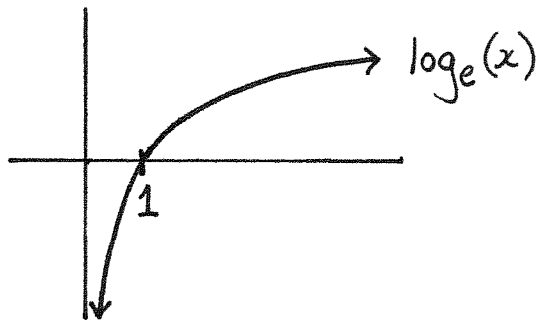
Equations in One Variable VI

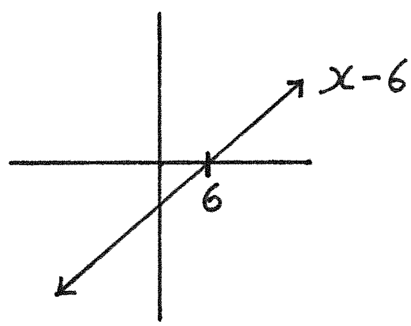
A *zero* of a function $f(x)$ is a number $z \in \mathbb{R}$ such that $f(z) = 0$.

Examples.

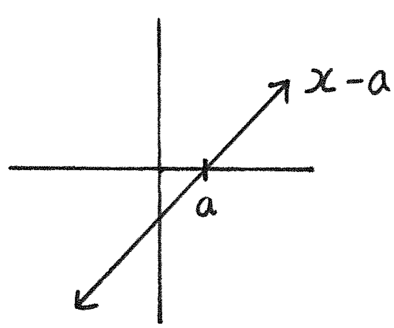
- The number 1 is a zero of the function $\log_e(x)$. That's because $\log_e(1) = 0$.



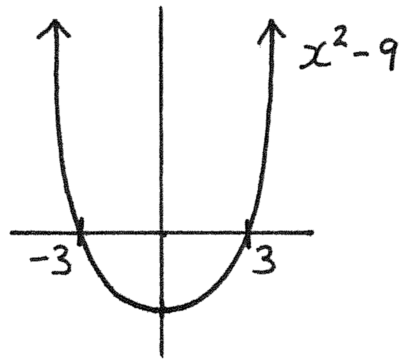
- 6 is a zero of the function $x - 6$. That's because $[6] - 6 = 0$.



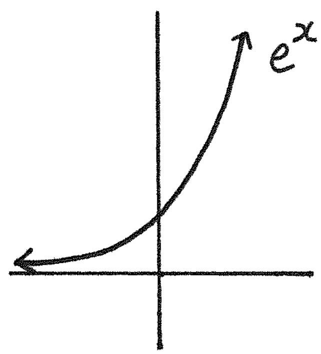
- To state a more general version of the previous example, a is a zero of $x - a$.



- The function $x^2 - 9$ has zeros 3 and -3.

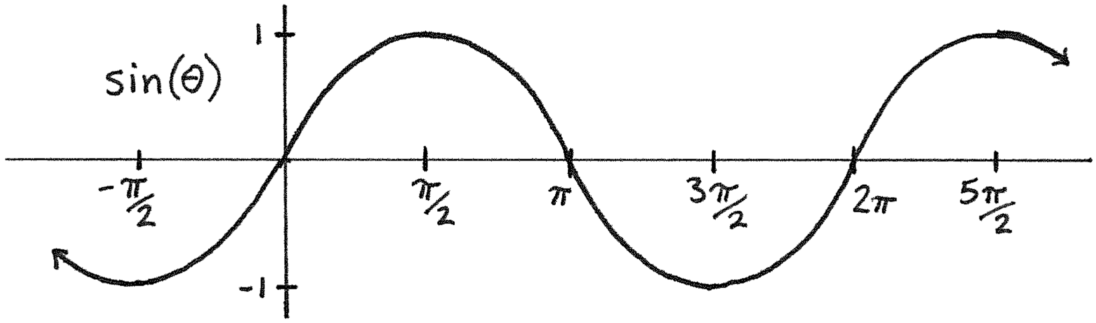


- The zeros of a polynomial are called roots.
- e^x has no zeros.



- The set of zeros of $\sin(x)$ is $\{\dots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots\}$.
  That is, the set of zeros of $\sin(x)$ is the set

$$\{\, n\pi \mid n \in \mathbb{Z} \,\}$$

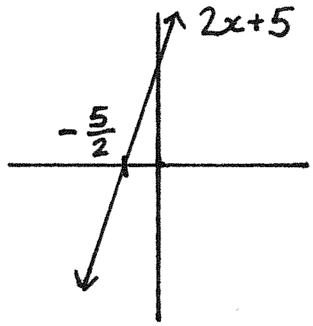


Finding zeros of functions

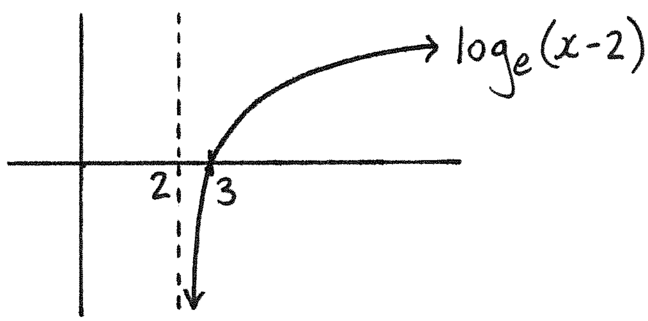
If $f(x)$ is a function, then the zeros of $f(x)$ are the solutions of the equation $f(x) = 0$.

Examples.

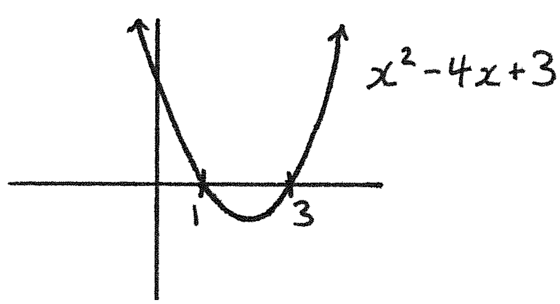
• The zeros of the linear polynomial $2x + 5$ are the solutions of the equation $2x+5 = 0$. Subtract 5: $2x = -5$. Then divide by 2: $x = -\frac{5}{2}$. Thus, $-\frac{5}{2}$ is the only zero of $2x + 5$.



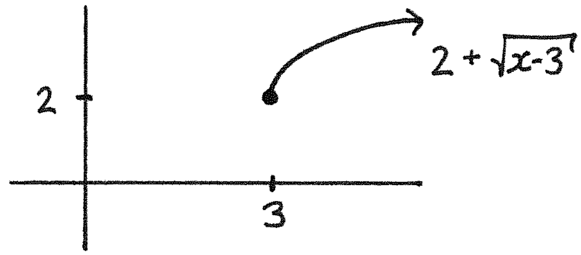
• The zeros of $\log_e(x - 2)$ are the solutions of $\log_e(x - 2) = 0$. This equation is equivalent to $x - 2 = e^0 = 1$, and thus is equivalent to $x = 3$. That is, 3 is the only zero of the function $\log_e(x - 2)$.



• The zeros of x^2-4x+3 are the solutions of the equation $x^2-4x+3 = 0$. Using the quadratic formula, we can find that the solutions are 1 and 3. Therefore, 1 and 3 are the zeros of $x^2 - 4x + 3$.



• The zeros of the function $2+\sqrt{x-3}$ are solutions of $2+\sqrt{x-3}=0$. This equation is equivalent to $\sqrt{x-3}=-2$, and thus it has no solutions since square-roots are never negative. Therefore, $2+\sqrt{x-3}$ has no zeros.



An illustration of when not to divide

If asked to find the solutions of the equation

$$(x-1)(x-2)=3(x-1)$$

you might be tempted to divide both sides of the equation by $(x-1)$ leaving you with $(x-2)=3$, a much simpler equation. However, while $(x-2)=3$ is certainly a much simpler equation (the solution to it is 5) it is not equivalent to the original equation $(x-1)(x-2)=3(x-1)$. That's because the function $(x-1)$ has a zero (the zero is 1), and multiplying or dividing an equation by a function that has a zero in the domain of the equation is not a valid way of obtaining an equivalent equation.

The steps below describe the general process for solving equations of the form $h(x)f(x)=h(x)g(x)$. After listing the general steps, we'll return to the example of $(x-1)(x-2)=3(x-1)$.

Steps for solving $h(x)f(x)=h(x)g(x)$

Step 1: Find the domain of the equation. Call this set D.

Step 2: Find the zeros of $h(x)$ that are in D. Call this set Z. The numbers in Z will be solutions of $h(x)f(x)=h(x)g(x)$ because if $h(z)=0$ then $h(z)f(z)=0f(z)=0=0g(z)=h(z)g(z)$.

Step 3: Find the solutions of the equation $h(x)f(x)=h(x)g(x)$ with the restricted domain $D-Z$. The function $h(x)$ has no zeros in $D-Z$, so you can start with dividing by $h(x)$ to obtain the equivalent equation $f(x)=g(x)$.

Step 4: Collect the zeros from Step 2 and the solutions from Step 3. These are the solutions of the original equation.

Examples.

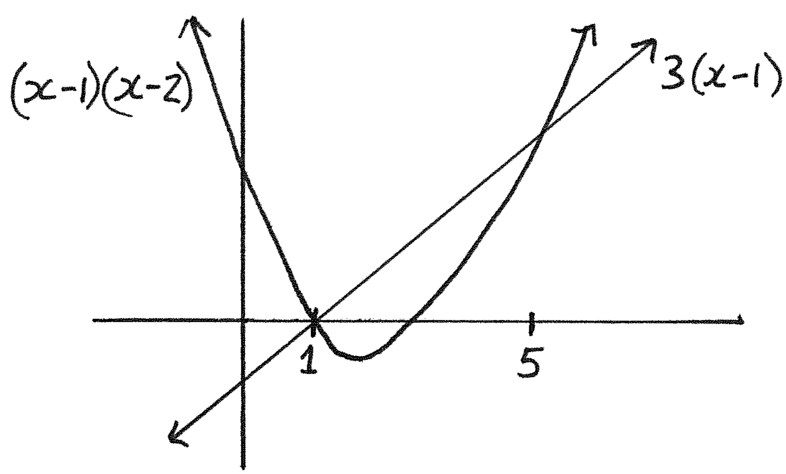
• Let's return to the equation $(x-1)(x-2) = 3(x-1)$ and proceed through the four steps outlined above.

(Step 1): $(x-1)(x-2) = 3(x-1)$ is a polynomial equation, so the domain is $\mathbb{R}$. Written in the notation of the general steps above, $D = \mathbb{R}$.

(Step 2): The function that we'd like to divide both sides of the equation by is $(x-1)$. Its zero is 1, which will be a solution of $(x-1)(x-2) = 3(x-1)$ since $([1]-1)([1]-2) = 0(-1) = 0 = 3(0) = 3([1]-1)$. Written in the notation of the general steps above, $Z = \{1\}$.

(Step 3): We need to find the solutions of the equation $(x-1)(x-2) = 3(x-1)$ with the restricted domain of $\mathbb{R} - \{1\}$. On this domain, the function $(x-1)$ has no zeros, so we can divide by $(x-1)$. The resulting equivalent equation is $(x-2) = 3$. We can add 2 to see that the solution is $x = 5$.

(Step 4): The zero of $(x-1)$—the number 1—and the solution of $(x-1)(x-2) = 3(x-1)$ with domain $\mathbb{R} - \{1\}$—the number 5—are the solutions of $(x-1)(x-2) = 3(x-1)$. To repeat, the set of solutions of the equation $(x-1)(x-2) = 3(x-1)$ is $\{1, 5\}$.



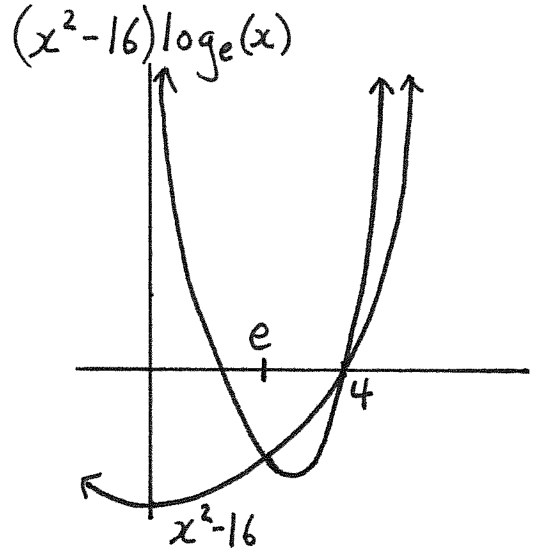
- Let's find the solutions of the equation $(x^2 - 16)\log_e(x) = (x^2 - 16)$.

(Step 1): The domain of the equation is $(0, \infty)$, because we can only take the logarithm of a positive number.

(Step 2): The zeros of $(x^2 - 16)$ are the solutions of $x^2 - 16 = 0$, or equivalently, the solutions of $x^2 = 16$, which are 4 and -4. However, of these two zeros, only 4 is in $(0, \infty)$, the domain of $(x^2 - 16) = (x^2 - 16)\log_e(x)$, so it's the only zero of $(x^2 - 16)$ that concerns us in this problem. With the notation of the general steps listed above, $Z = \{4\}$.

(Step 3): We need to find the solutions of $(x^2 - 16)\log_e(x) = (x^2 - 16)$ with the restricted domain $(0, \infty) - \{4\}$. The function $(x^2 - 16)$ has no zeros in the set $(0, \infty) - \{4\}$. Therefore, we can divide the equation $(x^2 - 16)\log_e(x) = (x^2 - 16)$ by $(x^2 - 16)$ to obtain the equivalent equation $\log_e(x) = 1$. The only solution of this equation is $x = e^1 = e$.

(Step 4): The solutions of $(x^2 - 16)\log_e(x) = (x^2 - 16)$ are 4 (from Step 2) and e (from Step 3).



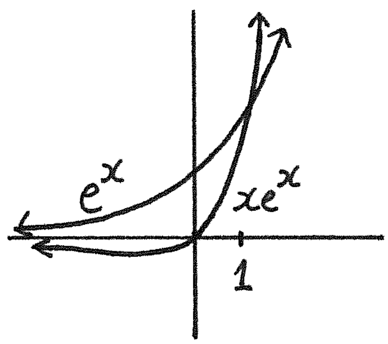
• Let's find the solutions of $xe^x = e^x$.

(Step 1): The implied domain is $\mathbb{R}$.

(Step 2): e^x has no zeros.

(Step 3): Dividing $xe^x = e^x$ by e^x yields the equivalent equation $x = 1$. Thus, 1 is the only solution.

(Step 4): There are no zeros from Step 2, so 1 is the only solution.



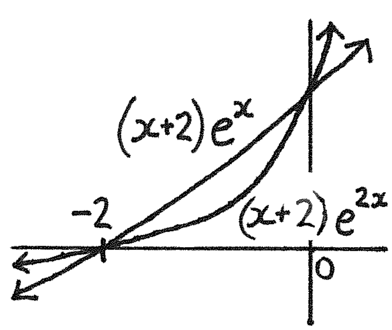
• Let's find the solutions of $(x + 2)e^{2x} = (x + 2)e^x$.

(Step 1): The implied domain is $\mathbb{R}$.

(Step 2): The zero of $(x + 2)$ is -2. It's in the domain of the equation, so it's a solution.

(Step 3): Dividing $(x + 2)e^{2x} = (x + 2)e^x$ by $(x + 2)$ gives us the equivalent equation $e^{2x} = e^x$. We can then divide by e^x to get $\frac{e^{2x}}{e^x} = 1$ which is the same equation as $e^{2x-x} = 1$ or more simply, $e^x = 1$. Applying the logarithm yields that $x = \log_e(1) = 0$.

(Step 4): -2 and 0 are the solutions of $(x + 2)e^{2x} = (x + 2)e^x$.



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Chapter Summary

The solutions of $h(x)f(x) = h(x)g(x)$
are the solutions of $f(x) = g(x)$
together with the solutions of $h(x) = 0$.

Exercises

For #1-6, find the zeros, if there are any, of the given functions.

1.) $x-2$

2.) $3x+4$

3.) x^2-81

4.) x^2+7

5.) $2\sqrt{x-7}$

6.) e^{3x-2}

Find the solutions of the following equations in one variable.

7.) $x^3\sqrt{x}=9x^3$

8.) $(x-2)(x^3-2x)^2=-5(x-2)$

9.) $(x-4)^2\log_e(x+2)=2(x-4)^2$

10.) $\log_e(x-1)e^x=-2\log_e(x-1)$

11.) $-8(x^2-25)=(x^2-25)[\log_e(x)^2-6\log_e(x)]$

12.) $(2x+1)^5=9(2x+1)^3$

13.) $(x^2-4)\sqrt{x}=-(x^2-4)$

14.) $e^x\left(\frac{1}{x}+x\right)=2e^x$

15.) $e^{3x}\sqrt{x}=e^{x-2}\sqrt{x}$

16.) $(x+3)\log_e(2x)=(x+3)\log_e(x+1)$

17.) $(x-4)^5=(x-4)^6$

18.) $(x-2)(x-3)=(x-2)(x+1)$

19.) $x=x^2$