Equations in One Variable V

This chapter is a review for solving logarithmic and exponential equations.

Inequalities

Here are four rules for inequalities that you should know.

If $a > b$, then:

- $a + d > b + d$
- $a - d > b - d$
- $ca > cb$ if $c > 0$
- $ca < cb$ if $c < 0$

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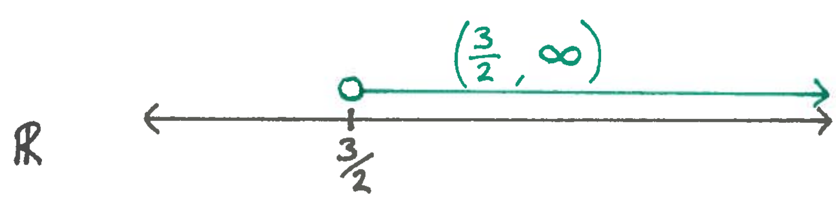
Implied domains for logarithmic functions

The domain of a logarithm is the set of positive numbers, $(0, \infty)$. A number has to be positive to take its logarithm.

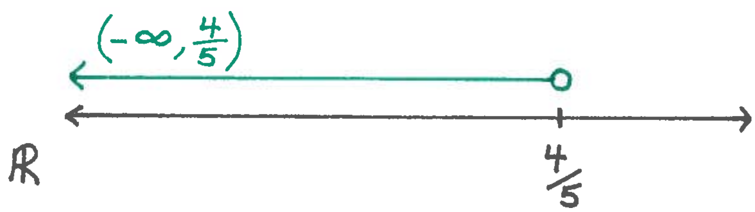
Examples.

- Let's find the implied domain of $\log_e(2x - 3)$. It only makes sense to take a logarithm of a number if it's positive, so the implied domain of $\log_e(2x - 3)$ is the set of all numbers x with the property that $2x - 3 > 0$.

Adding 3 to both sides of the inequality, we have $2x > 3$. Dividing by 2 we have $x > \frac{3}{2}$. That is, the implied domain of $\log_e(2x - 3)$ is the set of all numbers x with the property that $x > \frac{3}{2}$, or more succinctly, the implied domain of $\log_e(2x - 3)$ is $(\frac{3}{2}, \infty)$.



• The implied domain of $3\log_2(4-5x)$ is the set of all x with the property that $4-5x>0$. Writing that $4-5x>0$ is equivalent to writing $4>5x$ or that $\frac{4}{5}>x$. So the implied domain of $3\log_2(4-5x)$ is $(-\infty,\frac{4}{5})$.



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Implied domains for logarithmic equations

The implied domain of an equation is the set of all numbers that are in each of the domains of the functions involved in the equation.

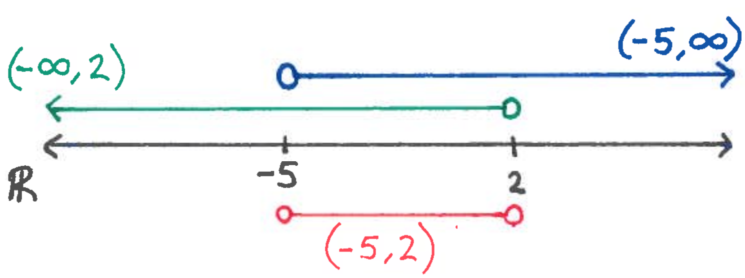
Examples.

• To find the implied domain of the equation

$$x=\log_e(x+5)+\log_e(2-x)$$

first list the implied domains of the functions in the equation. Those functions are x, $\log_e(x+5)$, and $\log_e(2-x)$, and their implied domains are $\mathbb{R}$, $(-5,\infty)$, and $(-\infty,2)$, respectively.

The implied domain of the equation is the set of numbers that are contained in each of the sets $\mathbb{R}$, $(-5,\infty)$, and $(-\infty,2)$, and that set is precisely the interval $(-5,2)$.



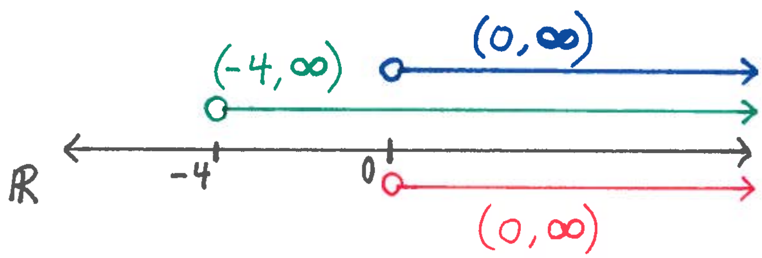
Thus, the implied domain of $x=\log_e(x+5)+\log_e(2-x)$ is $(-5,2)$.

- To find the implied domain of the equation

$$7\log_4(x) = \log_4(x+4)$$

first find the implied domains of the functions $7\log_4(x)$ and $\log_4(x+4)$. They are $(0,\infty)$ and $(-4,\infty)$ respectively.

The implied domain of the equation is the set of numbers that are in both $(0,\infty)$ and $(-4,\infty)$, which is the set $(0,\infty)$.



Therefore, $(0,\infty)$ is the implied domain of $7\log_4(x) = \log_4(x+4)$.

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Algebraic rules for exponentials and logarithms

These are the most important rules that you should know for the functions a^x and $\log_a(x)$. (Here a is a positive number, not equal to 1.)

exponentials	logarithms $(x, y > 0)$
• $a^x a^y = a^{x+y}$	• $\log_a(x) + \log_a(y) = \log_a(xy)$
• $\frac{a^x}{a^y} = a^{x-y}$	• $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$
• $(a^x)^y = a^{xy}$	• $\log_a(x^z) = z\log_a(x)$

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Solving exponential and logarithmic equations

Follow these three steps to solve an exponential or logarithmic equation.

First: find its domain.

Second: use the algebraic rules for exponentials and logarithms to make the equation look like either

$$a^{f(x)} = c \quad \text{ or } \quad \log_a(f(x)) = c$$

Third: use the techniques we've reviewed throughout the semester to solve the equation fully.

Problem. Find the set of solutions of the equation $5e^x = e^{3x}$.

Solution. First, the implied domain of this equation is $\mathbb{R}$.

Second, divide both sides by e^x:

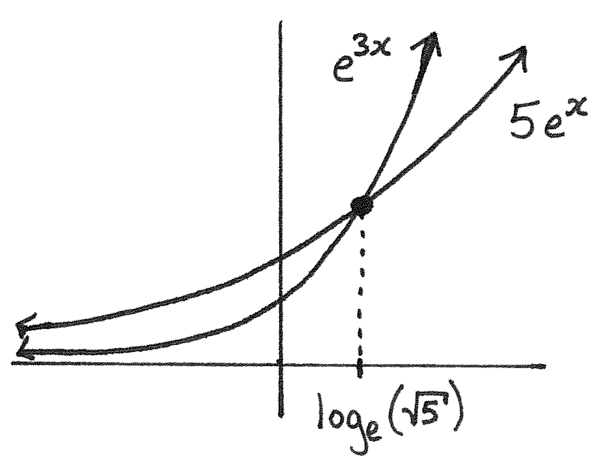
$$5 = \frac{e^{3x}}{e^x} = e^{3x-x} = e^{2x}$$

Third, we can erase the exponential base e from the right side of the equation if we apply a logarithm base e to the left side:

$$\log_e(5) = 2x$$

We can divide by 2 to see that the only solution is

$$x = \frac{1}{2}\log_e(5) = \log_e(5^{\frac{1}{2}}) = \log_e(\sqrt{5})$$



Problem. Find the set of solutions of the equation $(3^x)^5 = 81$.

Solution. First, the implied domain of this equation is $\mathbb{R}$.

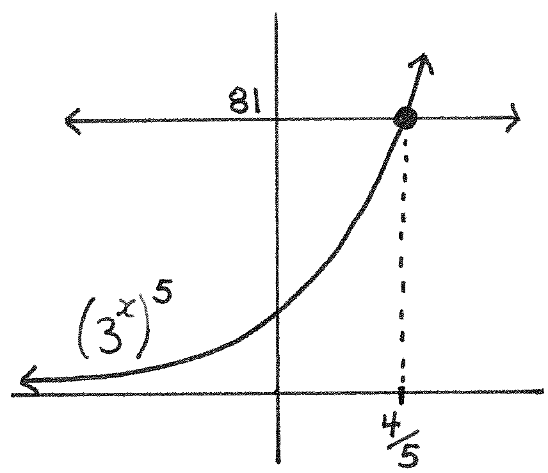
Second, rewrite $(3^x)^5$ as 3^{5x} so that we have the equation

$$3^{5x} = 81$$

Third, we can erase the exponential base 3 from the left side of the equation by applying the logarithm base 3 to the right side of the equation:

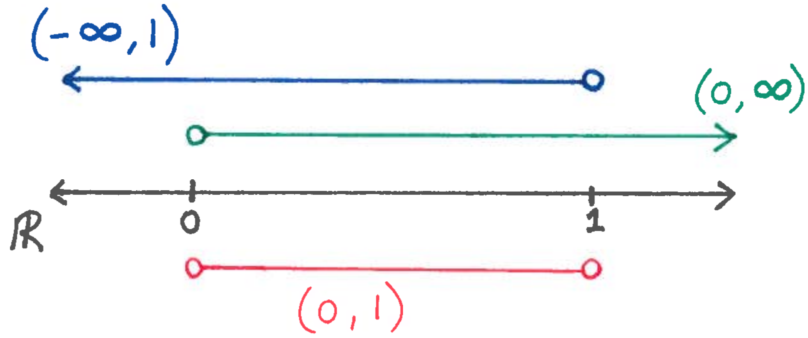
$$5x = \log_3(81) = 4$$

Thus, the solution is $x = \frac{4}{5}$.



Problem. Find the set of solutions of the equation $\log_e(x) - \log_e(1-x) = 0$.

Solution. First, we need to find the implied domain of the given equation. The implied domain of $\log_e(x)$ is $(0, \infty)$. The implied domain of $\log_e(1-x)$ is $(-\infty, 1)$. Thus, the implied domain of the equation is the set of numbers that are in both the interval $(0, \infty)$ and the interval $(-\infty, 1)$. Therefore, the implied domain of the equation is $(0, 1)$.



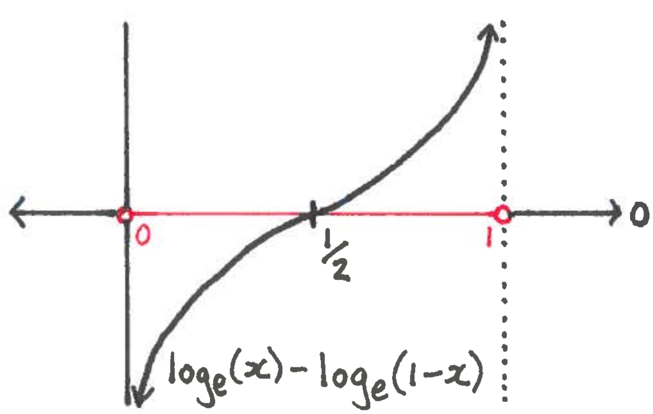
Second, we can use the algebraic rules for logarithms to rewrite $\log_e(x) - \log_e(1-x)$ as $\log_e\left(\frac{x}{1-x}\right)$. Thus, we are trying to find the solutions of

$$\log_e\left(\frac{x}{1-x}\right) = 0$$

Third, we can erase the logarithm base e on the left side of the equation by applying an exponential base e to the right side of the equation:

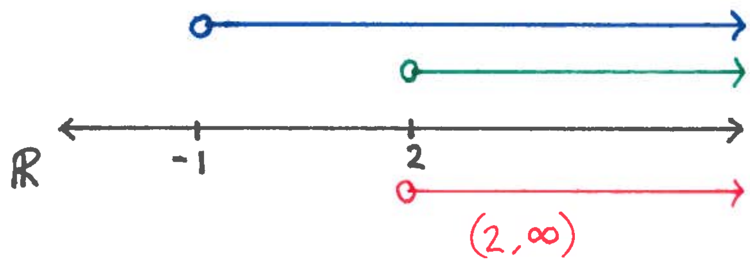
$$\frac{x}{1-x} = e^0 = 1$$

Multiply both sides by $1-x$ to get $x = 1-x$ and then add x to get $2x = 1$ so that $x = \frac{1}{2}$. Notice that $\frac{1}{2}$ is in the domain of our original equation, the set $(0, 1)$. Therefore, $\frac{1}{2}$ is indeed a solution.



Problem. Find the set of solutions of $\log_2(x-2) + \log_2(x+1) = 2$.

Solution. First, the implied domain of this equation is the set of all x with $x > 2$ and $x > -1$. That is, the implied domain is $(2, \infty)$.



Second, we know that

$$\log_2(x-2) + \log_2(x+1) = \log_2\big((x-2)(x+1)\big) = \log_2(x^2 - x - 2)$$

so we are trying to solve the equation

$$\log_2(x^2 - x - 2) = 2$$

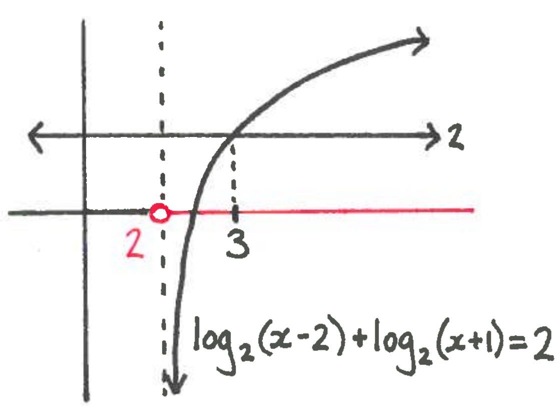
Third, the previous equation is equivalent to

$$x^2 - x - 2 = 2^2 = 4$$

or

$$x^2 - x - 6 = 0$$

Now the quadratic formula tells us that either $x = -2$ or $x = 3$. Only one of these solutions, 3, is in the domain of our original equation, $(2, \infty)$. Therefore, the only solution is $x = 3$.



Exercises

Find the set of solutions of the equations given in #1-10. (Remember that equations of the form $e^{f(x)} = c$ have no solution if $c \leq 0$.)

1.) $e^7 e^{x^2} = e^{3x}$

2.) $(e^{2x})^3 = 7$

3.) $\frac{e^{4x}}{e^x} = 2$

4.) $e^{3x-4} = -1$

5.) $e^{x^2}(e^x)^2 = e^3$

6.) $\log_e(3x+1) = 2$

7.) $\log_e(x-3) - \log_e(4x+2) = 0$

8.) $\log_3(x) + \log_3(x+2) = 1$

9.) $\log_e(x+1) + \log_e(x) = 0$

10.) $\log_e(2x-7) = \log_e(x+1)$

For #11-12, find the inverse of the given matrix.

11.) $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$

12.) $\begin{pmatrix} -3 & 4 \\ -1 & 2 \end{pmatrix}$