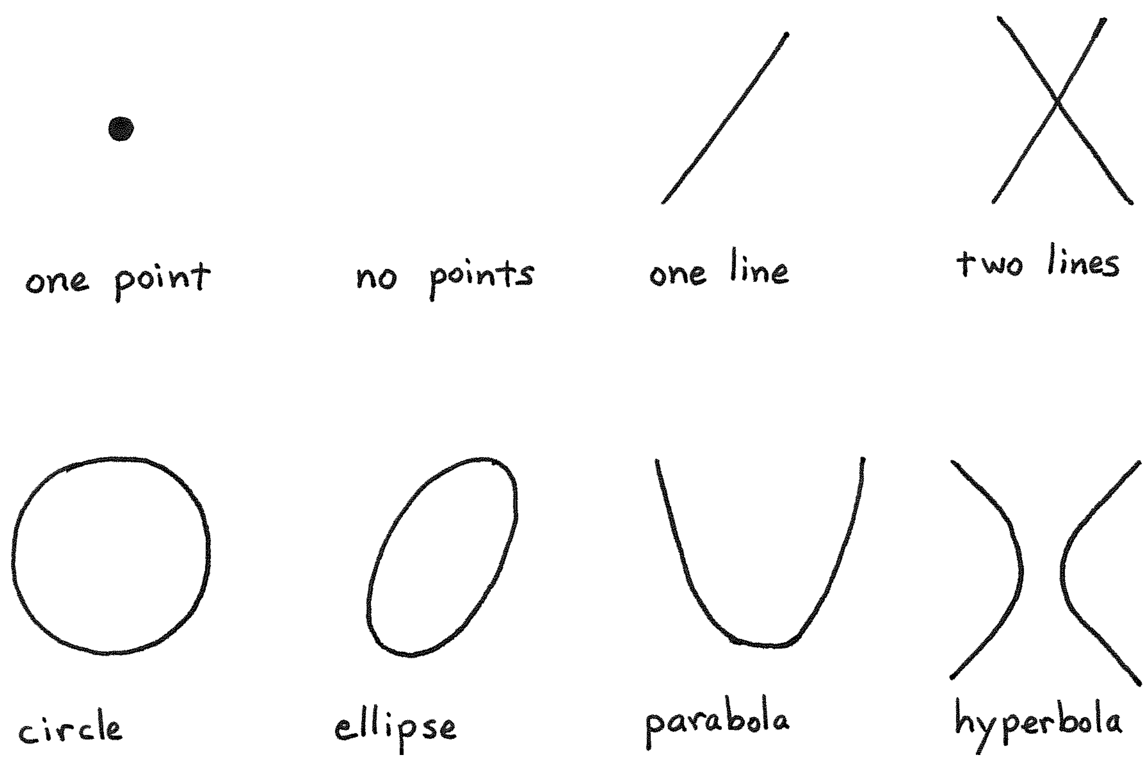
Linear algebra is not the primary interest of this course, so we'll keep our treatment of it brief, but it will be important for us to be comfortable with some basic linear algebra in order to better study the topics that this course is devoted to, namely, trigonometry, conics, and the complex numbers.

Trigonometry is the study of triangles. More precisely, it is the study of the relationship between the lengths of the sides of a triangle and the interior angles of a triangle.



As you've probably seen before, and as is drawn in the picture above, angles are usually drawn as circular arcs. This is because angles are naturally related to the geometry of the circle, and while the historical roots of trigonometry lie in the study of triangles, trigonometry now has as much to do with circular objects as it does with triangles. As an illustration of this, we'll see that the geometry of circular motions, called rotations, are described using trigonometry.

Conics are solutions to quadratic equations in two variables, such as $5x^2 - 2xy + y^2 - 6x + 2y + 3 = 0$. Conics are common objects in math and just about any field that uses math, so it's important to have a firm understanding of them. We'll see that there are 8 basic types of conics: those that consist of a single point, those that consist of no points, those that are a single straight line, those that are two straight lines, and those that are circles, ellipses, parabolas, or hyperbolas.



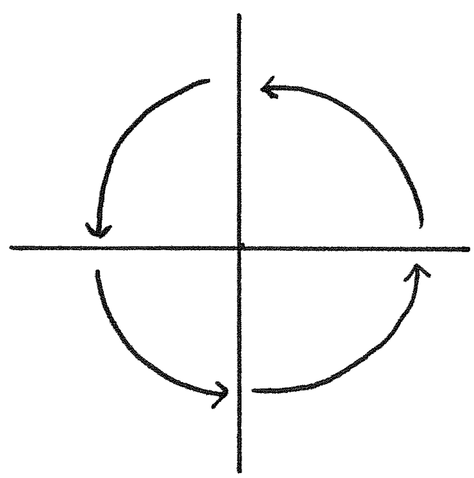
Trigonometry and the linear algebra relating to rotations will play an important role in leading us to see that any quadratic equation in two variables is one of the 8 types above.

Complex numbers were either invented or discovered, depending on your preference of language, in the 16th century as a means for finding real number roots to cubic polynomials with real number coefficients. More precisely, what was discovered in the 16th century was "the cubic formula", which is a counterpart to the quadratic formula that you already know. The cubic formula requires the use of complex numbers to work it, even if the formula is only used to find roots that are real numbers.

For example, the cubic polynomial $x^3 - 15x - 4$ has real number coefficients: 1, -15, and -4. Using the cubic formula, and the algebra of complex numbers, we can find that $x^3 - 15x - 4$ has three roots which are all real numbers: 4, $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$. [1]

Later in this course it will be explained what the complex numbers are exactly, but for now, the complex numbers are a set of numbers that contain the set of real numbers, and that contain many other numbers as well, including the most famous example of a complex number, the number i. The defining equation of the number i is that $i^2 = -1$. Since the square of any real number is positive, we see that i is not a real number.

The connection between the complex numbers and trigonometry is that, while addition of complex numbers is best understood using the geometry of vectors, multiplication of complex numbers is best understood using the geometry of rotations.



multiplication by i

[1] There's a nice account of this piece of history in the chapter "Cardano and the Solution of the Cubic" from the book "Journey Through Genius" by William Dunham.

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Preparation for calculus

This is a precalculus text, and it is meant to prepare you for calculus. In addition to learning the topics written about on the previous pages, that preparation will come from learning the algebra that's needed to succeed in a calculus course. Occasionally we'll take a break from the main themes of this course to practice solving equations, using exponentials and logarithms, and graphing functions, among other things. The most obvious examples of these digressions that can be seen from the table of contents are the chapters titled Equations in One Variable I-VII. These chapters are scattered throughout the text, and a mastery of their contents will make learning calculus much easier.

Exercises

The *discriminant* of a quadratic polynomial $p(x) = ax^2 + bx + c$ is $b^2 - 4ac$. What are the discriminants of the following quadratic polynomials?

1.) $3x^2 - 4x + 7$

2.) $x^2 + 5x$

3.) $-x^2 - 10$

4.) $-4x^2 + 2x - 5$

A number α is a *root* of the polynomial $p(x)$ if $p(\alpha) = 0$. If the discriminant of a quadratic polynomial (with real number coefficients) is positive, then the polynomial has two real numbers as roots. If it's 0, then there's only one root, and if it's negative, then there aren't any real numbers that are roots. How many real number roots do the following polynomials have?

5.) $-2x^2 - 3$

6.) $-x^2 + 2x - 1$

7.) $5x^2 - x + 1$

8.) $2x^2 + 3x$

If the quadratic polynomial $p(x) = ax^2 + bx + c$ has real number coefficients, and if the discriminant equals 0, then its only root is the number $-b/2a$. If the discriminant is positive, then there are two roots,

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

which is often abbreviated as

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What are the roots of the following polynomials?

9.) $x^2 + 4x + 1$

10.) $x^2 - x - 12$

11.) $-2x^2 + 4x - 2$

12.) $3x^2 - 2x - 2$