Functions

This is the second of three chapters that are meant primarily as a review of Math 1050. We will move quickly through this material.

A *function* is a way of describing a relationship between two sets.

To have a function we first need two sets, so let's suppose that D and T are sets. Then a *function* is something that assigns every $x \in D$ to a single object in T.

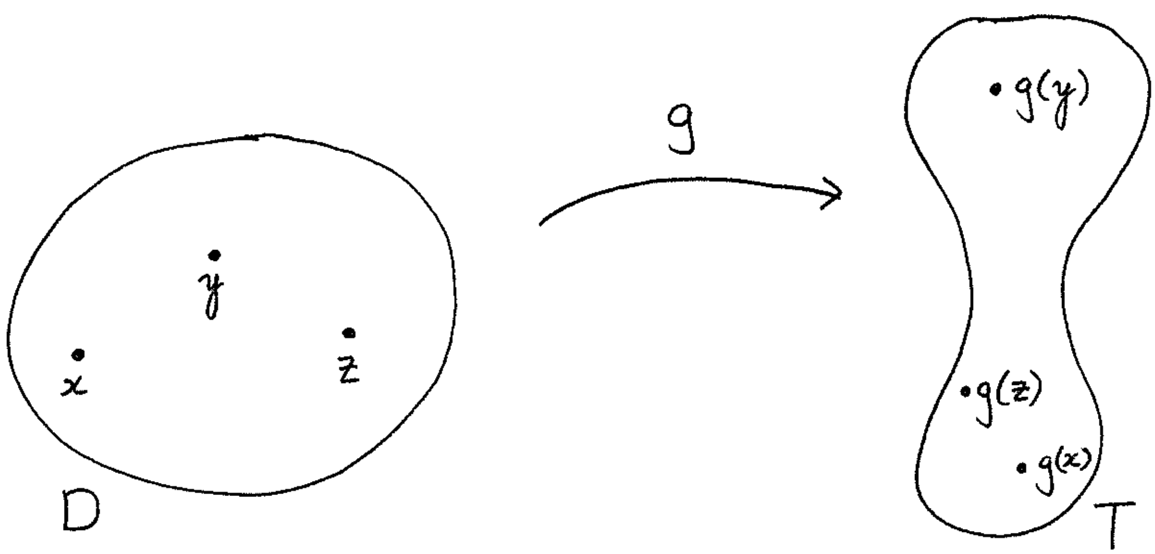
D is called the *domain* of the function, and T is called the *target* of the function. We usually assign names to our functions — though usually simple and generic names — like g, for example. Naming the function lets us give a specific name to the object in the target that the function assigns to an object in the domain as follows:

If $x \in D$, then $g(x) \in T$ is the object that g assigns to x.

Writing the symbols

$$g : D \to T$$

is a shorthand for writing that g is a function that assigns every $x \in D$ to a single object in T.



Polynomials. Some of the most important examples of functions in mathematics are *polynomials*. These are functions of the form $p : \mathbb{R} \to \mathbb{R}$ where $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for some fixed real numbers $a_n, a_{n-1}, \dots, a_0$.

For example, $q : \mathbb{R} \to \mathbb{R}$ where $q(x) = 5x^3 - 2x^2 + x - 1$ is a polynomial. The domain of a polynomial is $\mathbb{R}$. That is, you can put any real number into a polynomial function. If you do put a real number into a polynomial, then the output is another real number. In other words, the target of a polynomial is $\mathbb{R}$.

Identity function. The *identity function* is the function that assigns every object in the domain to itself. (To have an identity function, the domain and target have to be the same set.) Identity functions are important enough that they get to have a name that is reserved for identity functions only: *id*. In other words, if D is a set, then the identity function for the set D is described by

$$id : D \to D \qquad \text{where} \qquad id(x) = x.$$

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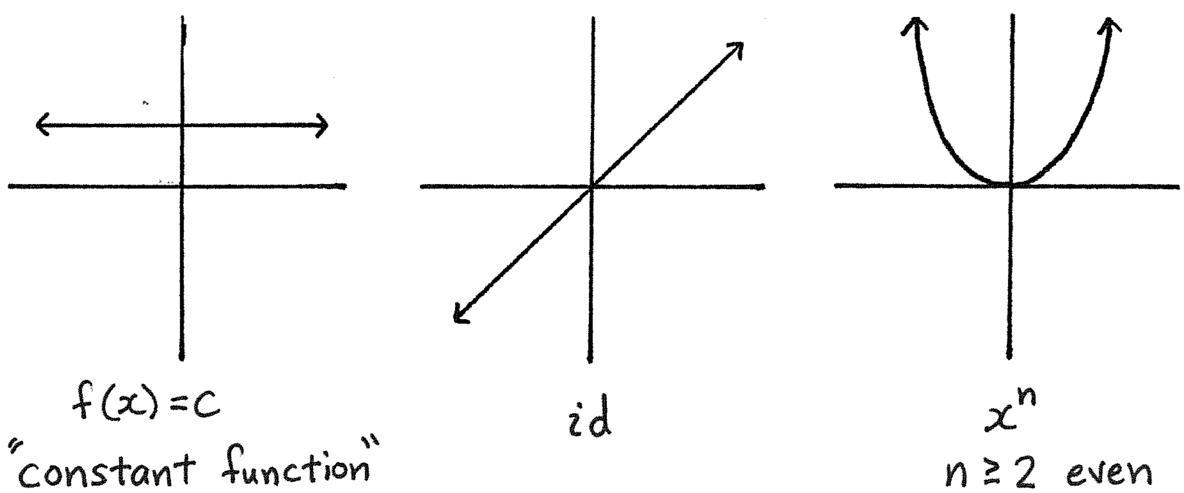
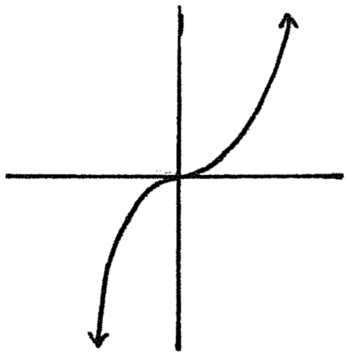
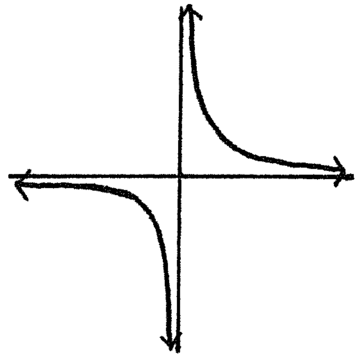
Some important functions

For much of this course, the functions we'll be interested in are functions whose domains are subsets of $\mathbb{R}$. Below are 12 very important examples of functions that should be familiar to you.

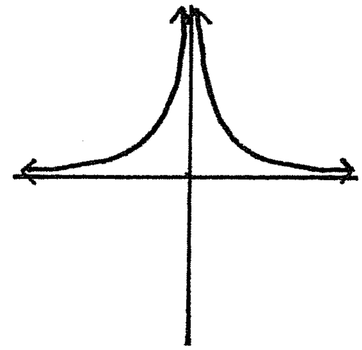




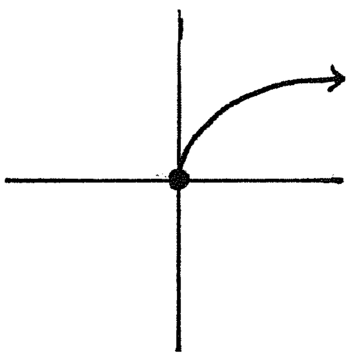
x^n
$n \geq 3$ odd



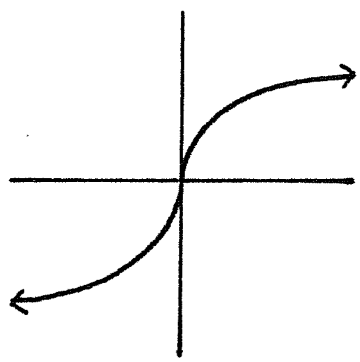
$1/x^n$
$n \geq 1$ odd



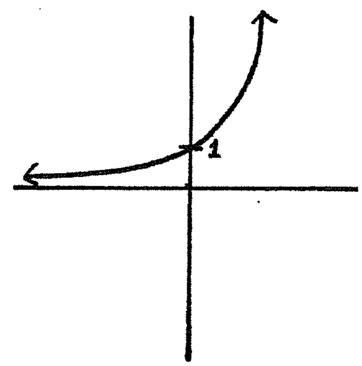
$1/x^n$
$n \geq 2$ even



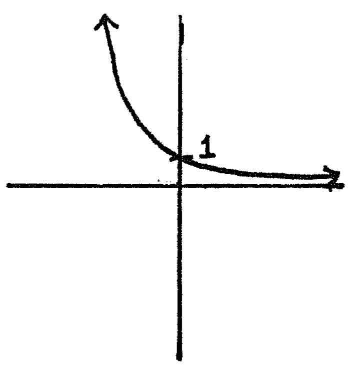
$\sqrt[n]{x}$
$n \geq 2$ even



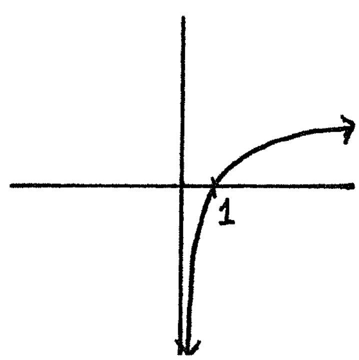
$\sqrt[n]{x}$
$n \geq 3$ odd



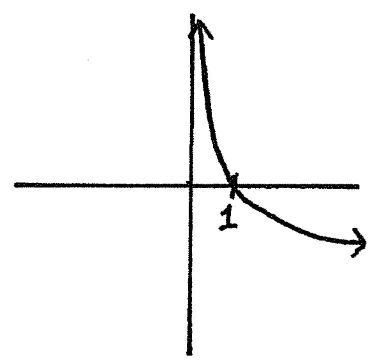
a^x
$a > 1$



a^x
$0 < a < 1$



$\log_a(x)$
$a > 1$



$\log_a(x)$
$0 < a < 1$

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Graphing functions with restricted domains

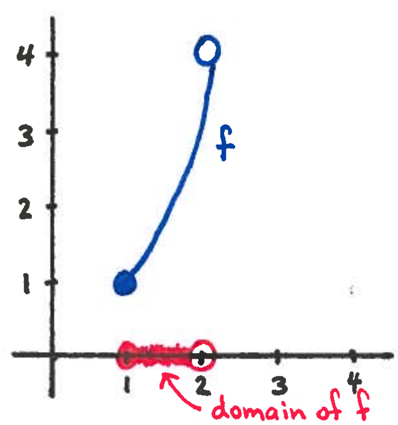
Sometimes we are interested in functions that have smaller domains than perhaps they possibly could have. These smaller domains are sometimes called *restricted domains*.

Examples.

- The function $f : [1, 2) \to \mathbb{R}$ where $f(x) = x^2$ has a restricted domain, because the domain is told to us as being the rather small interval $[1, 2)$ even though it makes sense to square any real number.

If someone gives us a function, and tells us that the domain of that function is $[1, 2)$, then that's its domain, regardless of what we think its domain should have been.

When graphing a function with a restricted domain, we must only draw the graph of the function above or below the restricted domain on the x-axis. That's because the restricted domain is the only set of inputs that are relevant for the function. You can see this in the picture below. The graph for $f : [1, 2) \to \mathbb{R}$ is the portion of the graph of x^2 that lies between the numbers 1 and 2 on the x-axis.

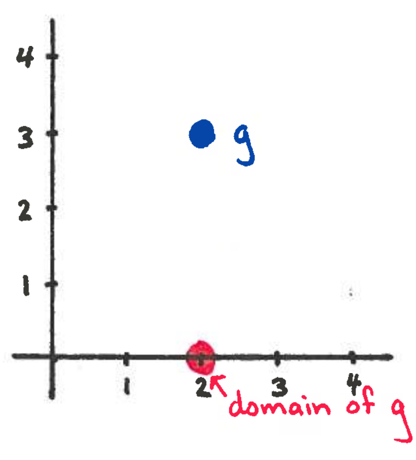


The number 3 is not an object of the domain $[1, 2)$, so it doesn't make sense to use 3 as an input for the function f, and therefore there is no point in the graph of f that lies above or below the number 3 on the x-axis. More generally, there are no points of the graph of f to the left of the number 1 on the x-axis, or to the right of the number 2 on the x-axis.

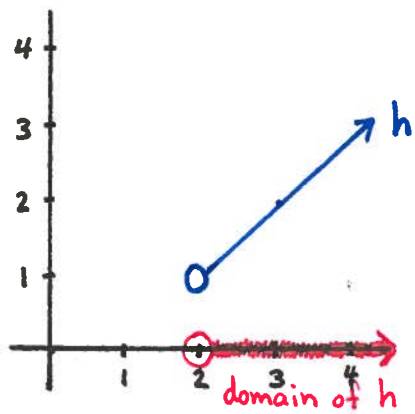
It's important to notice the endpoints of the interval $[1, 2)$ on the x-axis for this example. The number 1 is an object in the set $[1, 2)$, and $f(1) = 1^2 = 1$, so the point $(1, 1)$ is a point in the graph of f. To remind us that this is actually a point in the graph, we place a giant dot on the point $(1, 1)$.

In contrast, notice that instead of drawing a giant dot on the point $(2, 4)$ in the graph of f, instead we drew a little circle. That's because 2 is not an object in the domain of f, $[1, 2)$. Therefore, while $2^2 = 4$, the point $(2, 4)$ is not a point in the graph of f. We draw the little circle to remind us that the point $(2, 4)$ is not a point of the graph.

• The function $g : \{2\} \to \mathbb{R}$ where $g(2) = 3$ has a restricted domain that consists of the single number 2, so its graph should consist of a single point that lies directly above, below, or on the number 2 on the x-axis. In this case, the graph of g consists of the single point $(2, 3)$, since $g(2) = 3$, and we draw that point as a giant dot. We use a giant dot here for two reasons. First, to remind us that it actually is a point in the graph, unlike a little circle. Second, we make the dot big so that it's clear that the dot isn't a stray mark that we accidentally made with our pen.

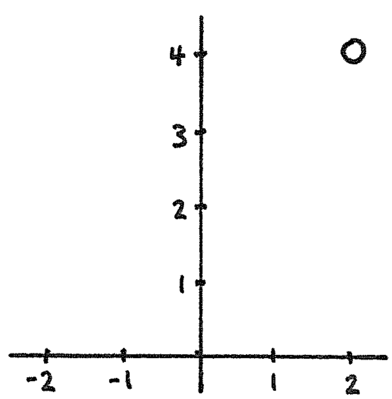


• The function $h : (2, \infty) \to \mathbb{R}$ where $h(x) = x - 1$ has a domain of $(2, \infty)$, so 2 is not an object in the domain. Thus, the point $(2, 2 - 1) = (2, 1)$ is not a point in the graph. So we draw a little circle on the point $(2, 1)$ to remind us of that. Otherwise, every point to the right of the number 2 on the x-axis is in the domain of h, so the graph of h stretches forever to the right. We put a little arrowhead on the rightmost part of the graph that we drew to indicate that the graph continues forever to the right.

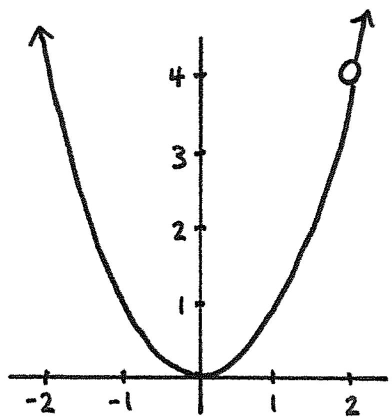


In each of the three examples above, we labeled the domain of the function on the x-axis. That isn't really part of the graph. It was only drawn in these three examples to illustrate the process of graphing functions with restricted domains. If someone asked you to draw the graph of a function, you don't have to label its domain on the x-axis.

• The function $q : \mathbb{R} - \{2\} \to \mathbb{R}$ where $q(x) = x^2$ is another example of a function with a restricted domain. We are used to graphing x^2 when the domain is $\mathbb{R}$. The function q will have a very similar graph, we just have to draw the graph of x^2 with a little circle on the point of the graph of x^2 that reminds us that 2 is not in the domain. More precisely, because 2 is not in the domain, and because $2^2 = 4$, we need to draw a little circle at the point $(2, 4)$.



Once we've drawn the little circle, we can draw the rest of the graph of x^2 making sure that we do not puncture the little circle. There shouldn't be any part of our graph that is inside of the little circle that we drew. That's how we know that $(2, 4)$ is not a point in the graph of q.



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Implied domains

Sometimes we won't go through the trouble of writing the entire name of a function as "$f : D \to T$ where $f(x) = x^5$". This is similar to how we usually call people by their first names, omitting their middle and last names, just because it's easier.

If we are introduced to a function that is given by an equation, and its domain is not specified, we will assume that the domain for that function is the largest subset of the real numbers possible. This set will be called the *implied domain* of the function.

Examples.

• Let $g(x) = \log_e(x)$. The positive real numbers are the only real numbers that we can take a logarithm of, so the implied domain of the function g is $(0, \infty)$.

• If p is a polynomial — for example if $p(x) = 2x - 3$, or $p(x) = 4x^7 - 17x^2 - 2$, or if $p(x) = 4$ — then we can use any real number as an input for p. That is, the implied domain is $\mathbb{R}$.

• Recall that a *rational function* is a function of the form $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. For any real number a, $p(a)$ and $q(a)$ are each real numbers, so the only problem here might be if the denominator of the fraction equals 0. We can never divide by zero, so the implied domain of $r(x) = \frac{p(x)}{q(x)}$ is the set of all real numbers a for which $q(a) \neq 0$. That is, the implied domain is the set of all real numbers except for the roots of $q(x)$.

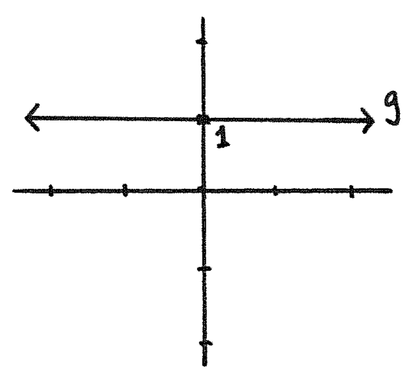
For example, the implied domain of $\frac{3x-2}{x-1}$ is $\mathbb{R} - \{1\}$, because 1 is the only root of $x - 1$. The implied domain of $\frac{x^3-7x^2+4x-2}{x^2-4}$ is $\mathbb{R} - \{-2, 2\}$, because -2 and 2 are the roots of $x^2 - 4$. The implied domain of $\frac{4x^5-17}{x^2+1}$ is $\mathbb{R}$, because $x^2 + 1$ is never equal to 0.

• If $n \in \mathbb{N}$ is an even number, such as 2, 4, 6, 8, etc., then the function $h(x) = \sqrt[n]{x}$ has an implied domain of $[0, \infty)$. We can't take an even root of a negative number.

On the other hand, we can take an odd root of any real number, so the implied domain of $f(x) = \sqrt[3]{x}$ or $g(x) = \sqrt[5]{x}$ is $\mathbb{R}$.

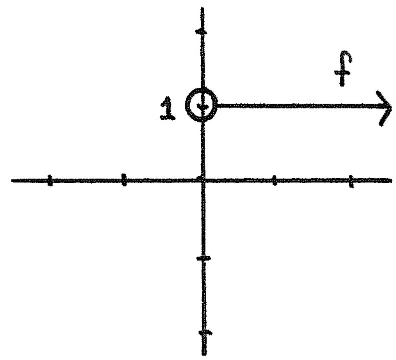
Examples to be careful with.

• The implied domain of the constant function $g(x) = 1$ is $\mathbb{R}$.



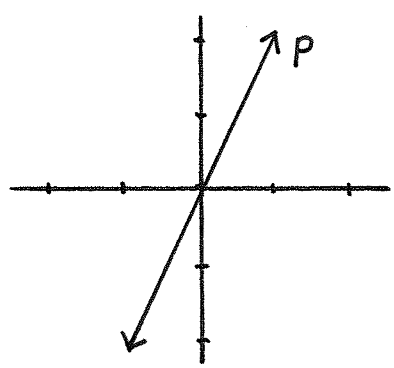
Now suppose that $f(x) = 1 + \log_e(x) - \log_e(x)$. Because the function f asks us to take a logarithm, and because we can only take the logarithm of a positive number, the implied domain of f is $(0, \infty)$.

You must find the implied domain of a given function <u>before</u> you simplify it. The function f above can be simplified as $f(x) = 1$, but the implied domain is still $(0, \infty)$.



Because they have different implied domains, g and f are not the same function.

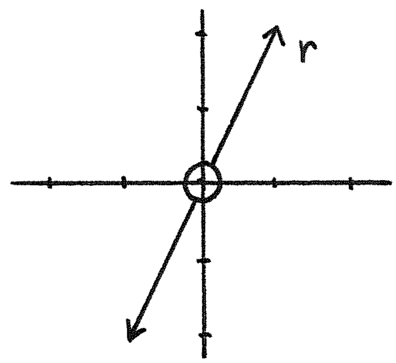
- The implied domain of the polynomial $p(x) = 2x$ is $\mathbb{R}$.



Let r be the rational function $r(x) = \frac{2x^2}{x}$. We must never divide by 0, so the implied domain of r is $\mathbb{R} - \{0\}$.

Now that we know that $r(x) = \frac{2x^2}{x}$ has a domain of $\mathbb{R} - \{0\}$, we are free to divide by x, because x can't be 0 if $x \in \mathbb{R} - \{0\}$. Then the function r

simplifies as $r(x) = 2x$. This doesn't change the implied domain though. We already said that the implied domain is $\mathbb{R} - \{0\}$, and it has to stay that way.



In particular, p is not the same function as r. They have different domains.

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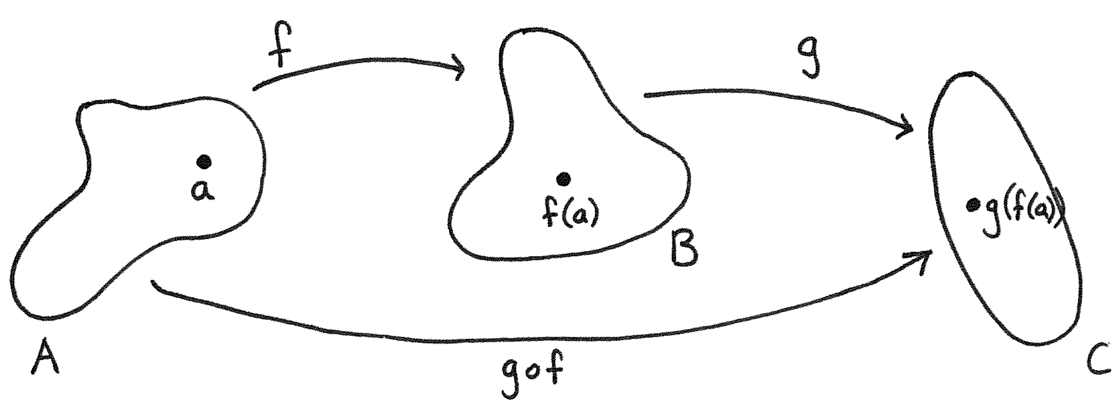
Composition

Suppose that $f : A \to B$ and $g : B \to C$ are functions.

If $a \in A$, then a is in the domain of f and $f(a) \in B$. Since $f(a) \in B$, we have that $f(a)$ is in the domain of g, so $g(f(a))$ is an object in C.

This process defines a third function, named $g \circ f : A \to C$ that is defined by

$$g \circ f(a) = g(f(a))$$



The function $g \circ f$ is pronounced "g composed with f".

Examples.

• Suppose $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are functions and that $g(1) = 3$ and $h(3) = 7$. Then

$$h \circ g(1) = h(g(1)) = h(3) = 7$$

• Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions that are defined by $f(x) = x^2$ and $g(x) = x - 1$.

Then $g \circ f : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$g \circ f(x) = g(f(x)) = g(x^2) = x^2 - 1$$

And $f \circ g : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f \circ g(x) = f(g(x)) = f(x - 1) = (x - 1)^2$$

Important: Notice in the previous example that $g \circ f(2) = 3$ and $f \circ g(2) = 1$. That means that $g \circ f$ is not the same function as $f \circ g$. In other words, $g \circ f \neq f \circ g$.

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Piecewise defined functions

A *piecewise defined function* is a single function that is described using multiple functions that are more common to us.

Examples.

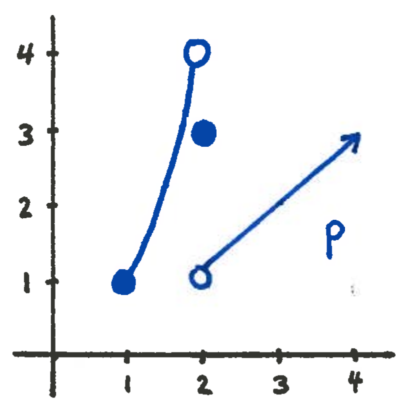
• The function $p : [1, \infty) \to \mathbb{R}$ where

$$p(x) = \begin{cases} x^2 & \text{if } x \in [1, 2); \\ 3 & \text{if } x = 2; \quad \text{and} \\ x - 1 & \text{if } x \in (2, \infty). \end{cases}$$

Here's how the function p works. If you are given a number in the domain of p, $[1, \infty)$, and you want to see which number p assigns it to in the target, first check which piece of the domain the number is in. Is the number in the interval $[1, 2)$, or does it equal 2, or is the number in the interval $(2, \infty)$? If the number is in $[1,2)$, then use the function $p(x) = x^2$; if the number equals 2, then use the function $p(x) = 3$; and if the number is in $(2,\infty)$, then use the function $p(x) = x - 1$.

Let's give it a try. The number 4 is in the domain of p, that is, $4 \in [1,\infty)$. To find $p(4)$, notice that 4 is in the domain of the third piece of p because $4 \in (2,\infty)$. Therefore, $p(4) = 4-1 = 3$. To check your understanding, verify that $p(1) = 1$, $p(2) = 3$, and $p(3) = 2$.

The function $p(x)$ comes in three pieces, and to graph $p(x)$, we just have to graph each of the three pieces. That is, we have to graph $p : [1,2) \to \mathbb{R}$ where $p(x) = x^2$, as well as $p : \{2\} \to \mathbb{R}$ where $p(x) = 3$, and also $p : (2,\infty) \to \mathbb{R}$ where $p(x) = x - 1$. We already graphed each of these three pieces earlier in this chapter. (Earlier, they had the names f, g, and h, respectively.) So now we'll just draw all three pieces together, and the result is the graph of $p : [1,\infty) \to \mathbb{R}$.



Sometimes, depending on who is writing it, the function $p : [1,\infty) \to \mathbb{R}$ above might also be written as

$$p(x) = \begin{cases} x^2 & \text{if } 1 \leq x < 2; \\ 3 & \text{if } x = 2; \quad \text{and} \\ x - 1 & \text{if } x > 2. \end{cases}$$

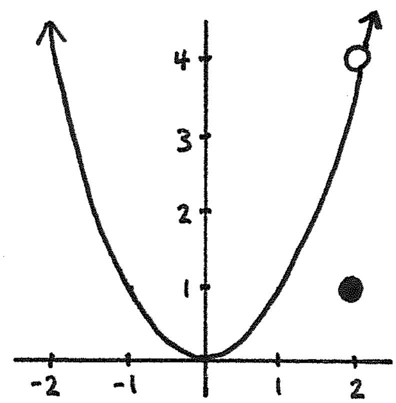
It doesn't matter which of these two ways that you'd prefer to write p. Both are correct. You should be used to reading both though, because both are common ways of writing piecewise defined functions.

• Let's look at the piecewise defined function $q : \mathbb{R} \to \mathbb{R}$ where

$$q(x) = \begin{cases} x^2 & \text{if } x \neq 2; \quad \text{and} \\ 1 & \text{if } x = 2. \end{cases}$$

There are two pieces here, $q : \mathbb{R} - \{2\} \to \mathbb{R}$ where $q(x) = x^2$, and $q : \{2\} \to \mathbb{R}$ where $q(2) = 1$. So $q(-1) = (-1)^2 = 1$, $q(0) = 0^2 = 0$, $q(1) = 1^2 = 1$, $q(3) = 3^2 = 9$, and $q(2) = 1$.

We drew the graph of the first piece earlier in this chapter. The second piece has as its graph a single giant dot on the point $(2, 1)$, since $q(2) = 1$, and the two pieces drawn together make up the graph for $q : \mathbb{R} \to \mathbb{R}$.

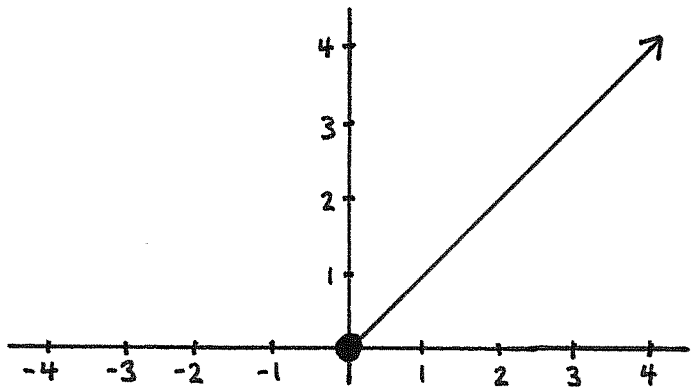


• The single most important piecewise defined function is the *absolute value function*. It is the function defined by

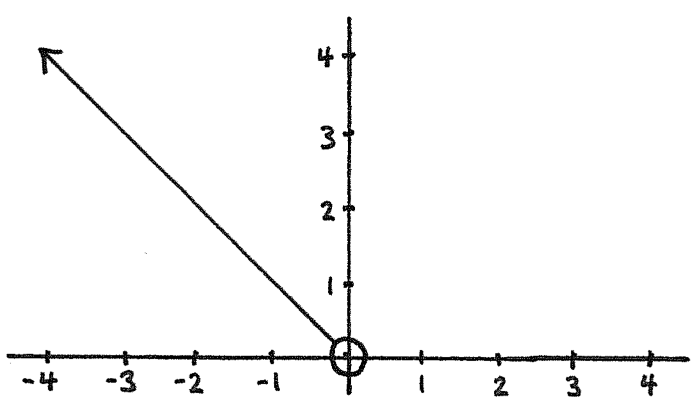
$$|x| = \begin{cases} x & \text{if } x \geq 0; \quad \text{and} \\ -x & \text{if } x < 0. \end{cases}$$

The domain of the absolute value function is $\mathbb{R}$.

If x is a number that's positive or zero, then $|x| = x$. This first piece of the absolute value function is the identity function, with a domain of $[0, \infty)$. If you take the absolute value of a positive number, or of 0, nothing happens. You are returned the same number that you put in to the absolute value function. For example, $|0| = 0$, $|5| = 5$, and $|\sqrt{2}| = \sqrt{2}$. The graph of the first piece of the absolute value function is below. Notice that there is a giant dot on the point $(0, 0)$ since 0 is an object in the domain of this first piece, $[0, \infty)$.

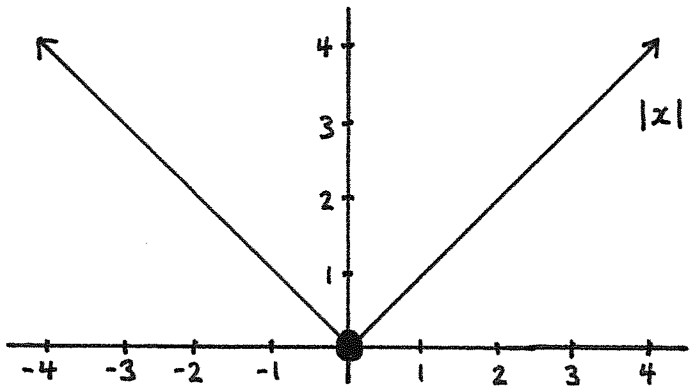


The domain of the second piece of the absolute value function is the set of negative numbers, $(-\infty, 0)$. If x is a negative number, then $|x| = -x$. For example, $|-3| = -(-3) = 3$ and $|-\pi| = -(-\pi) = \pi$. Notice that for this second piece of the absolute value function, the effect of the function is to erase the minus sign that is in front of the negative number. Therefore, the absolute value of a negative number is always positive. The graph of this second piece is drawn below. Notice that there's a little circle drawn at the point $(0, 0)$. That because 0 is not negative, and therefore it is not a number in the domain of this second piece, $(-\infty, 0)$.



To draw the graph of the absolute value function, just graph both of its pieces at once. Notice that the giant dot of the first piece at the point $(0, 0)$

fills up the little circle of the second piece at the point $(0, 0)$. The result is that there is no little circle in the graph of the absolute value function



Exercises

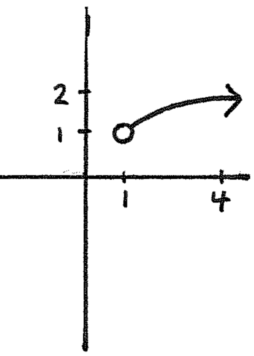
For #1-18, match the numbered functions on the left to the lettered implied domains on the right.

1.) $3x^2 - 7x + 4$	10.) $\log_e(-e^x)$	A.) $\mathbb{R}$
2.) e^x	11.) $\frac{2x^3-4x+2}{x+5}$	B.) $(0, \infty)$
3.) $\frac{2x+3}{x^7}$	12.) $e^{\sqrt[2]{x}}$	C.) $\mathbb{R} - \{0\}$
4.) $\log_e(x) - \sqrt[2]{x}$	13.) $\frac{4x}{x^2-9}$	D.) $\mathbb{R} - \{-3, 3\}$
5.) 11	14.) e^{3x^7-5}	E.) $[0, \infty)$
6.) $\frac{x^2}{5}$	15.) $\frac{\log_e(x)}{x+5}$	F.) $\emptyset$
7.) e^{x^2}	16.) $x + 5$	G.) $\mathbb{R} - \{-5\}$
8.) $\frac{\sqrt[2]{x}}{\sqrt[2]{x}}$	17.) $\sqrt[2]{\frac{1}{x}}$	
9.) $\sqrt[9]{x^3 - 13x^2 + 2x - 7}$	18.) $\log_e(\,\lvert x\rvert\,)$	

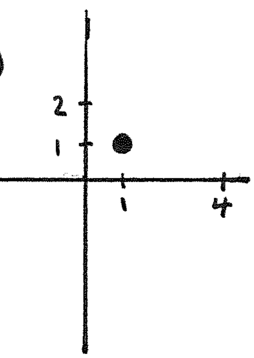
For #19-27, match the numbered functions below to their lettered graphs on the next page.

19.) $\sqrt[2]{} : (1, 4) \to \mathbb{R}$	22.) $\sqrt[2]{} : [1, 4] \to \mathbb{R}$	25.) $\sqrt[2]{} : [1, \infty) \to \mathbb{R}$
20.) $\sqrt[2]{} : (1, \infty) \to \mathbb{R}$	23.) $\sqrt[2]{} : \{1\} \to \mathbb{R}$	26.) $\sqrt[2]{} : (1, \infty) - \{4\} \to \mathbb{R}$
21.) $\sqrt[2]{} : [1, 4) \to \mathbb{R}$	24.) $\sqrt[2]{} : \{4\} \to \mathbb{R}$	27.) $\sqrt[2]{} : (1, 4] \to \mathbb{R}$

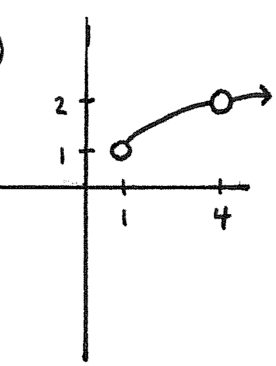
A.)



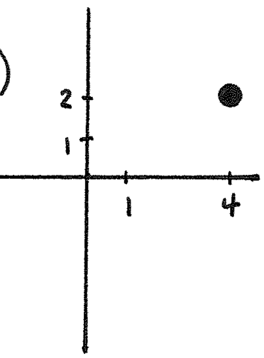
B.)



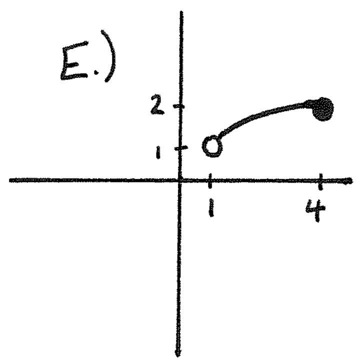
C.)



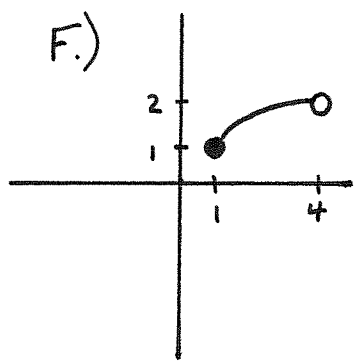
D.)



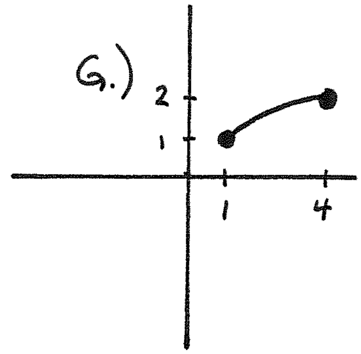
E.)



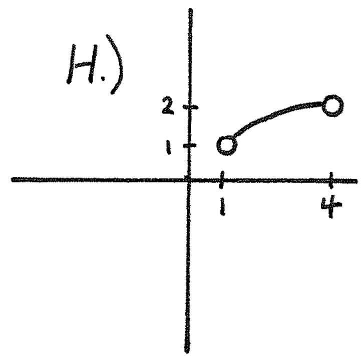
F.)



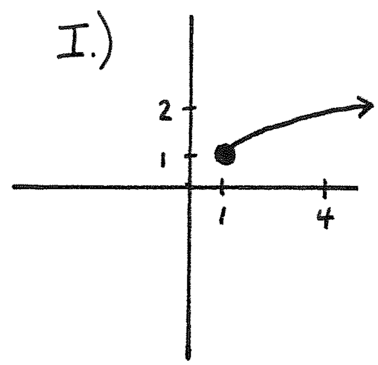
G.)



H.)



I.)



Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ be the funtions $f(x) = x^3$, $g(x) = x^2$, and $h(x) = (x - 1)$. For #28-33, match the numbered functions on the left to the lettered functions on the right that they equal.

28.) $f \circ g(x)$	A.) $x^3 - 1$
29.) $g \circ f(x)$	B.) x^6
30.) $f \circ h(x)$	C.) $x^2 - 2x + 1$
31.) $h \circ g(x)$	D.) $x^2 - 1$
32.) $g \circ h(x)$	E.) $x^3 - 3x^2 + 3x - 1$
33.) $h \circ f(x)$	

For #34-38, identify the given absolute value.

34.) $|5|$ 35.) $|0|$ 36.) $|-7|$ 37.) $|\frac{3}{8}|$ 38.) $|-\frac{2}{3}|$

Match the numbered piecewise defined functions below with their lettered graphs on the next page.

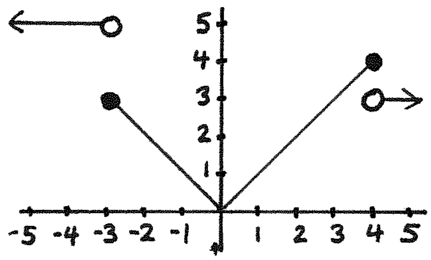
39.) $f(x) = \begin{cases} e^x & \text{if } x \neq 1; \quad \text{and} \\ 1 & \text{if } x = 1. \end{cases}$

40.) $g(x) = \begin{cases} \log_e(x) & \text{if } x \in (0, e); \quad \text{and} \\ 1 & \text{if } x \in [e, \infty). \end{cases}$

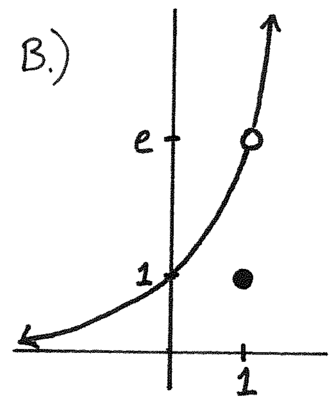
41.) $h(x) = \begin{cases} 5 & \text{if } x < -3; \\ |x| & \text{if } -3 \leq x \leq 4; \quad \text{and} \\ 3 & \text{if } x > 4. \end{cases}$

42.) $p(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1); \\ 2 & \text{if } x = 1; \quad \text{and} \\ 5 - x & \text{if } x \in (1, 4]. \end{cases}$

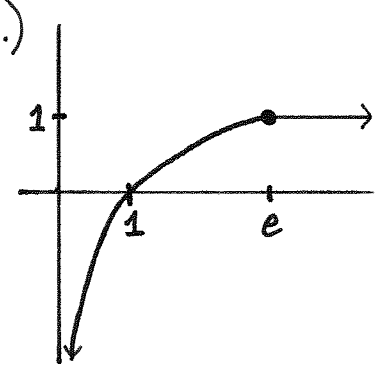
A.)



B.)



C.)



D.)