Equations in One Variable III

The functions x^3 and $\sqrt[3]{x}$ are inverses of each other. Thus, if $f(x)$ and $g(x)$ are functions in one variable, then the equations

$$f(x)^3 = g(x) \quad \text{and} \quad f(x) = \sqrt[3]{g(x)}$$

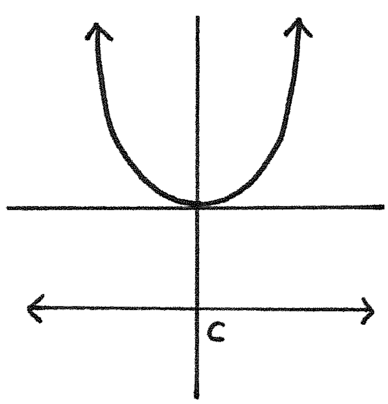
are equivalent by invertible function. As a consequence, many of the equations involving cubes or cube-roots that you'll come across can be solved using the methods described in the chapter "Equations in One Variable I".

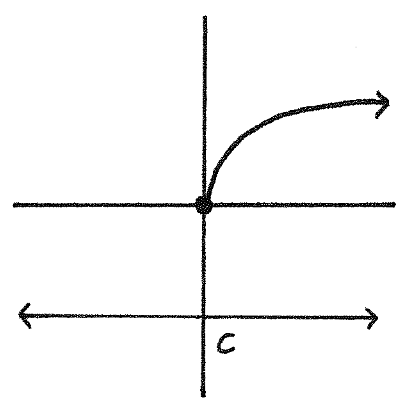
In contrast, x^2 and $\sqrt[2]{x}$ are not inverse functions, and finding solutions of equations involving squares and square-roots requires a special set rules. These rules are the topic of this chapter.

Squares and square-roots can't be negative

We saw the following rule in the chapter "Equations in One Variable II"

Equations of the form $f(x)^2 = c$ or $\sqrt{f(x)} = c$
have no solution if c is a negative number.

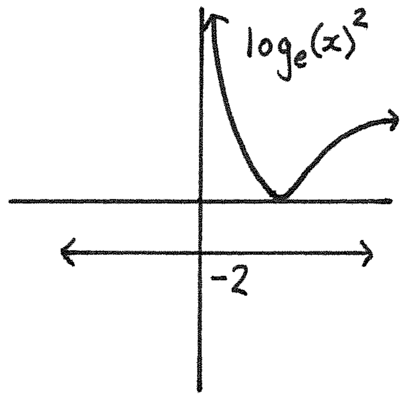


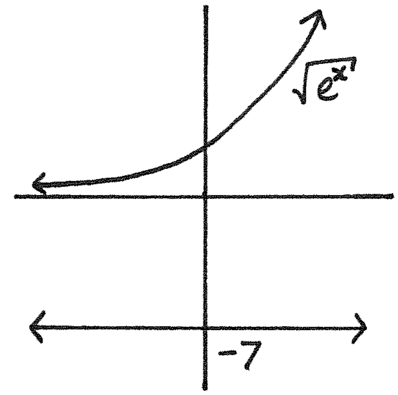


Examples.

- $\log_e(x)^2 = -2$ has no solutions, because -2 is negative and squares can't be negative.

- $\sqrt{e^x} = -7$ also has no solutions, because -7 is negative and square-roots can't be negative.





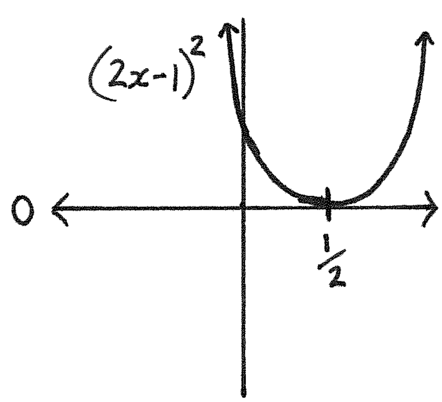
Squaring zero

There is only one number that you can square to obtain 0. That number is 0. $0^2 = 0$, and if we ever have that $x^2 = 0$, then we know that $x = 0$. It doesn't matter what x is in the previous sentence. It's just a variable. We could just as easily replace x with $f(x)$, and that's what is done in our second rule below.

If $f(x)^2 = 0$, then $f(x) = 0$.

Example.

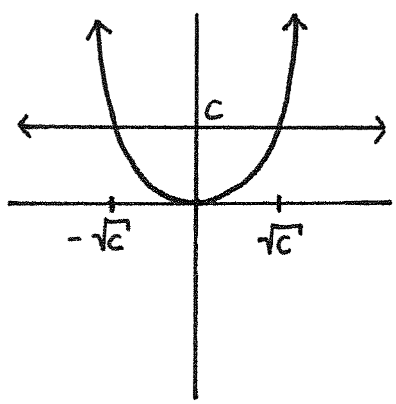
To find the solutions of $(2x-1)^2 = 0$, we first use the rule above which tells us that $2x - 1 = 0$. Add 1: $2x = 1$. Divide by 2: $x = \frac{1}{2}$. Thus, $\frac{1}{2}$ is the only solution of the equation $(2x-1)^2 = 0$.



Two solutions when squaring

The graph of the squaring function is a parabola. If you draw a horizontal line above the x-axis, it will intersect the parabola in two points. That means there are always two numbers you can square to obtain a given positive number. This observation is the foundation of our third rule.

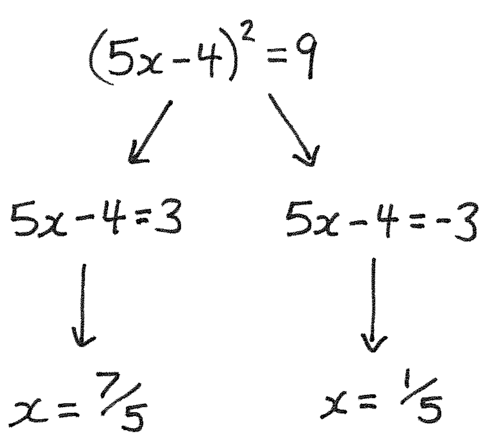
If $c > 0$ and $f(x)^2 = c$, then either
$$f(x) = \sqrt[2]{c} \quad \text{or} \quad f(x) = -\sqrt[2]{c}.$$

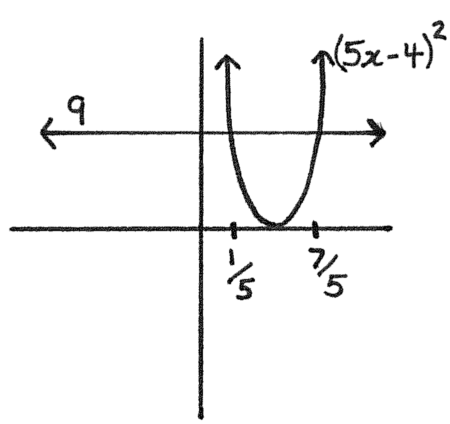


Example.

There are two numbers you can square to get 9: 3 and -3. So if we have the equation $(5x-4)^2 = 9$, then we know that either $5x - 4 = 3$ or $5x - 4 = -3$.

If $5x - 4 = 3$ then $x = \frac{7}{5}$. If $5x - 4 = -3$ then $x = \frac{1}{5}$. Therefore, $\frac{7}{5}$ and $\frac{1}{5}$ are both solutions of the equation $(5x-4)^2 = 9$.

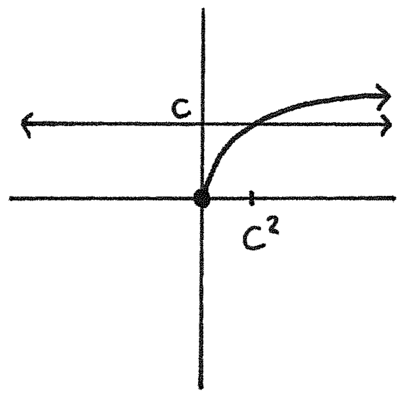




One solution when square-rooting

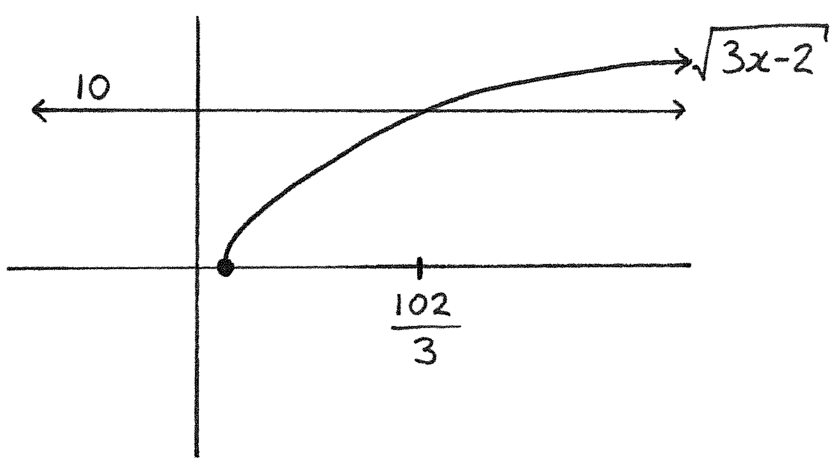
The rule on the previous page exists because the squaring function is not one-to-one. It is two-to-one, so equations involving squares often have two solutions. On the other hand, the square-root function is one-to-one. Equations involving square-roots will often have just a single solution. This is the last rule from this chapter.

If $c \geq 0$ and $\sqrt[2]{f(x)} = c$, then
$f(x) = c^2$.



Example.

To solve the equation $\sqrt{3x-2} = 10$, we use the rule above to see that $3x - 2 = 10^2 = 100$. Then we add 2 and divide by 3 to see that $x = \frac{102}{3}$. This is the only solution.



Exercises

Find the set of solutions of the following equations in one variable. Notice that each of these equations are of the form $f(x)^2 = c$ or $\sqrt{f(x)} = c$.

1.) $(e^x)^2 = 9$

2.) $\sqrt{e^{3x+2}} = 5$

3.) $\log_e(x)^2 = 5$

4.) $(3x+5)^2 = -1$

5.) $\sqrt{x^2 - 2x - 1} = 2$

6.) $\sqrt{\log_e(3x^2) + 27e^x} = -2$

7.) $(2x^2 - 3x - 2)^2 = 1$

8.) $(4 - 7x)^2 = 0$