Equations in One Variable VII

In this chapter we'll look at solutions of equations involving the three basic trigonometric functions, sine, cosine, and tangent. For example, we'll find solutions of equations such as $\sin(x) = \frac{1}{3}$ or $\tan(x) = 2$.

Periodic families of solutions

Recall that the period of tangent is π. That means that $\tan(x+\pi) = \tan(x)$.

Let c be a constant. If the number α is a solution of the equation $\tan(x) = c$, then $\tan(\alpha) = c$. Because tangent has a period of π, we know that $\alpha + \pi$ is also a solution of the equation $\tan(x) = c$. That's because

$$\tan(\alpha + \pi) = \tan(\alpha) = c$$

This is worth repeating: if a number is a solution of $\tan(x) = c$, then so is that number plus π. Therefore, if α is a solution, then $\alpha + \pi$ is a solution. Because $\alpha + \pi$ is a solution, we can add π to it to find another solution, namely $(\alpha + \pi) + \pi = \alpha + 2\pi$. Then we could add π to find another solution, the number $(\alpha + 2\pi) + \pi = \alpha + 3\pi$. We could add π again to obtain the solution $\alpha + 4\pi$, and then add π again to obtain the solution $\alpha + 5\pi$. We could repeat this process forever to see that if α is a solution of the equation $\tan(x) = c$, then so are the numbers

$$\alpha + \pi,\ \alpha + 2\pi,\ \alpha + 3\pi,\ \alpha + 4\pi,\ \alpha + 5\pi,\ \alpha + 6\pi,\ \dots$$

That's an infinite list of solutions, and there are more solutions still. Instead of just moving forward in the line of numbers, we can also move backwards. That is, $\alpha - \pi$ is also a solution. That's because

$$\tan(\alpha - \pi) = \tan((\alpha - \pi) + \pi) = \tan(\alpha) = c$$

So we can subtract the period, π, from any solution to find another solution. Then we could subtract another π to find one more solution. Then subtract π to find yet another solution, and this process would continue on forever, so that if α was a solution then so too would be

$$\alpha - \pi,\ \alpha - 2\pi,\ \alpha - 3\pi,\ \alpha - 4\pi,\ \dots$$

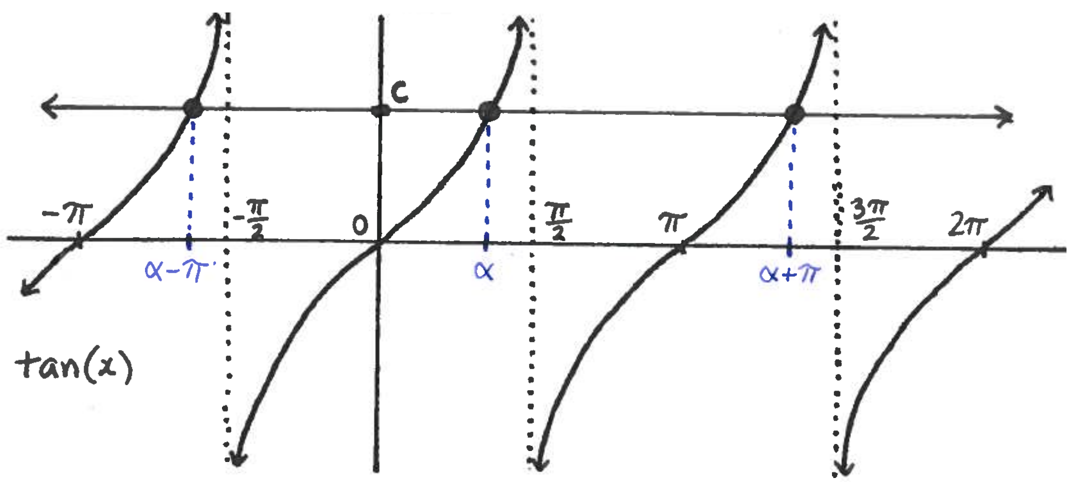
Altogether we have seen that if α is a solution of $\tan(x) = c$, then so is every number in the infinite family of numbers

$$\dots,\ \alpha - 3\pi,\ \alpha - 2\pi,\ \alpha - \pi,\ \alpha,\ \alpha + \pi,\ \alpha + 2\pi,\ \alpha + 3\pi,\ \alpha + 4\pi,\ \dots$$

Notice that every number on the list above is a number of the form $\alpha + n\pi$ for some $n \in \mathbb{Z}$, so our discussion can be summarized succinctly as follows:

Periodic solutions for tangent

Tangent has period π,
so if α is a solution of $\tan(x) = c$,
then any number of the form $\alpha + n\pi$
is a solution of $\tan(x) = c$.



The above discussion would essentially apply to any periodic function. For example, cosine is periodic with period 2π, so if α was a solution to the equation $\cos(x) = c$, then $\alpha + 2\pi$ would be a solution. So to would be $\alpha + 4\pi$, $\alpha + 6\pi$, $\alpha + 8\pi$, and $\alpha - 2\pi$. These are numbers of the form $\alpha + n2\pi$, where $n \in \mathbb{Z}$.

Periodic solutions for cosine

Cosine has period 2π,
so if α is a solution of $\cos(x) = c$,
then any number of the form $\alpha + n2\pi$
is a solution of $\cos(x) = c$.

A similar observation applies to the sine function as well.

Periodic solutions for sine

Sine has period 2π,
so if α is a solution of $\sin(x) = c$,
then any number of the form $\alpha + n2\pi$
is a solution of $\sin(x) = c$.

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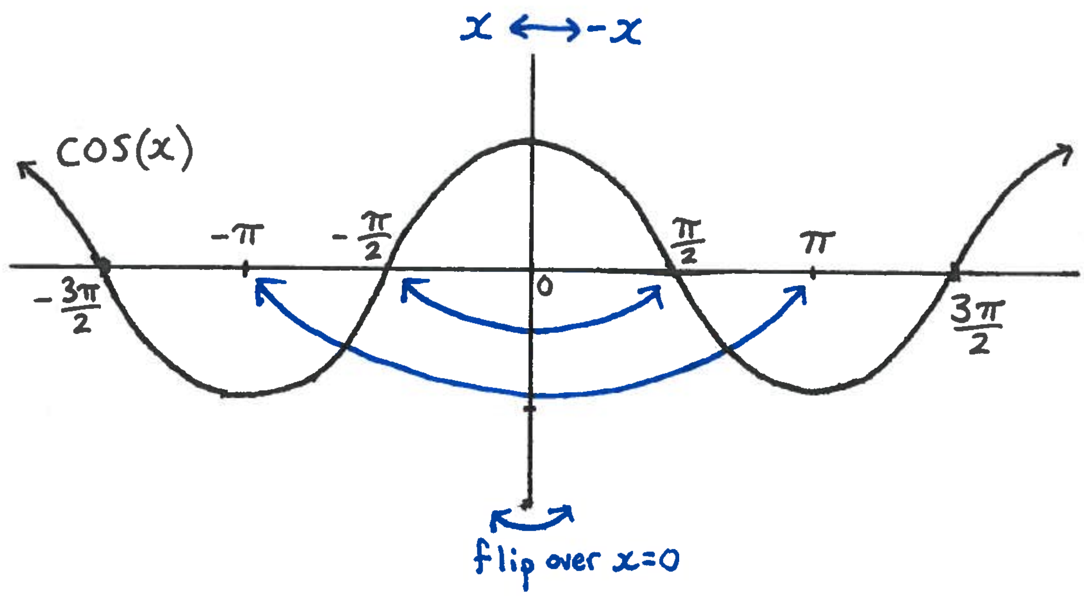
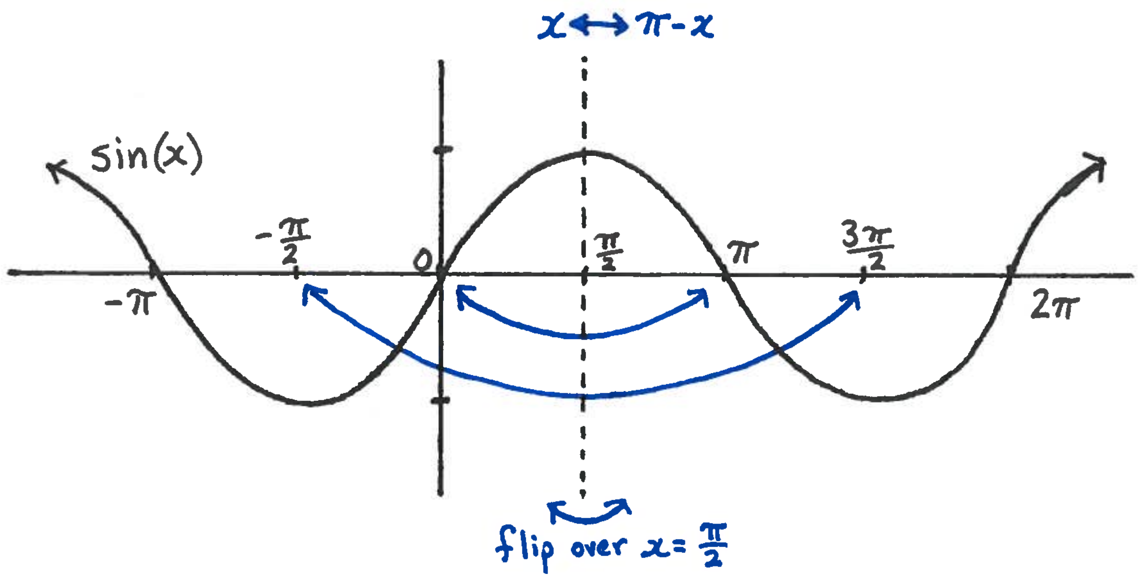
An Important identity for sine

Cosine is an even function, meaning $\cos(x) = \cos(-x)$. Geometrically, that means we can flip the graph of $\cos(x)$ over the y-axis, the line $x = 0$, and we'd have the same picture that we started with.



Sine is not an even function, so we can not flip its graph over the line $x = 0$ without considerably changing the picture. We can however, flip the graph of $\sin(x)$ over the line $x = \frac{\pi}{2}$ without changing the picture in any way. This geometric description is encoded in the identity

$$\sin(x) = \sin(\pi - x)$$



Notice that the value $\pi - x$ is obtained by flipping the plane over the line $x = \frac{\pi}{2}$. For example, 0 is $\frac{\pi}{2}$ to the left of $\frac{\pi}{2}$, while $\pi - (0) = \pi$ is $\frac{\pi}{2}$ to the right of $\frac{\pi}{2}$. The number $\frac{3\pi}{2}$ is π to the right of $\frac{\pi}{2}$, while $\pi - \left(\frac{3\pi}{2}\right) = -\frac{\pi}{2}$ is π to the left of $\frac{\pi}{2}$.

To check that the identity $\sin(x) = \sin(\pi - x)$ is true, we'll use two identities that we already know, that $-\sin(\theta) = \sin(-\theta)$ and that $-\sin(\theta) = \sin(\pi + \theta)$.

$$-\sin(x) = \sin(-x) = -[-\sin(-x)] = -[\sin(\pi + (-x))] = -\sin(\pi - x)$$

Now dividing by -1 gives us our desired identity that

$$\sin(x) = \sin(\pi - x)$$

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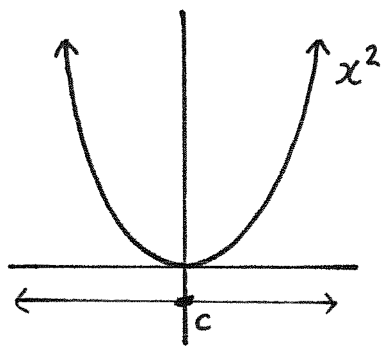
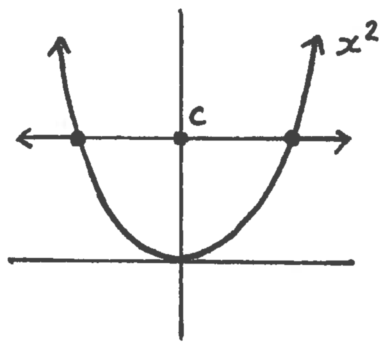
Solving $x^2 = c$

Solving trigonometric equations requires use of the partial inverse functions arctangent, arccosine, and arcsine. Before examining trigonometric equations, it will help to review how we use the partial inverse function that we are most comfortable with, the square-root.

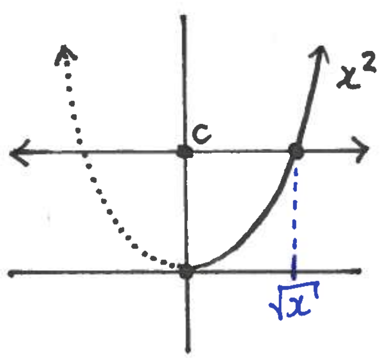
If c is a constant, then $x^2 = c$ has no solution if $c < 0$.



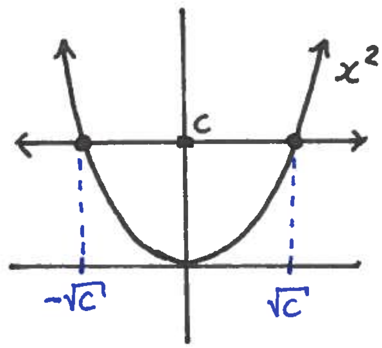
If $c \geq 0$, then $x^2 = c$ will typically have two solutions.



The partial inverse $\sqrt{x}$ gives us a single solution to $x^2 = c$, namely $\sqrt{c}$. This is the only solution in the interval $[0, \infty)$. We'll call $\sqrt{c}$ the *primary solution.*



The function x^2 is even, meaning that $x^2 = (-x)^2$. Therefore, if $\sqrt{c}$ is a solution, then $-\sqrt{c}$ is also a solution. We'll call this solution the *secondary solution.* These are the only solutions of the equation $x^2 = c$.



Example.

- If $x^2 = 1/7$, then $x = \sqrt{\frac{1}{7}}$ or $x = -\sqrt{\frac{1}{7}}$.

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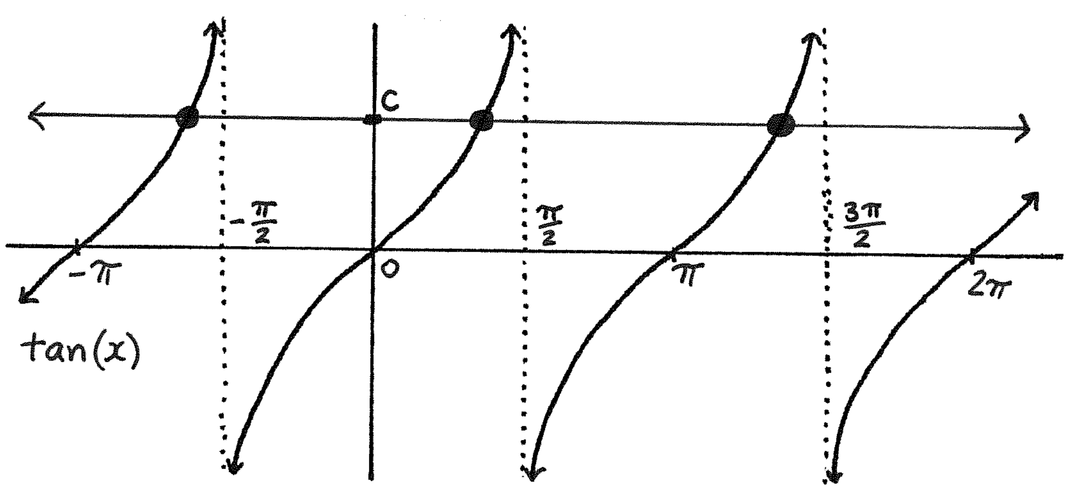
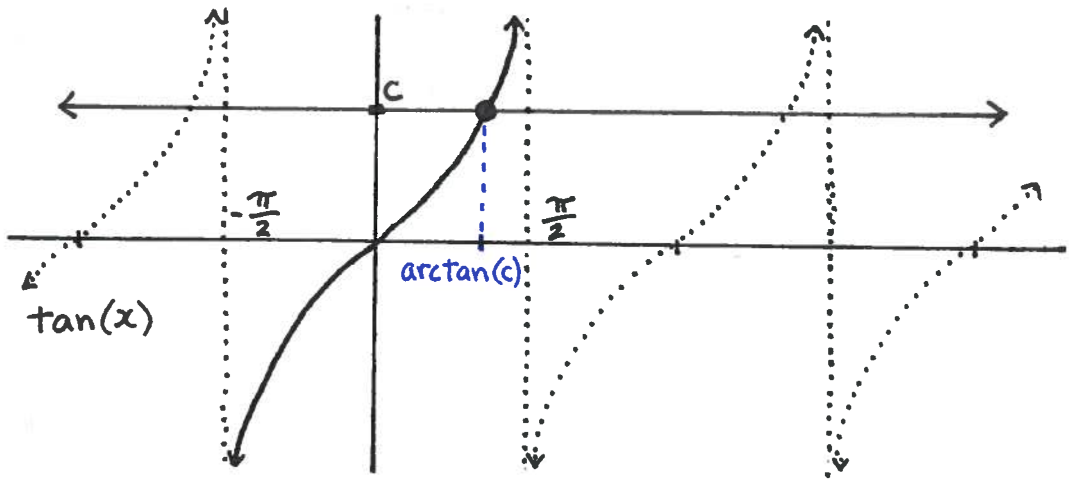
Solving $\tan(x) = c$

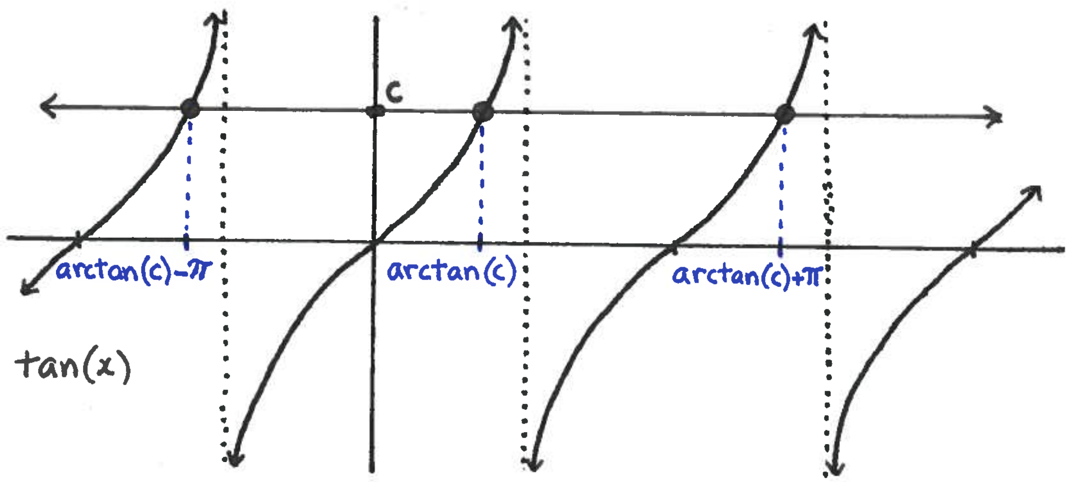
If c is a constant, then $\tan(x) = c$ has infinitely many solutions. It doesn't matter which number c is.



The *primary solution* is given by the partial inverse of tangent, namely the arctangent function. This primary solution is $x = \arctan(c)$. The primary solution is the only number in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ that is a solution. This interval has length π, the period of tan.



Once we've identified the primary solution $\arctan(c)$, we know that any number in the periodic family $\arctan(c) + n\pi$ where $n \in \mathbb{Z}$ is a solution.



Example.

- If $\tan(x) = 1/7$, then $x = \arctan(\frac{1}{7}) + n\pi$ for some $n \in \mathbb{Z}$.

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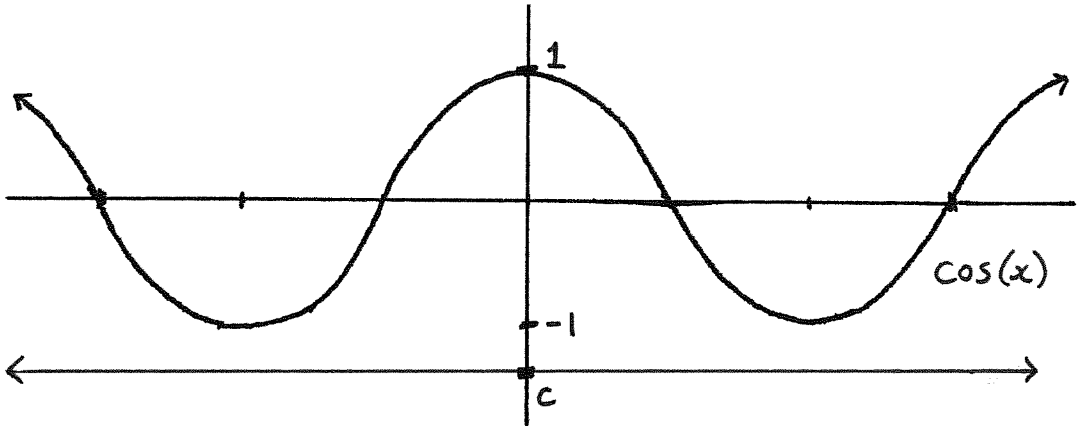
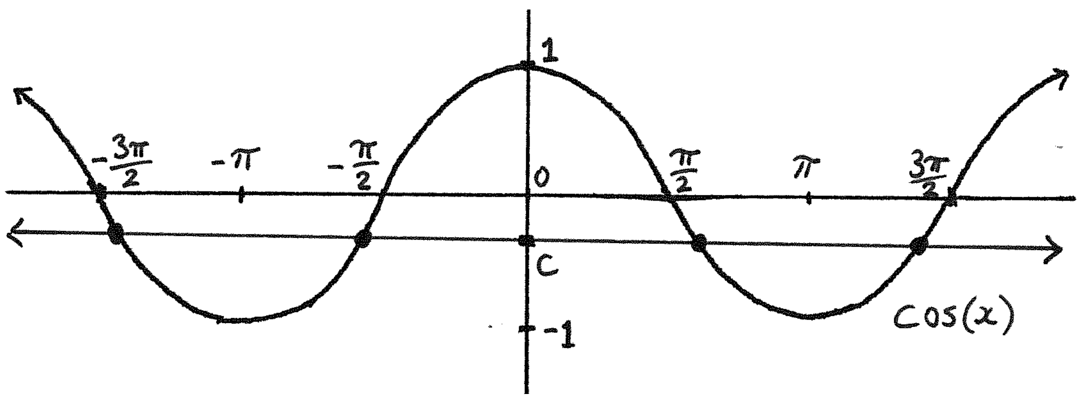
Solving $\cos(x) = c$

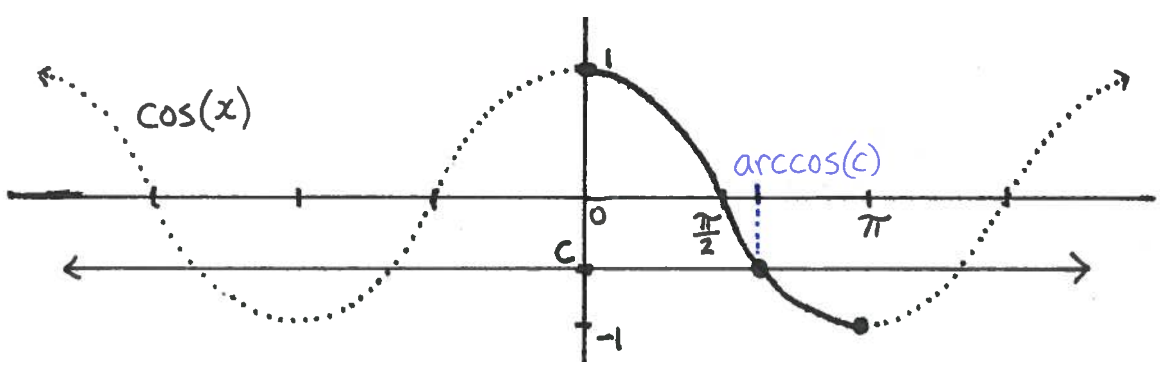
If c is a constant, then $\cos(x) = c$ has no solution if $c \notin [-1, 1]$.



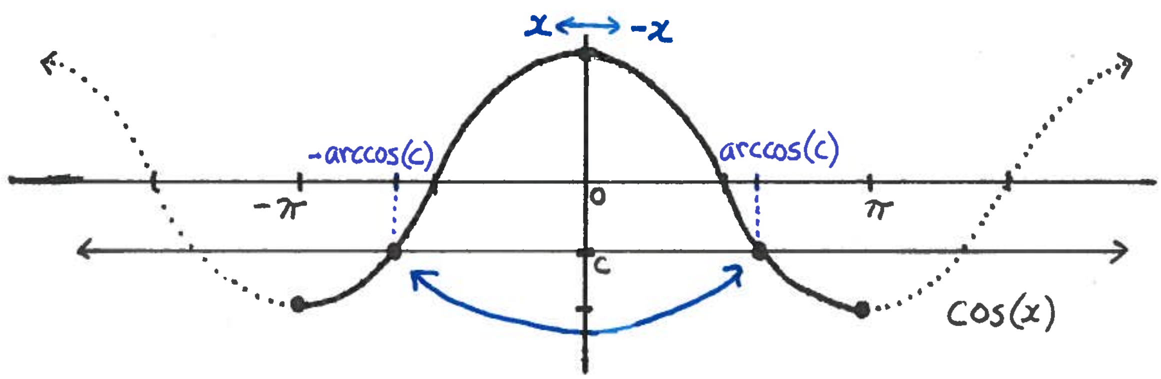
If $c \in [-1, 1]$, then $\cos(x) = c$ will have infinitely many solutions.



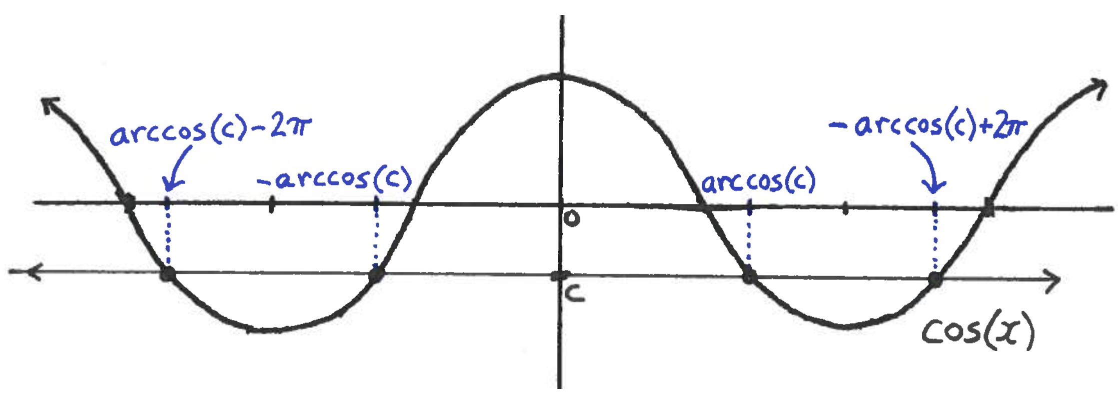
The *primary solution* is given by the partial inverse of $\cos(x)$, namely the arccosine function. This primary solution is $x = \arccos(c)$. It's the only solution in the interval $[0, \pi]$.



After we've identified the primary solution $\arccos(c)$, we can use that cosine is an even function to find another solution. Recall that cosine being an even function means that $\cos(x) = \cos(-x)$, so that $-\arcsin(c)$ must also be a solution. We'll call this the *secondary solution*. The primary and secondary solutions are the only solutions in the interval $[-\pi, \pi]$. This is an interval of length 2π, which is the period of cosine.



Both the primary and secondary solutions give us infinite families of periodic solutions. Namely numbers of the form $\arccos(c) + n2\pi$ or $-\arccos(c) + n2\pi$ where $n \in \mathbb{Z}$ are solutions.



Example.

- If $\cos(x) = 1/7$, then $x = \arccos(\frac{1}{7}) + n2\pi$ or $x = -\arccos(\frac{1}{7}) + n2\pi$ for some $n \in \mathbb{Z}$.

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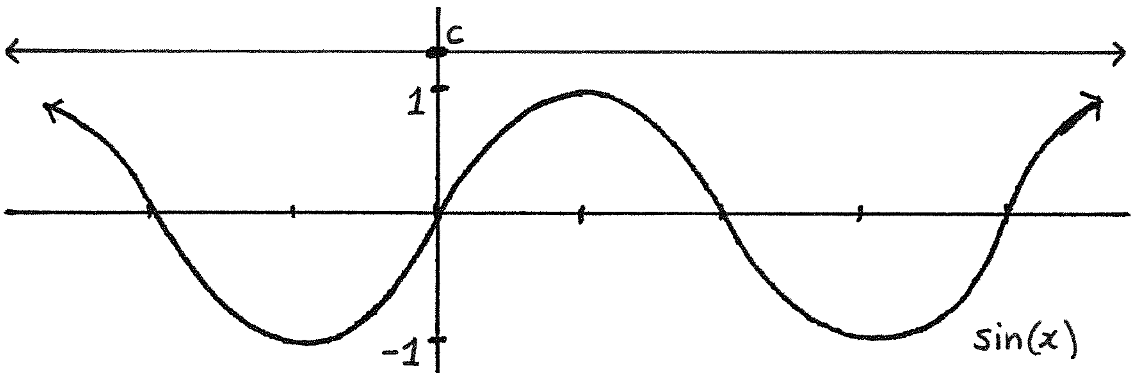
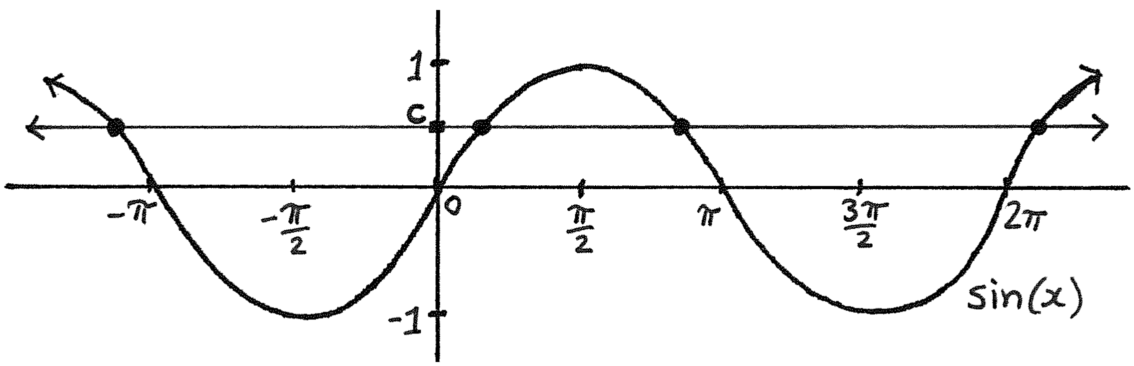
Solving $\sin(x) = c$

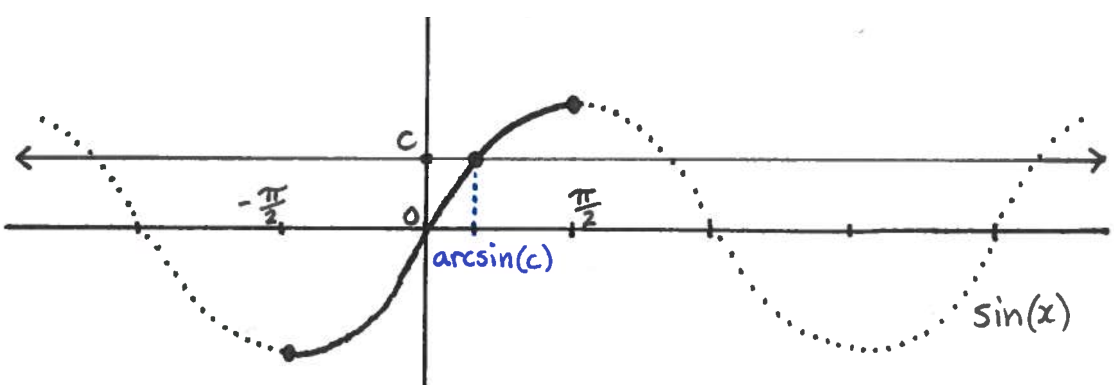
If c is a constant, then $\sin(x) = c$ has no solution if $c \notin [-1, 1]$.



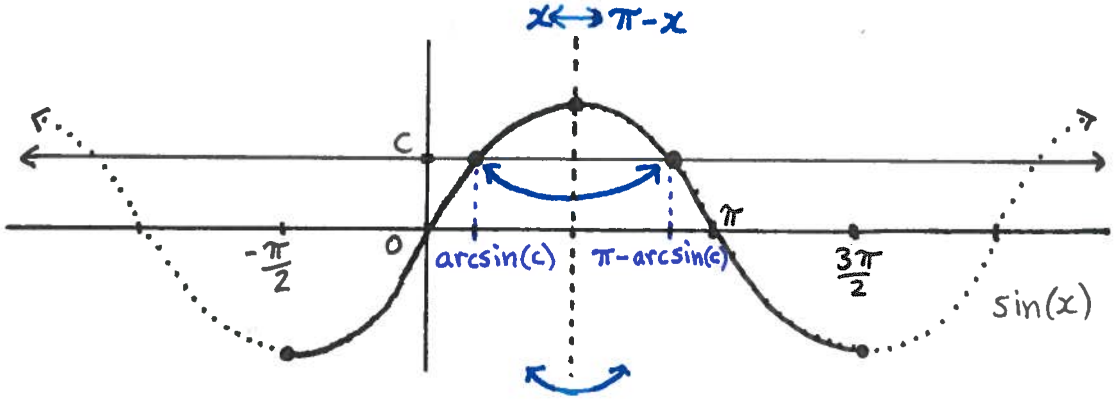
If $c \in [-1, 1]$, then $\sin(x) = c$ will have infinitely many solutions.



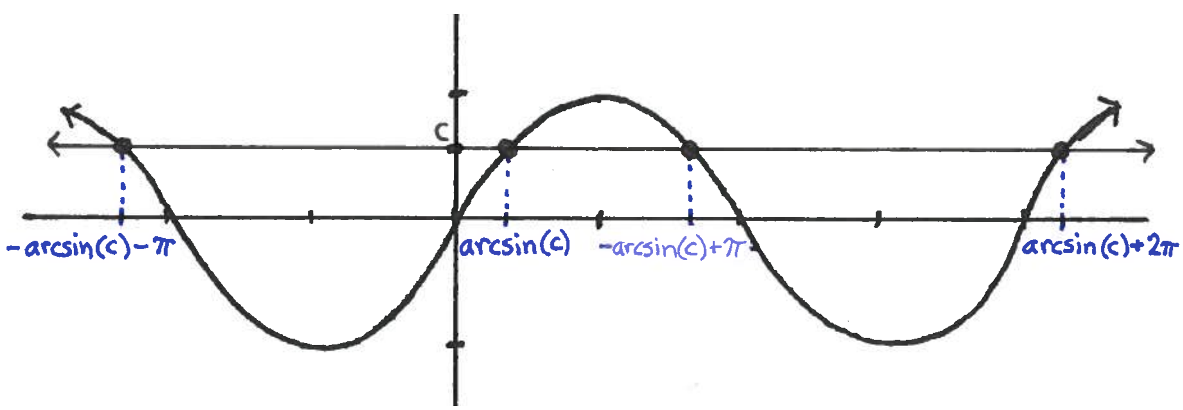
The *primary solution* is given by the partial inverse of $\sin(x)$, the arcsine function. The primary solution is $x = \arcsin(c)$. It is the only solution in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



The identity $\sin(x) = \sin(\pi - x)$ tells us that $\pi - \arcsin(c)$ is also a solution. We'll call this the *secondary solution*. The primary and secondary solutions are the only solutions in the interval $[-\frac{\pi}{2}, \frac{3\pi}{2}]$. This interval has length 2π, which is the period of sin.



The primary and secondary solutions give us two infinite periodic families of solutions, $\arcsin(c) + n2\pi$ where $n \in \mathbb{Z}$ and $\pi - \arcsin(c) + n2\pi$ where $n \in \mathbb{Z}$. Because $\pi + n2\pi = (n2 + 1)\pi$, the latter family can be simplified. In the simplified form, the two families of solutions are $\arcsin(c) + n2\pi$ where $n \in \mathbb{Z}$ and $-\arcsin(c) + (n2 + 1)\pi$ where $n \in \mathbb{Z}$.



Example.

- If $\sin(x) = 1/7$, then $x = \arcsin(\frac{1}{7}) + n2\pi$ or $x = -\arcsin(\frac{1}{7}) + (n2+1)\pi$ for some $n \in \mathbb{Z}$.

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Chapter Summary

Solutions of $\tan(x) = c$

$$x = \arctan(c) + n\pi \quad \text{where } n \in \mathbb{Z}.$$

Solutions of $\cos(x) = c$

None if $c \notin [-1, 1]$.

If $c \in [-1, 1]$, then the solutions are

$$x = \arccos(c) + n2\pi \quad \text{and} \quad x = -\arccos(c) + n2\pi \quad \text{where } n \in \mathbb{Z}.$$

Solutions of $\sin(x) = c$

None if $c \notin [-1, 1]$.

If $c \in [-1, 1]$, then the solutions are

$$x = \arcsin(c) + n2\pi \quad \text{and} \quad x = -\arcsin(c) + (n2+1)\pi \quad \text{where } n \in \mathbb{Z}.$$

Exercises

Find the set of solutions of the following equations. For #1-12, you just need to use the information from the chapter summary on the previous page.

1.) $\tan(x) = 5$ 2.) $\tan(x) = -4$ 3.) $\tan(x) = \frac{1}{3}$ 4.) $\tan(x) = -\frac{3}{5}$

5.) $\cos(x) = \frac{1}{3}$ 6.) $\cos(x) = -\frac{1}{5}$ 7.) $\cos(x) = 7$ 8.) $\cos(x) = -\frac{2}{3}$

9.) $\sin(x) = \frac{2}{3}$ 10.) $\sin(x) = \frac{1}{5}$ 11.) $\sin(x) = -2$ 12.) $\sin(x) = -\frac{1}{6}$

The equations in #13-17 are quadratic equations in either $\cos(x)$ or $\sin(x)$. To solve for x, you'll first use the quadratic formula to solve for $\cos(x)$ or $\sin(x)$. Then to solve for x, you'll use the same method that was used in #5-12 above.

13.) $9\cos(x)^2 - 12\cos(x) + 4 = 0$

14.) $2\sin(x)^2 - 7\sin(x) + 3 = 0$

15.) $3\cos(x)^2 + 7\cos(x) + 2 = 0$

16.) $\sin(x)^2 - \sin(x) - 6 = 0$

17.) $\cos(x)^2 + 4\cos(x) + 4 = 0$